Challenge:

Find necessary and sufficient conditions for a numerical scheme to be derivable as a Godunov scheme (for some reconstruction)!

Challenge:

Find necessary and sufficient conditions for a numerical scheme to be derivable as a Godunov scheme (for some reconstruction)!

Notes:

- Multi-d Cartesian grids
- Systems of hyperbolic PDEs
- Assume that the exact solution is known for all times and all kinds of initial data
- The reconstruction can be discontinuous and vary arbitrarily inside the cell

Vorticity preservation and low Mach number

Wasilij Barsukow

DAAD PRIME postdoctoral fellow

U Zurich

May 23, 2018

Acoustic equations ($c > 0$):

$$\begin{aligned} \partial_t \mathbf{v} + \nabla p &= 0 \\ \partial_t p + c^2 \nabla \cdot \mathbf{v} &= 0 \end{aligned}$$

Acoustic equations ($c > 0$):

$$\partial_t \mathbf{v} + \nabla p = 0$$
$$\partial_t p + c^2 \nabla \cdot \mathbf{v} = 0$$

Stationary vorticity:

$$\Rightarrow \partial_t (\nabla \times \mathbf{v}) = 0$$

Acoustic equations ($c > 0$):

$$\partial_t \mathbf{v} + \nabla p = 0$$
$$\partial_t p + c^2 \nabla \cdot \mathbf{v} = 0$$

Stationary vorticity:

$$\Rightarrow \partial_t (\nabla \times \mathbf{v}) = 0$$

$$\partial_t^2 \mathbf{v} - c^2 \nabla (\nabla \cdot \mathbf{v}) = 0$$
$$\partial_t^2 p - c^2 \nabla \cdot \nabla p = 0$$

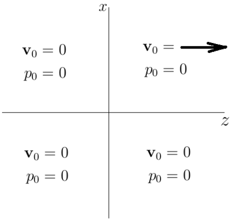
Acoustic equations $(c > 0)$:

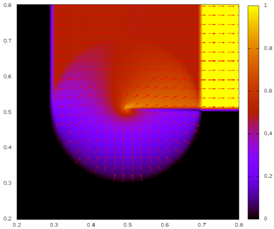
$$\partial_t \mathbf{v} + \nabla p = 0$$
$$\partial_t p + c^2 \nabla \cdot \mathbf{v} = 0$$

$$\partial_t^2 \mathbf{v} - c^2 \nabla(\nabla \cdot \mathbf{v}) = 0$$
$$\partial_t^2 p - c^2 \nabla \cdot \nabla p = 0$$

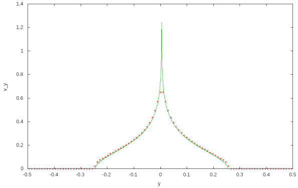
Stationary vorticity:

$$\Rightarrow \partial_t(\nabla \times \mathbf{v}) = 0$$





Logarithmic singularity: $\mathbf{v} \sim -\log|\mathbf{x}|$



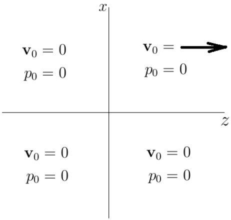
Acoustic equations ($c > 0$):

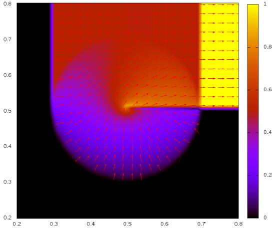
$$\partial_t \mathbf{v} + \nabla p = 0$$
$$\partial_t p + c^2 \nabla \cdot \mathbf{v} = 0$$

$$\partial_t^2 \mathbf{v} - c^2 \nabla(\nabla \cdot \mathbf{v}) = 0$$
$$\partial_t^2 p - c^2 \nabla \cdot \nabla p = 0$$

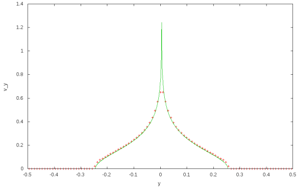
Stationary vorticity:

$$\Rightarrow \partial_t(\nabla \times \mathbf{v}) = 0$$

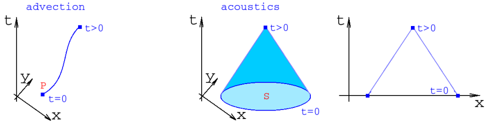




Logarithmic singularity: $\mathbf{v} \sim -\log|\mathbf{x}|$



Multi-d: characteristics become characteristic cones:



Characteristic cones do not only transport values: derivatives of initial data are involved!
(Discontinuous initial data?)

