DEPARTAMENTO DE MATEMÁTICA APLICADA

SHARK-FV

Grupo de Investigación en Ingeniería Matemática

# Time and space dependant advection coefficients: a new upwind term

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# Incompressible Newtonian fluids. $k-\varepsilon$ turbulence model

$$\mathrm{div}\,(\rho\mathbf{u}) = 0,$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) + \mathrm{grad}\,\pi - \mathrm{div}\,\tau = \rho\mathbf{g},$$

$$\tau = (\mu+\mu_t)\left(\mathrm{grad}\,\mathbf{u} + \mathrm{grad}\,\mathbf{u}^T\right) - \frac{2}{3}\rho k I, \qquad \mu_t = \rho C_\mu \frac{k^2}{\varepsilon},$$

$$\frac{\partial\rho k}{\partial t} + \mathrm{div}\,(\mathbf{u}\rho k) - \mathrm{div}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\mathrm{grad}\,k\right] - G_k + \rho\varepsilon = 0,$$

$$\frac{\partial\rho\varepsilon}{\partial t} + \mathrm{div}\,(\mathbf{u}\rho\varepsilon) - \mathrm{div}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\mathrm{grad}\,\varepsilon\right] = C_{1\varepsilon}\frac{\varepsilon}{k}G_k - C_{2\varepsilon}\rho\frac{\varepsilon^2}{k}.$$

- $\rho \in \mathbb{R}$: density.
- $\mathbf{u} = \mathbf{u}(x,y,z,t)$: velocity vector.
- $\pi = \pi(x,y,z,t)$: pressure.
- $\mathbf{g} = \mathbf{g}(x,y,z,t)$: gravity force.
- $\tau$: viscous stress tensor.
- $\mu$: laminar viscosity.
- $\mu_t$: turbulent viscosity.
- $k$: turbulent kinetic energy.
- $\varepsilon$: dissipation rate.
- $G_k$: turbulent production.
- $\sigma_k$, $\sigma_\varepsilon$: Prandtl numbers.
- $C_\mu$, $C_{1,\varepsilon}$, $C_{2,\varepsilon}$: closure constants.

# Compressible low Mach number flows

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{u}) = 0, \quad \overline{\pi} = \rho R \theta, \quad R = \mathcal{R} \sum_{i=1}^{N_e} \frac{\mathrm{y}_i}{\mathcal{M}_i}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \mathrm{div}\,(\rho \mathbf{u} \otimes \mathbf{u}) + \mathrm{grad}\,\pi - \mathrm{div}\,\tau = \rho \mathbf{g},$$

$$\tau = \mu \left(\mathrm{grad}\,\mathbf{u} + \mathrm{grad}\,\mathbf{u}^T\right) - \frac{2}{3}\mu \mathrm{div}\,(\mathbf{u})\, I,$$

$$\frac{\partial (\rho \mathrm{y}_i)}{\partial t} + \mathrm{div}\,(\rho \mathrm{y}_i \mathbf{u}) - \mathrm{div}\,(\rho \mathcal{D}_i \mathrm{grad}\,\mathrm{y}_i) = 0, \quad i = 1, \ldots, N_e,$$

$$\frac{\partial (\rho h)}{\partial t} + \mathrm{div}\,(\rho h \mathbf{u}) - \mathrm{div}\,(\rho \mathcal{D} \mathrm{grad}\,h) = 0.$$

- $\rho = \rho(x, y, z, t)$: density.
- $p = p(x, y, z, t)$: pressure,

  $p(x, y, z, t) = \overline{\pi}(t) + \pi(x, y, z, t)$.
- $y_i = y_i(x,y,z,t)$: mass fraction.
- $\mathcal{D}$: mass diffusivity coefficient.
- $h$: enthalpy.
- $\theta$: temperature.
- $\mathcal{R}$: universal constant for perfect gases.
- $\mathcal{M}_i$: molecular mass of species $y_i$.

## Navier-Stokes momentum equation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \mathrm{div}\left(F(\mathbf{u}, \rho)\right) = \mathrm{div}\tau - \mathrm{grad}\pi + \mathbf{f_u}$$

$$\partial_t q(x,t) + \partial_x f\left(q(x,t), \lambda(x,t)\right) = \partial_x\left(\alpha(x,t)\partial_x q(x,t)\right) + \beta q(x,t)$$

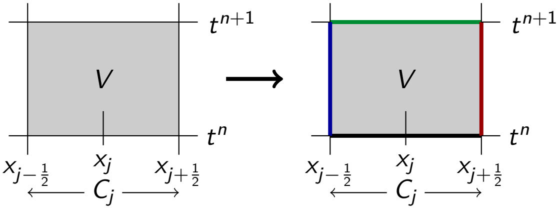
$$\beta \in \mathbb{R}^-$$

## Advection-diffusion-reaction equation

# Advection-diffusion-reaction equation

# The finite volume framework

$$\partial_t q(x,t) + \partial_x f(q(x,t), \lambda(x,t)) = \partial_x \left(\alpha(x,t)\partial_x q(x,t)\right) + \beta q(x,t).$$




Exact integration in the control volume $V$ gives

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left[q(x,t^{n+1}) - q(x,t^n)\right] dx + \int_{t^n}^{t^{n+1}} \left[f(q(x_{j+\frac{1}{2}},t), \lambda(q(x_{j+\frac{1}{2}},t)) - f(q(x_{j-\frac{1}{2}},t), \lambda(x_{j-\frac{1}{2}},t))\right] dt$$

$$= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left[\int_{t^n}^{t^{n+1}} \partial_x \left(\alpha \partial_x q\right)(x,t) \ dt\right] dx + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left[\int_{t^n}^{t^{n+1}} \beta q(x,t)\, dt\right] dx.$$

Let us introduce the following notation

$$q_j^{n+1} = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} q(x, t^{n+1})\,dx, \quad q_j^n = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} q(x, t^n)\,dx,$$

$$f_{j-\frac{1}{2}}^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(q(x_{j-\frac{1}{2}}, t), \lambda(x_{j-\frac{1}{2}}, t)\right) dt,$$

$$f_{j+\frac{1}{2}}^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(q(x_{j+\frac{1}{2}}, t), \lambda(x_{j+\frac{1}{2}}, t)\right) dt,$$

$$g_j^n = \frac{1}{\Delta t \Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left[\int_{t^n}^{t^{n+1}} \partial_x\left(\alpha \partial_x q\right)(x, t)\,dt\right] dx,$$

$$s_j^n = \frac{1}{\Delta t \Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left[\int_{t^n}^{t^{n+1}} \beta q(x, t)\,dt\right] dx.$$

Then, we get the exact relation

$$q_j^{n+1} = q_j^n - \frac{\Delta t}{\Delta x}\left(f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n\right) + \Delta t g_j^n + \Delta t s_j^n.$$

# Rusanov flux and Godunov's theorem

Assuming that the advection coefficient is constant, the physical flux is approximated by means of the Rusanov numerical flux function[1]:

$$f^n_{j+\frac{1}{2}} \simeq \phi(q^n_j, q^n_{j+1}) = \frac{f(q^n_{j+1}) + f(q^n_j)}{2} - \frac{1}{2}\alpha_{RS}(q^n_j, q^n_{j+1})(q^n_{j+1} - q^n_j).$$

## Godunov's theorem[2]

There are no monotone, linear schemes for the advection equation of second or higher order of accuracy.