Amadori & Gosse 2015; WB & Klingenberg 2017, subm.

Euler equations

The acoustic equations are contained in the Euler equations:

$$\begin{aligned} & & \partial_t \varrho + \mathbf{v} \cdot \nabla \varrho + \varrho \nabla \cdot \mathbf{v} &= 0 \\ \partial_t \mathbf{v} + \nabla p &= 0 & \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\nabla p}{\varrho} &= 0 \\ \partial_t p + c^2 \nabla \cdot \mathbf{v} &= 0 & \partial_t p + \mathbf{v} \cdot \nabla p + \varrho c^2 \nabla \cdot \mathbf{v} &= 0 \end{aligned}$$

They capture the behaviour of acoustics and leave aside advection.

They also govern the (**Lagrangian**) evolution of a fluid element.

Numerics:

$q := (\mathbf{v}, p) \in \mathbb{R}^{d+1}$ E.g. in 2-d $\partial_t q_{ij} + \frac{f^x_{i+\frac12,j} - f^x_{i-\frac12,j}}{\Delta x} + \frac{f^y_{i,j+\frac12} - f^y_{i,j-\frac12}}{\Delta y} = 0$

Numerics:

$q := (\mathbf{v}, p) \in \mathbb{R}^{d+1}$ E.g. in 2-d $\partial_t q_{ij} + \frac{f^x_{i+\frac{1}{2},j} - f^x_{i-\frac{1}{2},j}}{\Delta x} + \frac{f^y_{i,j+\frac{1}{2}} - f^y_{i,j-\frac{1}{2}}}{\Delta y} = 0$

Linear systems: $f^x = J^x q$, $f^y = J^y q$.

$$J^x = \begin{pmatrix} & & 1 \\ & 0 & \\ c^2 & & \end{pmatrix} \quad J^y = \begin{pmatrix} 0 & & \\ & & 1 \\ & c^2 & \end{pmatrix}$$

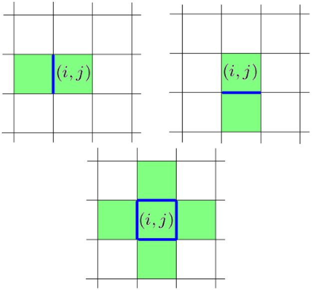
Numerics:

$$q := (\mathbf{v}, p) \in \mathbb{R}^{d+1} \qquad \text{E.g. in 2-d} \qquad \partial_t q_{ij} + \frac{f^x_{i+\frac{1}{2},j} - f^x_{i-\frac{1}{2},j}}{\Delta x} + \frac{f^y_{i,j+\frac{1}{2}} - f^y_{i,j-\frac{1}{2}}}{\Delta y} = 0$$

Linear systems: $f^x = J^x q$, $f^y = J^y q$.

$$J^x = \begin{pmatrix} & & 1 \\ & 0 & \\ c^2 & & \end{pmatrix} \qquad J^y = \begin{pmatrix} 0 & & \\ & & 1 \\ & c^2 & \end{pmatrix}$$

Upwind scheme (directionally split)

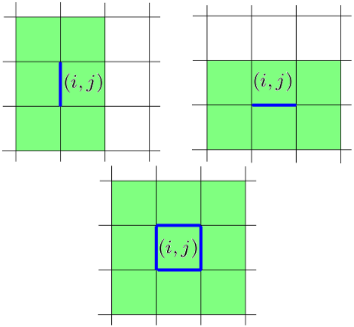


$$f^x_{i+\frac{1}{2},j} = \frac{1}{2} J^x (q_{i+1,j} + q_{ij}) - \frac{1}{2} D^x (q_{i+1,j} - q_{ij})$$

$$D_x = |J_x| \qquad D_y = |J_y|$$

$$D^x = \begin{pmatrix} c & & \\ & 0 & \\ & & c \end{pmatrix} \qquad D^y = \begin{pmatrix} 0 & & \\ & c & \\ & & c \end{pmatrix} \tag{1}$$