To attain a second order accuracy scheme we will consider Local ADER methodology.

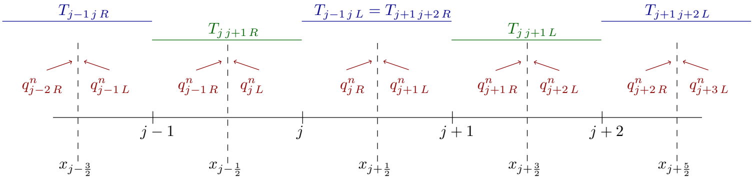
[1] V.V. Rusanov. "The calculation of the interaction of non-stationary shock waves and obstacles". USSR Comp. Math. Math. Phys., 1 304–320, 1962.

[2] S.K. Godunov. "A finite difference method for the computation of discontinuous solutions of the equations of fluid dynamics.". Mat. Sb., 47, 357–393, 1959.

# LADER methodology

Step 1. **Polynomial reconstruction**

$$p_j(x) = \begin{cases} p_{j\,L}(x) = q_j^n + \Delta_{j\,L}^n (x - x_j), & \text{if } x \in \left(x_{j-\frac{1}{2}}, x_j\right], \\ p_{j\,R}(x) = q_j^n + \Delta_{j\,R}^n (x - x_j), & \text{if } x \in \left[x_j, x_{j+\frac{1}{2}}\right). \end{cases}$$



# LADER methodology

Step 2. Solution of the generalized Riemann problem

$$\begin{cases} \partial_t q\left(x,t\right) + \lambda \partial_x q\left(x,t\right) = \partial_x\left(\alpha \partial_x q\right)\left(x,t\right) + \beta q\left(x,t\right), \\ q(x,0) = \begin{cases} p_{j\,R}(x), & \text{if } x < 0, \\ p_{j+1\,L}(x), & \text{if } x > 0. \end{cases} \end{cases}$$

Step 3. Computation of diffusion and reaction terms

## Flux term

Considering a Taylor series expansion in time and Cauchy-Kovalevskaya procedure, we get the expression of the evolved conservative variable

$$\overline{q}^n_{j+\frac{1}{2}} = q(0,0_+) + \tau \left[ -\lambda \partial_x q(0,0_+) + \partial_x \left( \alpha \partial_x q \right) (0,0_+) + \beta q(0,0_+) \right].$$

where the leading term and the spatial derivatives are approximated as

$$q\left(0,0_+\right) = \begin{cases} q^n_{j\,R} = q^n_j + \frac{1}{2}\left(q^n_j - q^n_{j-1}\right), & \text{if } \lambda > 0, \\ q^n_{j+1\,L} = q^n_{j+1} - \frac{1}{2}\left(q^n_{j+2} - q^n_{j+1}\right), & \text{if } \lambda < 0. \end{cases}$$

$$\partial_x q(0,0_+) = \Delta^n_{j+\frac{1}{2}} \approx \frac{1}{\Delta x}\left(q^n_{j+1} - q^n_j\right),$$

$$\partial_x \left( \alpha \partial_x q \right)\left(0,0_+\right) = \left(\Delta \alpha \Delta\right)^n_{j+\frac{1}{2}} \approx \frac{1}{\Delta x^2}\left[\alpha^n_{j+1}\left(q^n_{j+2} - q^n_{j+1}\right) - \alpha^n_j\left(q^n_j - q^n_{j-1}\right)\right].$$

Performing exact integration, the numerical flux becomes ($\lambda > 0$)

$$
\begin{aligned}
f^n_{j+\frac{1}{2}} =& \lambda \overline{q^n_{j+\frac{1}{2}}} = \lambda \left\{ {q^n_j}_R + \frac{\Delta t}{2} \left[ -\lambda \Delta^n_{j+\frac{1}{2}} + (\Delta \alpha \Delta)^n_{j+\frac{1}{2}} + \beta q^n_j \right] \right\} \\
=& \lambda \left\{ q^n_j + \frac{1}{2} \left( q^n_j - q^n_{j-1} \right) - \frac{\lambda \Delta t}{2 \Delta x} \left( q^n_{j+1} - q^n_j \right) \right. \\
& \left. + \frac{\Delta t}{2 \Delta x^2} \left[ \alpha^n_{j+1} \left( q^n_{j+2} - q^n_{j+1} \right) - \alpha^n_j \left( q^n_j - q^n_{j-1} \right) \right] + \beta \frac{\Delta t}{2} q^n_j \right\} .
\end{aligned}
$$

## Diffusion term

For the diffusion term computation we build new evolved variables depending on diffusion and reaction terms:

$$\overline{(\Delta\alpha\Delta)}_j^n = \frac{\overline{\alpha}_{j+\frac{1}{2}}^n \overline{\Delta}_{j+\frac{1}{2}}^n - \overline{\alpha}_{j-\frac{1}{2}}^n \overline{\Delta}_{j-\frac{1}{2}}^n}{\Delta x} = \frac{1}{\Delta x^2}\left[\overline{\alpha}_{j+\frac{1}{2}}^n \left(\overline{q}_{j+1}^n - \overline{q}_j^n\right) - \overline{\alpha}_{j-\frac{1}{2}}^n \left(\overline{q}_j^n - \overline{q}_{j-1}^n\right)\right]$$
$$= \frac{1}{\Delta x^2}\left\{\left[\alpha_{j+\frac{1}{2}}^n + \frac{\Delta t}{2}\partial_t \alpha_{j+\frac{1}{2}}^n\right]\left[q_{j+1}^n - q_j^n + \frac{\Delta t}{2}\left((\Delta\alpha\Delta)_{j+1}^n - (\Delta\alpha\Delta)_j^n + \beta\left(q_{j+1}^n - q_j^n\right)\right)\right]\right.$$
$$\left. + \left[\alpha_{j-\frac{1}{2}}^n + \frac{\Delta t}{2}\partial_t \alpha_{j-\frac{1}{2}}^n\right]\left[q_{j-1}^n - q_j^n + \frac{\Delta t}{2}\left((\Delta\alpha\Delta)_{j-1}^n - (\Delta\alpha\Delta)_j^n + \beta\left(q_{j-1}^n - q_j^n\right)\right)\right]\right\}.$$

## Reaction term

The reaction term is calculated like for ADER scheme,

$$\beta\overline{q}_j^n = \beta\left[q_j^n + \frac{\Delta t}{2}\left(-\lambda\Delta_j^n + (\Delta\alpha\Delta)_j^n + \beta q_j^n\right)\right].$$

# Flux term with variable coefficient

$$f(q(x,t),\lambda(x,t)) = \lambda(x,t)q(x,t)$$

Two main issues must be taken into account with respect to the advection equation with constant coefficient:

- A new numerical viscosity related to the spatial derivative of $\lambda(x,t)$ should be included.[3]
- To build a second-order in time and space scheme using LADER methodology, the extrapolation and the half in time evolution of $\lambda(x,t)$ need to be performed.

[3] A. Bermúdez, X. López, M.E. Vázquez-Cendón "Finite volume methods for multi-component Euler equations with source terms". Comput. Fluids, 156, 113–134, 2017.

New numerical viscosity related to the spatial derivative of $\lambda(x,t)$.

Rusanov flux function is divided into two terms:

$$\phi(q_j^n, q_{j+1}^n, \lambda_j^n, \lambda_{j+1}^n) = \frac{1}{2}\left(\lambda_j^n q_j^n + \lambda_{j+1}^n q_{j+1}^n\right) - \frac{1}{2}\alpha_{RS}\left(q_j^n, q_{j+1}^n, \lambda_j^n, \lambda_{j+1}^n\right)\left(q_{j+1}^n - q_j^n\right).$$

The second one is supposed to introduce the numerical viscosity needed for the stability of the scheme. However, splitting the spatial derivative of the flux into two terms,

$$\partial_x\left(\lambda q\right)(x,t) = \lambda(x,t)\partial_x q(x,t) + q(x,t)\partial_x \lambda(x,t),$$

we notice that Rusanov flux only adds the artificial viscosity related to the first one:

$$\alpha_{RS}\left(q_j^n, q_{j+1}^n, \lambda_j^n, \lambda_{j+1}^n\right) = \max\left\{|\lambda_j^n|, |\lambda_{j+1}^n|\right\}.$$

To correct this lack of upwind, we propose to introduce a new artificial viscosity term,

$$-\left[\partial_\lambda\left(\lambda q\right)\Delta\lambda\right]\big|_{j+\frac{1}{2}} \approx -\frac{1}{2}\mathrm{sign}\left(\breve{\alpha}_{RS}\right) q^n_{j+\frac{1}{2}}\left(\lambda^n_{j+1}-\lambda^n_j\right)$$

with $\breve{\alpha}_{RS}$ the value of the eigenvalue used to compute $\alpha_{RS}$, that is, $\breve{\alpha}_{RS}=\lambda^n_j$ or $\breve{\alpha}_{RS}=\lambda^n_{j+1}$. Then, the new numerical flux on the boundary $x_{j+\frac{1}{2}}$ reads

$$\begin{aligned}\phi(q^n_j, q^n_{j+1}, \lambda^n_j, \lambda^n_{j+1}) =&\frac{1}{2}\left(\lambda^n_j q^n_j + \lambda^n_{j+1} q^n_{j+1}\right)\\ &-\frac{1}{2}\alpha_{RS}\left(q^n_j, q^n_{j+1}, \lambda^n_j, \lambda^n_{j+1}\right)\left(q^n_{j+1}-q^n_j\right)\\ &-\frac{1}{2}\mathrm{sign}\left(\breve{\alpha}_{RS}\left(q^n_j, q^n_{j+1}, \lambda^n_j, \lambda^n_{j+1}\right)\right) q^n_{j+\frac{1}{2}}\left(\lambda^n_{j+1}-\lambda^n_j\right).\end{aligned}$$

To build a second-order in time and space scheme using LADER methodology.

The evolved values of the conservative variable and the advection coefficient read

$$\overline{q}^n_{j-1R} = q^n_{j-1R} - \frac{\Delta t}{2\Delta x}\left(\lambda^n_j q^n_j - \lambda^n_{j-1} q^n_{j-1}\right) + \frac{\Delta t}{2\Delta x^2}\left[\alpha^n_j\left(q^n_{j+1} - q^n_j\right) - \alpha^n_{j-1}\left(q^n_{j-1} - q^n_{j-2}\right)\right] + \frac{\Delta t}{2} s^n_{j-\frac{1}{2}},$$

$$\overline{q}^n_{jL} = q^n_{jL} - \frac{\Delta t}{2\Delta x}\left(\lambda^n_j q^n_j - \lambda^n_{j-1} q^n_{j-1}\right) + \frac{\Delta t}{2\Delta x^2}\left[\alpha^n_j\left(q^n_{j+1} - q^n_j\right) - \alpha^n_{j-1}\left(q^n_{j-1} - q^n_{j-2}\right)\right] + \frac{\Delta t}{2} s^n_{j-\frac{1}{2}},$$

$$\overline{q}^n_{jR} = q^n_{jR} - \frac{\Delta t}{2\Delta x}\left(\lambda^n_{j+1} q^n_{j+1} - \lambda^n_j q^n_j\right) + \frac{\Delta t}{2\Delta x^2}\left[\alpha^n_{j+1}\left(q^n_{j+2} - q^n_{j+1}\right) - \alpha^n_j\left(q^n_j - q^n_{j-1}\right)\right] + \frac{\Delta t}{2} s^n_{j+\frac{1}{2}},$$

$$\overline{q}^n_{j+1L} = q^n_{j+1L} - \frac{\Delta t}{2\Delta x}\left(\lambda^n_{j+1} q^n_{j+1} - \lambda^n_j q^n_j\right) + \frac{\Delta t}{2\Delta x^2}\left[\alpha^n_{j+1}\left(q^n_{j+2} - q^n_{j+1}\right) - \alpha^n_j\left(q^n_j - q^n_{j-1}\right)\right] + \frac{\Delta t}{2} s^n_{j+\frac{1}{2}},$$

On the other hand, we extrapolate and compute the half in time evolved values of the advection coefficient, it is necessary to extrapolate and compute the half in time evolved values of the advection coefficient,

$$
\begin{aligned}
\overline{\lambda}_{j-1R}^{n} &= \lambda_{j-1R}^{n} + \frac{\Delta t}{2}\partial_t \lambda_{j-\frac{1}{2}}^{n} = \lambda_j^n + \frac{1}{2}\left(\lambda_{j-1}^{n} - \lambda_{j-2}^{n}\right) + \frac{\Delta t}{2}\partial_t \lambda_{j-\frac{1}{2}}^{n},\\
\overline{\lambda}_{jL}^{n} &= \lambda_{jL}^{n} + \frac{\Delta t}{2}\partial_t \lambda_{j-\frac{1}{2}}^{n} = \lambda_j^n - \frac{1}{2}\left(\lambda_{j+1}^{n} - \lambda_{j}^{n}\right) + \frac{\Delta t}{2}\partial_t \lambda_{j-\frac{1}{2}}^{n},\\
\overline{\lambda}_{jR}^{n} &= \lambda_{jR}^{n} + \frac{\Delta t}{2}\partial_t \lambda_{j+\frac{1}{2}}^{n} = \lambda_j^n + \frac{1}{2}\left(\lambda_{j}^{n} - \lambda_{j-1}^{n}\right) + \frac{\Delta t}{2}\partial_t \lambda_{j+\frac{1}{2}}^{n},\\
\overline{\lambda}_{j+1L}^{n} &= \lambda_{j+1L}^{n} + \frac{\Delta t}{2}\partial_t \lambda_{j+\frac{1}{2}}^{n} = \lambda_j^n - \frac{1}{2}\left(\lambda_{j+2}^{n} - \lambda_{j+1}^{n}\right) + \frac{\Delta t}{2}\partial_t \lambda_{j+\frac{1}{2}}^{n}.
\end{aligned}
$$

Finally, the scheme for the advection-diffusion-reaction equation with variable advection and diffusion coefficients is

$$
\begin{aligned}
q_j^{n+1} = & q_j^n - \frac{\Delta t}{2\Delta x}\left\{\left[\left(\overline{\lambda}_{jR}^n \overline{q}_{jR}^n + \overline{\lambda}_{j+1L}^n \overline{q}_{j+1L}^n\right) - \overline{\alpha}_{RS,\,j+\frac{1}{2}}^n \left(\overline{q}_{j+1L}^n - \overline{q}_{jR}^n\right)\right.\right. \\
& -\mathrm{sign}\left(\breve{\overline{\alpha}}_{RS,\,j+\frac{1}{2}}^n\right)\overline{q}_{j+\frac{1}{2}}^n\left(\overline{\lambda}_{j+1L}^n - \overline{\lambda}_{jR}^n\right)\Big] - \Big[\left(\overline{\lambda}_{j-1R}^n \overline{q}_{j-1R}^n + \overline{\lambda}_{jL}^n \overline{q}_{jL}^n\right) \\
& \left.\left. -\overline{\alpha}_{RS,\,j-\frac{1}{2}}^n \left(\overline{q}_{jL}^n - \overline{q}_{j-1R}^n\right) - \mathrm{sign}\left(\breve{\overline{\alpha}}_{RS,\,j-\frac{1}{2}}^n\right)\overline{q}_{j-\frac{1}{2}}^n\left(\overline{\lambda}_{jL}^n - \overline{\lambda}_{j-1R}^n\right)\right]\right\} \\
& + \frac{\Delta t}{\Delta x^2}\left\{\left[\alpha_{j+\frac{1}{2}}^n + \frac{\Delta t}{2}\partial_t \alpha_{j+\frac{1}{2}}^n\right]\left[q_{j+1}^n - q_j^n + \frac{\Delta t}{2\Delta x^2}\left(\alpha_{j+\frac{3}{2}}^n\left(q_{j+2}^n - q_{j+1}^n\right)\right.\right.\right. \\
& \left.\left. -2\alpha_{j+\frac{1}{2}}^n\left(q_{j+1}^n - q_j^n\right) + \alpha_{j-\frac{1}{2}}^n\left(q_j^n - q_{j-1}^n\right)\right) + \frac{\Delta t}{2}\left(s_{j+1}^n - s_j^n\right)\right] \\
& + \left[\alpha_{j-\frac{1}{2}}^n + \frac{\Delta t}{2}\partial_t \alpha_{j-\frac{1}{2}}^n\right]\left[q_{j-1}^n - q_j^n + \frac{\Delta t}{2\Delta x^2}\left(-\alpha_{j+\frac{1}{2}}^n\left(q_{j+1}^n - q_j^n\right)\right.\right. \\
& \left.\left.\left. +2\alpha_{j-\frac{1}{2}}^n\left(q_j^n - q_{j-1}^n\right) - \alpha_{j-\frac{3}{2}}^n\left(q_{j-1}^n - q_{j-2}^n\right)\right) + \frac{\Delta t}{2}\left(s_{j-1}^n - s_j^n\right)\right]\right\} + s_j^{n+\frac{1}{2}}
\end{aligned}
$$

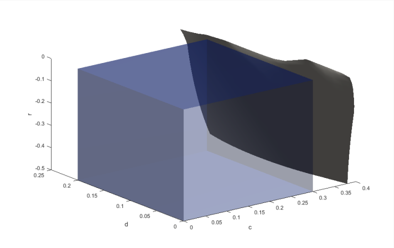
with $s_j^n = s\left(x_j, t^n\right)$ and $s_j^{n+\frac{1}{2}} = s\left(x_j, t^n + \frac{\Delta t}{2}\right)$.

# Stability analysis

LADER scheme for the advection equation is conditionally stable with stability condition $|c| \leq \sqrt{2} - 1$.

There exist $c_M,\ d_M \in \mathbb{R}^+,\ r_m \in \mathbb{R}^-$ such that LADER scheme for the advection-diffusion-reaction equation is stable in the 4-orthotopes

$$O_{c_M,d_M,r_m} = \{(\theta, c, r, d)\,|\,\theta \in [-\pi,\pi],\ c \in [0, c_M],\ d \in [0, d_M],\ r \in [r_m, 0],\ c_M, d_M \in \mathbb{R}^+,\ r_m \in \mathbb{R}^-\}\,.$$




S. Busto, J.L. Ferrín, E.F. Toro, M.E. Vázquez-Cendón. "A projection hybrid high order finite volume/finite element method for incompressible turbulent flows". J. Comput. Phys., 353, 169–192, 2018.

# Accuracy analysis

LADER scheme is second-order in space and time.

A. Bermúdez, S. Busto, J.L. Ferrín, M.E. Vázquez-Cendón. "A high order projection method for low Mach number flows". Submitted.

# Test 1. Advection-diffusion-reaction equation

- Computational domain: $\Omega = [0, 2]$.
- Time interval: $[0, 1]$.
- Flow definition:

$$\partial_t q(x,t) + \partial_x \left[\lambda(x,t) q\left(x,t\right)\right] = s(x,t),$$

$$q(x,0) = e^{-2x^2},$$

$$\lambda(x,t) = x + 2,$$

$$s(x,t) = 4(x-t)(-1-x)e^{-2(x-t)^2-t},$$

$$q(x,t) = e^{-2(x-t)^2-t}.$$

| Cells | $\mathrm{Err}_{\mathcal{L}^1}$ | $\mathcal{O}_{\mathcal{L}^1}$ | $\mathrm{Err}_{\mathcal{L}^2}$ | $\mathcal{O}_{\mathcal{L}^2}$ | $\mathrm{Err}_{\mathcal{L}^\infty}$ | $\mathcal{O}_{\mathcal{L}^\infty}$ |
|---|---|---|---|---|---|---|
| 8 | $8.87E-02$ | | $8.71E-02$ | | $1.13E-01$ | |
| 16 | $5.36E-02$ | 0.73 | $5.26E-02$ | 0.73 | $6.98E-02$ | 0.69 |
| 32 | $2.95E-02$ | 0.86 | $2.88E-02$ | 0.87 | $3.87E-02$ | 0.85 |
| 64 | $1.55E-02$ | 0.93 | $1.50E-02$ | 0.94 | $2.03E-02$ | 0.93 |
| 128 | $7.98E-03$ | 0.96 | $7.69E-03$ | 0.97 | $1.04E-02$ | 0.96 |
| 256 | $4.04E-03$ | 0.98 | $3.89E-03$ | 0.98 | $5.28E-03$ | 0.98 |
| 512 | $2.03E-03$ | 0.99 | $1.96E-03$ | 0.99 | $2.65E-03$ | 0.99 |

Test 1. Errors and convergence rates obtained by using the first order scheme.
$\Omega = [0, 2]$, $t_{\mathrm{end}} = 1$, $c = c_M = 0.5$.

| Cells | $\mathrm{Err}_{\mathcal{L}^1}$ | $\mathcal{O}_{\mathcal{L}^1}$ | $\mathrm{Err}_{\mathcal{L}^2}$ | $\mathcal{O}_{\mathcal{L}^2}$ | $\mathrm{Err}_{\mathcal{L}^\infty}$ | $\mathcal{O}_{\mathcal{L}^\infty}$ |
|---|---|---|---|---|---|---|
| 8 | $8.34E-02$ | | $8.20E-02$ | | $1.07E-01$ | |
| 16 | $5.13E-02$ | 0.70 | $5.03E-02$ | 0.71 | $6.65E-02$ | 0.68 |
| 32 | $2.88E-02$ | 0.83 | $2.79E-02$ | 0.85 | $3.73E-02$ | 0.83 |
| 64 | $1.53E-02$ | 0.91 | $1.47E-02$ | 0.92 | $1.98E-02$ | 0.92 |
| 128 | $7.92E-03$ | 0.95 | $7.55E-03$ | 0.96 | $1.02E-02$ | 0.96 |
| 256 | $4.02E-03$ | 0.98 | $3.83E-03$ | 0.98 | $5.17E-03$ | 0.98 |
| 512 | $2.03E-03$ | 0.99 | $1.93E-03$ | 0.99 | $2.60E-03$ | 0.99 |

Test 1. Errors and convergence rates obtained by using the first order scheme without the new numerical viscosity term. $\Omega = [0, 2]$, $t_{\mathrm{end}} = 1$, $c = c_M = 0.5$.

| Cells | $\mathrm{Err}_{\mathcal{L}^1}$ | $\mathcal{O}_{\mathcal{L}^1}$ | $\mathrm{Err}_{\mathcal{L}^2}$ | $\mathcal{O}_{\mathcal{L}^2}$ | $\mathrm{Err}_{\mathcal{L}^\infty}$ | $\mathcal{O}_{\mathcal{L}^\infty}$ |
|---|---|---|---|---|---|---|
| 8 | $5.73E-02$ | | $5.80E-02$ | | $8.73E-02$ | |
| 16 | $2.10E-02$ | 1.45 | $2.02E-02$ | 1.52 | $3.54E-02$ | 1.30 |
| 32 | $6.58E-03$ | 1.67 | $6.42E-03$ | 1.66 | $1.20E-02$ | 1.56 |
| 64 | $2.57E-03$ | 1.36 | $2.61E-03$ | 1.30 | $8.93E-03$ | 0.43 |
| 128 | $1.44E-03$ | 0.83 | $1.55E-03$ | 0.75 | $8.08E-03$ | 0.14 |
| 256 | $7.62E-04$ | 0.92 | $9.26E-04$ | 0.74 | $7.71E-03$ | 0.07 |
| 512 | $3.92E-04$ | 0.96 | $5.74E-04$ | 0.69 | $7.54E-03$ | 0.03 |

Test A1. Errors and convergence rates obtained by using LADER scheme without the new numerical viscosity term. $\Omega = [0, 2]$, $t_{\mathrm{end}} = 1$, $c = c_M = 0.5$.

| Cells | $\mathrm{Err}_{\mathcal{L}^1}$ | $\mathcal{O}_{\mathcal{L}^1}$ | $\mathrm{Err}_{\mathcal{L}^2}$ | $\mathcal{O}_{\mathcal{L}^2}$ | $\mathrm{Err}_{\mathcal{L}^\infty}$ | $\mathcal{O}_{\mathcal{L}^\infty}$ |
|---|---|---|---|---|---|---|
| 8 | $5.05E-02$ | | $5.59E-02$ | | $8.69E-02$ | |
| 16 | $1.73E-02$ | 1.55 | $1.93E-02$ | 1.53 | $3.43E-02$ | 1.34 |
| 32 | $5.17E-03$ | 1.74 | $5.93E-03$ | 1.71 | $1.17E-02$ | 1.55 |
| 64 | $1.27E-03$ | 2.02 | $1.54E-03$ | 1.94 | $3.48E-03$ | 1.75 |
| 128 | $3.16E-04$ | 2.01 | $3.90E-04$ | 1.98 | $9.63E-04$ | 1.85 |
| 256 | $7.88E-05$ | 2.00 | $9.86E-05$ | 1.99 | $2.57E-04$ | 1.91 |
| 512 | $1.97E-05$ | 2.00 | $2.48E-05$ | 1.99 | $6.63E-05$ | 1.95 |

Test 1. Errors and convergence rates obtained by using LADER scheme. $\Omega = [0, 2]$, $t_{\mathrm{end}} = 1$, $c = c_M = 0.5$.

| Cells | $\mathrm{Err}_{\mathcal{L}^1}$ | $\mathcal{O}_{\mathcal{L}^1}$ | $\mathrm{Err}_{\mathcal{L}^2}$ | $\mathcal{O}_{\mathcal{L}^2}$ | $\mathrm{Err}_{\mathcal{L}^\infty}$ | $\mathcal{O}_{\mathcal{L}^\infty}$ |
|---|---|---|---|---|---|---|
| 8 | $2.69E-02$ | | $2.83E-02$ | | $3.82E-02$ | |
| 16 | $4.86E-03$ | 1.75 | $5.60E-03$ | 1.65 | $8.76E-03$ | 1.48 |
| 32 | $1.95E-03$ | 1.33 | $1.76E-03$ | 1.59 | $2.39E-03$ | 1.64 |
| 64 | $1.81E-03$ | 1.00 | $1.62E-03$ | 1.18 | $2.15E-03$ | 1.53 |
| 128 | $1.18E-03$ | 0.68 | $1.05E-03$ | 0.62 | $1.35E-03$ | 0.67 |
| 256 | $6.64E-04$ | 0.83 | $5.90E-04$ | 0.83 | $7.49E-04$ | 0.85 |
| 512 | $3.50E-04$ | 0.92 | $3.12E-04$ | 0.92 | $3.93E-04$ | 0.93 |

Test 1. Errors and convergence rates obtained by using LADER scheme without applying LADER methodology to the advection coefficient. $\Omega = [0, 2]$, $t_{\mathrm{end}} = 1$, $c = c_M = 0.5$.

# Test 2. Advection-diffusion-reaction equation

- Computational domain: $\Omega = [0, 2]$.
- Time interval: $[0, 1]$.
- Flow definition:

$$\partial_t q(x,t) + \partial_x \left[\lambda(x,t) q\left(x,t\right)\right] + \partial_x \left[\alpha(x,t)\partial_x q(x,t)\right] = s(x,t),$$

$$\lambda(x,t) = 2 + x + t^2, \quad \alpha(x,t) = 10^{-5} e^{x(t-1)^2},$$

$$s(x,t) = 4e^{-2(x-t)^2-t}(x-t)(-1-x-t^2)$$
$$+10^{-5}(t-1)^2 e^{x(t-1)^2}(-4(x-t)e^{-2(x-t)^2-t})$$
$$+10^{-5} e^{x(t-1)^2}(-4+16(x-t)^2)e^{-2(x-t)^2-t},$$

$$q\left(x,t\right) = e^{-2(x-t)^2-t}.$$

| Cells | $\mathrm{Err}_{\mathcal{L}^1}$ | $\mathcal{O}_{\mathcal{L}^1}$ | $\mathrm{Err}_{\mathcal{L}^2}$ | $\mathcal{O}_{\mathcal{L}^2}$ | $\mathrm{Err}_{\mathcal{L}^\infty}$ | $\mathcal{O}_{\mathcal{L}^\infty}$ |
|---|---|---|---|---|---|---|
| 8 | $1.40E-01$ | | $1.35E-01$ | | $1.61E-01$ | |
| 16 | $9.46E-02$ | 0.56 | $8.43E-02$ | 0.67 | $9.65E-02$ | 0.74 |
| 32 | $5.37E-02$ | 0.82 | $4.64E-02$ | 0.86 | $5.28E-02$ | 0.87 |
| 64 | $2.86E-02$ | 0.91 | $2.43E-02$ | 0.93 | $2.76E-02$ | 0.94 |
| 128 | $1.47E-02$ | 0.95 | $1.24E-02$ | 0.97 | $1.41E-02$ | 0.97 |
| 256 | $7.50E-03$ | 0.98 | $6.30E-03$ | 0.98 | $7.14E-03$ | 0.98 |

Test 2. Errors and convergence rates obtained by using the first order scheme. $\Omega = [0,2]$, $t_{\mathrm{end}} = 1$, $c = c_M = 0.5$.

| Cells | $\mathrm{Err}_{\mathcal{L}^1}$ | $\mathcal{O}_{\mathcal{L}^1}$ | $\mathrm{Err}_{\mathcal{L}^2}$ | $\mathcal{O}_{\mathcal{L}^2}$ | $\mathrm{Err}_{\mathcal{L}^\infty}$ | $\mathcal{O}_{\mathcal{L}^\infty}$ |
|---|---|---|---|---|---|---|
| 8 | $7.80E-02$ | | $7.49E-02$ | | $9.51E-02$ | |
| 16 | $3.18E-02$ | 1.30 | $2.68E-02$ | 1.48 | $3.88E-02$ | 1.29 |
| 32 | $1.12E-02$ | 1.50 | $9.57E-03$ | 1.48 | $1.59E-02$ | 1.28 |
| 64 | $5.29E-03$ | 1.08 | $4.25E-03$ | 1.17 | $7.00E-03$ | 1.19 |
| 128 | $2.59E-03$ | 1.03 | $2.15E-03$ | 0.99 | $3.22E-03$ | 1.12 |
| 256 | $1.30E-03$ | 1.00 | $1.09E-03$ | 0.98 | $1.53E-03$ | 1.07 |

Test 2. Errors and convergence rates obtained by using LADER scheme without the new numerical viscosity term. $\Omega = [0, 2]$, $t_{\text{end}} = 1$, $c = c_M = 0.5$.

- Numerical results:

| Cells | $\mathrm{Err}_{\mathcal{L}^1}$ | $\mathcal{O}_{\mathcal{L}^1}$ | $\mathrm{Err}_{\mathcal{L}^2}$ | $\mathcal{O}_{\mathcal{L}^2}$ | $\mathrm{Err}_{\mathcal{L}^\infty}$ | $\mathcal{O}_{\mathcal{L}^\infty}$ |
|---|---|---|---|---|---|---|
| 8 | $6.86E-02$ | | $6.67E-02$ | | $9.31E-02$ | |
| 16 | $2.44E-02$ | 1.49 | $2.27E-02$ | 1.56 | $3.62E-02$ | 1.36 |
| 32 | $6.57E-03$ | 1.89 | $6.56E-03$ | 1.79 | $1.20E-02$ | 1.59 |
| 64 | $1.83E-03$ | 1.84 | $1.67E-03$ | 1.97 | $3.50E-03$ | 1.78 |
| 128 | $6.23E-04$ | 1.56 | $5.11E-04$ | 1.71 | $9.66E-04$ | 1.86 |
| 256 | $2.38E-04$ | 1.38 | $2.02E-04$ | 1.34 | $3.48E-04$ | 1.47 |

Test 2. Errors and convergence rates obtained by using LADER scheme with an ENO-base reconstruction. $\Omega = [0, 2]$, $t_{\mathrm{end}} = 1$, $c = c_M = 0.5$.

# Compressible Navier-Stokes equations

# Compressible low Mach number flows. Governing equations

The system of equations to be solved reads

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \, \mathbf{u} = 0,$$

$$\frac{\partial \rho \, \mathbf{u}}{\partial t} + \mathcal{F}_i^{\mathbf{w_u}}(\mathbf{w_u}, \mathbf{u}) + \nabla \pi - \operatorname{div} \tau = 0,$$

$$\tau = \mu \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T \right) - \frac{2}{3} \mu \operatorname{div} \mathbf{u} \, I,$$

$$\bar{\pi} = \rho R \theta,$$

$$R = \mathcal{R} \sum_{i=1}^{N_e} \frac{\mathrm{y}_i}{\mathcal{M}_i}.$$

# Numerical discretization

Let us consider $\rho^n$, $Q^{n+1}$ the approximations of $\rho\left(x,y,z,t^n\right)$ and $q\left(x,y,z,t^{n+1}\right)$ and $\mathbf{Y}^{n+1}$, $\theta^{n+1}$ the evaluations of $\mathbf{y}\left(x,y,z,t^{n+1}\right)$ and $\theta\left(x,y,z,t^{n+1}\right)$. Then $\mathbf{W}_{\mathbf{u}}^{n+1}$, $\rho^{n+1}$ and $\pi^{n+1}$ are defined from the following system of equations:

$$\frac{1}{\Delta t}\left(\widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1}-\mathbf{W}_{\mathbf{u}}^{n}\right)+\operatorname{div}\mathcal{F}^{\mathbf{W}_{\mathbf{u}}}\left(\mathbf{W}_{\mathbf{u}}^{n},\rho^{n}\right)+\nabla\pi^{n}-\operatorname{div}\tau^{n}=0, \tag{1}$$

$$\rho^{n+1}=\frac{\overline{\pi}}{\mathcal{R}\theta^{n+1}\displaystyle\sum_{i=1}^{N_e}\frac{Y_i^{n+1}}{\mathcal{M}_i}}, \tag{2}$$

$$\frac{1}{\Delta t}\left(\mathbf{W}_{\mathbf{u}}^{n+1}-\widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1}\right)+\nabla\left(\pi^{n+1}-\pi^{n}\right)=0, \tag{3}$$

$$\operatorname{div}\mathbf{W}_{\mathbf{u}}^{n+1}=Q^{n+1}. \tag{4}$$

# Overall method

- Transport-diffusion stage

  Equation (1) is solved by a FVM.

- Pre-projection stage
  - $\rho^{n+1}$ is computed from (2).
  - $Q^{n+1}$ is approximated as the time derivative of the density.

- Projection stage

  A FEM is applied to (3)-(4) in order to obtain the pressure correction.

- Post-projection stage

  $\widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1}$ is updated by using the pressure correction.

A. Bermúdez, S. Busto, J.L. Ferrín, M.E. Vázquez-Cendón. "A high order projection method for low Mach number flows". Submitted.

# Transport-diffusion stage

$\phi_{\mathbf{u}}$ is the numerical flux function and $\varphi_{\mathbf{u}}$ is the diffusion flux function

$$\frac{1}{\Delta t}\left(\widetilde{\mathbf{W}}^{n+1}_{\mathbf{u},\,i}-\mathbf{W}^{n}_{\mathbf{u},\,i}\right)+\frac{1}{|C_i|}\sum_{\mathcal{N}_j\in\mathcal{K}_i}\phi_{\mathbf{u}}\left(\mathbf{W}^{n}_{\mathbf{u},\,i},\mathbf{W}^{n}_{\mathbf{u},j},\rho^n_i,\rho^n_j,\boldsymbol{\eta}_{ij}\right)$$
$$+\frac{1}{|C_i|}\int_{C_i}\nabla\pi^n\mathrm{dV}-\frac{1}{|C_i|}\sum_{\mathcal{N}_j\in\mathcal{K}_i}\varphi_{\mathbf{u}}\left(\mathbf{U}^n_i,\mathbf{U}^n_j,\boldsymbol{\eta}_{ij}\right)=0.$$

**Artificial viscosity related to the density**

$$\phi_{\mathbf{u}}\left(\mathbf{W}^{n}_{\mathbf{u},\,i},\mathbf{W}^{n}_{\mathbf{u},j},\rho^n_i,\rho^n_j,\boldsymbol{\eta}_{ij}\right)$$
$$=\frac{1}{2}\left[\mathcal{Z}(\mathbf{W}^{n}_{\mathbf{u},\,i},\rho^n_i,\boldsymbol{\eta}_{ij})+\mathcal{Z}(\mathbf{W}^{n}_{\mathbf{u},j},\rho^n_j,\boldsymbol{\eta}_{ij})\right]-\frac{1}{2}\alpha^{\mathbf{w}_{\mathbf{u}},\,n}_{RS,\,ij}\left(\mathbf{W}^n_j-\mathbf{W}^n_i\right)$$
$$-\frac{1}{3}\mathrm{sign}\left(\breve{\alpha}^{\mathbf{w}_{\mathbf{u}},\,n}_{RS,\,ij}\right)\left(\mathbf{W}^{n}_{\mathbf{u},\,i}+\mathbf{W}^{n}_{\mathbf{u},j}\right)\left(\mathbf{W}^{n}_{\mathbf{u},\,i}+\mathbf{W}^{n}_{\mathbf{u},j}\right)\cdot\boldsymbol{\eta}_{ij}\left(\rho^n_i+\rho^n_j\right)^{-2}\left(\rho^n_j-\rho^n_i\right).$$

**LADER scheme**

To attain a second order in time and space scheme LADER methodology is also applied to the density. To apply Cauchy-Kovalevskaya procedure, the mass conservation equation is considered.

# Pre-projection, projection and post-projection stages

## Pre-projection stage

At the pre-projection stage the source term for the projection stage is obtained from the following relations:

$$Q_i^{n+1} = \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t}, \quad \rho_i^{n+1} = \frac{\overline{\pi}^{n+1}}{\mathcal{R}\theta_i^{n+1} \sum_{l=1}^{N_e} \frac{Y_{l,\,i}^{n+1}}{\mathcal{M}_l}}.$$

## Projection stage

FEM is applied to compute the pressure correction, $\delta^{n+1}$. The weak problem to be solved reads:
*Find* $\delta^{n+1} \in V_0 := \left\{ z \in H^1(\Omega) : \int_\Omega z = 0 \right\}$ *verifying*

$$\int_\Omega \mathrm{grad}\delta^{n+1} \cdot \mathrm{grad} z \, dV = \frac{1}{\Delta t} \int_\Omega \widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1} \cdot \mathrm{grad} z \, dV + \frac{1}{\Delta t} \int_\Omega Q^{n+1} z dV - \frac{1}{\Delta t} \int_{\partial\Omega} G^{n+1} z \, dA$$

*for all* $z \in V_0$.

## Post-projection stage

The conservative variables related to the velocity are updated using the pressure correction,

$$\mathbf{W}_{\mathbf{u},i}^{n+1} = \widetilde{\mathbf{W}}_{\mathbf{u},i}^{n+1} + \Delta t \ \mathrm{grad} \ \delta_i^{n+1}.$$

# Test 1. Euler flow

- Computational domain: $\Omega = [0,1]^3$.
- Flow definition:

$$\rho(x,y,z,t) = \cos(t) + x + 1, \quad \pi(x,y,z,t) = 1,$$

$$\mathbf{u}(x,y,z,t) = \left( \frac{x\sin(t)+1}{\cos(t)+x+1}, 0, 0 \right)^T,$$

$$\mathrm{y}(x,y,z,t) = 1, \quad \theta(x,y,z,t) = \frac{10^3}{\cos(t)+x+1}$$

with $\mu = 0$ and

$$f_{u_1} = f_{u_2} = 0,$$

$$f_{u_3} = x\cos(t) - \frac{(x\sin(t)+1)^2}{(x+\cos(t)+1)^2} + \frac{(2\sin(t)(x\sin(t)+1))}{x+\cos(t)+1}$$

the source terms related to the momentum equation.

- $CFL = 1$.
- Dirichlet boundary conditions are set on the boundary.

- Mesh features:

| Mesh | $N$ | Elements | Vertices | Nodes | $v_h^m$ $(m^3)$ | $v_h^M$ $(m^3)$ |
|---|---|---|---|---|---|---|
| $M_1$ | 4 | 384 | 125 | 864 | $6.51E-04$ | $1.30E-03$ |
| $M_2$ | 8 | 3072 | 729 | 6528 | $8.14E-05$ | $1.63E-04$ |
| $M_3$ | 16 | 24576 | 4913 | 50688 | $1.02E-05$ | $2.03E-05$ |

- We have denoted by $N+1$ the number of nodes along the edges of the domain, $h = 1/N$, $v_h^m$ the minimum volume of the finite volumes and $v_h^M$ the maximum volume of the finite volumes.

We have considered three different schemes:

| Method | Variable | $E_{M_1}$ | $E_{M_2}$ | $E_{M_3}$ | $o_{M_1/M_2}$ | $o_{M_2/M_3}$ |
|---|---|---|---|---|---|---|
| Order 1 | $\pi$ | $3.73E-03$ | $1.44E-03$ | $5.29E-04$ | 1.37 | 1.44 |
| | $\mathbf{w_u}$ | $7.30E-03$ | $4.05E-03$ | $2.13E-03$ | 0.85 | 0.93 |
| LADER | $\pi$ | $4.13E-03$ | $1.60E-03$ | $6.15E-04$ | 1.37 | 1.38 |
| without $\overline{\rho}_{iN_{ij}}$ | $\mathbf{w_u}$ | $6.11E-03$ | $3.23E-03$ | $1.71E-03$ | 0.92 | 0.91 |
| LADER | $\pi$ | $4.64E-04$ | $1.93E-04$ | $9.21E-05$ | 1.26 | 1.07 |
| | $\mathbf{w_u}$ | $6.31E-04$ | $1.62E-04$ | $4.15E-05$ | 1.97 | 1.96 |

Test 1. Euler flow. Observed errors and convergence rates. $CFL = 1$.

- Applying LADER methodology to compute the density is crucial to achieve a second order scheme.
- If the new viscosity term is not considered, spurious oscillations arise when applying LADER methodology.

# Test 2. Navier-Stokes flow

- Computational domain: $\Omega = [0,1]^3$.

- Flow definition:

$$\mu = 10^{-2},$$
$$\rho(x,y,z,t) = \sin(\pi y t) + 2,$$
$$\pi(x,y,z,t) = \exp(xyz)\cos(t),$$
$$\mathbf{u}(x,y,z,t) = \left((\cos(\pi x t))^2, \exp(-2\pi y t), -\cos(\pi x y t)\right)^T,$$
$$\mathrm{y}(x,y,z,t) = 1,$$
$$\theta(x,y,z,t) = \frac{10^3}{\sin(\pi y t) + 2}.$$

- $CFL = 5$.

- Numerical results:

| Method | Variable | $E_{M_1}$ | $E_{M_2}$ | $E_{M_3}$ | $o_{M_1/M_2}$ | $o_{M_2/M_3}$ |
|---|---|---|---|---|---|---|
| Order 1 | $\pi$ | $3.32E-01$ | $1.52E-01$ | $6.76E-02$ | 1.13 | 1.17 |
| | $\mathbf{w_u}$ | $1.45E-01$ | $7.80E-02$ | $4.27E-02$ | 0.89 | 0.87 |
| Order 1 | $\pi$ | $3.33E-01$ | $1.52E-01$ | $6.76E-02$ | 1.13 | 1.17 |
| $(\partial_t\rho)_{\text{exact}}$ | $\mathbf{w_u}$ | $1.45E-01$ | $7.80E-02$ | $4.27E-02$ | 0.89 | 0.87 |
| LADER | $\pi$ | $8.65E-02$ | $1.72E-02$ | $4.40E-03$ | 2.33 | 1.97 |
| | $\mathbf{w_u}$ | $7.43E-02$ | $1.76E-02$ | $4.33E-03$ | 2.08 | 2.02 |
| LADER | $\pi$ | $8.55E-02$ | $1.66E-02$ | $3.77E-03$ | 2.36 | 2.14 |
| $(\partial_t\rho)_{\text{exact}}$ | $\mathbf{w_u}$ | $7.40E-02$ | $1.75E-02$ | $4.30E-03$ | 2.08 | 2.02 |
| LADER | $\pi$ | $1.02E-01$ | $2.68E-02$ | $1.06E-02$ | 1.93 | 1.33 |
| $\rho_{i_{\text{exact}}}^{n+\frac{1}{2}\Delta t}$ | $\mathbf{w_u}$ | $8.29E-02$ | $2.78E-02$ | $1.23E-02$ | 1.58 | 1.18 |
| LADER | $\pi$ | $8.81E-02$ | $1.74E-02$ | $4.43E-03$ | 2.34 | 1.98 |
| $\overline{\rho_{iN_{ij}}}_{\text{exact}}$ | $\mathbf{w_u}$ | $7.44E-02$ | $1.76E-02$ | $4.33E-03$ | 2.08 | 2.02 |

# Conclusions

Main differences between ADER and LADER methodologies:

- At the polynomial reconstruction step, LADER uses piecewise linear polynomials whereas ADER considers linear polynomials.
- Within LADER, the evolved variables obtained for computing the diffusion term neglect the presence of the advection term.
- Applying LADER, advection, diffusion and reaction terms need to be computed using the proper evolved variables, which will be different for each of them.
- The resulting schemes have diverse stability regions.

Advantages of LADER:

- LADER profits from the dual mesh structure build using the face-type volumes.
- Performing the 3D extension of LADER is easier than that of ADER.

# Conclusions

- The space dependence of density may produce spurious oscillations on the solution of the momentum equation.
- The numerical flux function has been modified by adding a new upwind term.
- Willing to obtain a high order scheme, LADER methodology has also been used.
- In order to get insight on the effects that the variable density has on the accuracy of the scheme, the unidimensional advection-diffusion-reaction equation with space and time dependent advection coefficient has been examined.
- The corresponding accuracy analysis together with the empirical convergence rate studies reveal the necessity of reconstructing and evolving both the conservative variable and the advection coefficient to attain a second order scheme.

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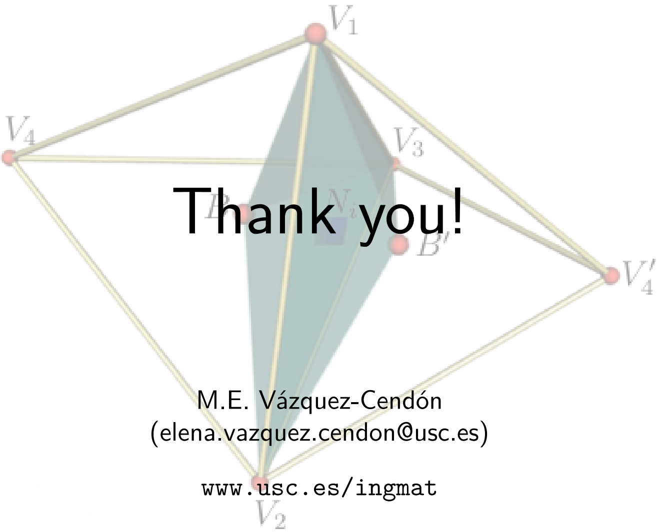


# Thank you!

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