You can also use multi-d information:



Multi-dimensional schemes can appear very naturally!
E.g. Morton, Roe 2001: Define discrete derivative operator δ_x, δ_y. Rough idea:

$$
\begin{aligned}
\partial_t u + \partial_x p &= 0 \\
\partial_t v + \partial_y p &= 0 \\
\partial_t p + c^2 \nabla \cdot \mathbf{v} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\partial_t u + \delta_x p &= \delta_x^2 u + \delta_x \delta_y v \\
\partial_t v + \delta_y p &= \delta_x \delta_y u + \delta_y^2 v \\
\partial_t p + c^2 (\delta_x u + \delta_y v) &= \dots
\end{aligned}
$$

Note the gradient on the right hand side!

Multi-dimensional schemes can appear very naturally!
E.g. Morton, Roe 2001: Define discrete derivative operator δ_x, δ_y. Rough idea:

$$
\begin{aligned}
\partial_t u + \partial_x p &= 0 \\
\partial_t v + \partial_y p &= 0 \\
\partial_t p + c^2 \nabla \cdot \mathbf{v} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\partial_t u + \delta_x p &= \delta_x^2 u + \delta_x \delta_y v \\
\partial_t v + \delta_y p &= \delta_x \delta_y u + \delta_y^2 v \\
\partial_t p + c^2 (\delta_x u + \delta_y v) &= \dots
\end{aligned}
$$

Note the gradient on the right hand side!

Multi-dimensional schemes can appear very naturally!
E.g. Morton, Roe 2001: Define discrete derivative operator δ_x, δ_y. Rough idea:

$$\begin{aligned} \partial_t u + \partial_x p &= 0 \\ \partial_t v + \partial_y p &= 0 \\ \partial_t p + c^2 \nabla \cdot \mathbf{v} &= 0 \end{aligned}$$

$$\begin{aligned} \partial_t u + \delta_x p &= \delta_x^2 u + \delta_x \delta_y v \\ \partial_t v + \delta_y p &= \delta_x \delta_y u + \delta_y^2 v \\ \partial_t p + c^2 (\delta_x u + \delta_y v) &= \ldots \end{aligned}$$

Note the gradient on the right hand side!

One might hope that the discrete vorticity $\delta_y u - \delta_x v$ might be stationary...

Multi-dimensional schemes can appear very naturally!
E.g. Morton, Roe 2001: Define discrete derivative operator δ_x, δ_y. Rough idea:

$$\begin{aligned} \partial_t u + \partial_x p &= 0 \\ \partial_t v + \partial_y p &= 0 \\ \partial_t p + c^2 \nabla \cdot \mathbf{v} &= 0 \end{aligned}$$

$$\begin{aligned} \partial_t u + \delta_x p &= \delta_x^2 u + \delta_x \delta_y v \\ \partial_t v + \delta_y p &= \delta_x \delta_y u + \delta_y^2 v \\ \partial_t p + c^2(\delta_x u + \delta_y v) &= \dots \end{aligned}$$

Note the gradient on the right hand side!

One might hope that the discrete vorticity $\delta_y u - \delta_x v$ might be stationary...

With the discrete averaging operator μ_x, μ_y one can actually show that for

$$\begin{aligned} \partial_t u + \delta_x \mu_x \mu_y^2 p &= \delta_x^2 \mu_y^2 u + \delta_x \mu_x \delta_y \mu_y v \\ \partial_t v + \delta_y \mu_y \mu_x^2 p &= \delta_x \mu_x \delta_y \mu_y u + \delta_y^2 \mu_x^2 v \\ \partial_t p + c^2(\delta_x \mu_x \mu_y^2 u + \delta_y \mu_y \mu_x^2 v) &= \delta_x^2 \mu_y^2 p + \delta_y^2 \mu_x^2 p \end{aligned}$$

the vorticity is stationary: $\partial_t(\delta_x \mu_y v - \delta_y \mu_x u) = 0$.

Jeltsch & Torrilhon 2006; Mishra & Tadmor 2009; Lung & Roe 2014; . . .

Actually, you don't need to be multi-d for vorticity preservation:

Directionally split Roe-type scheme (central + diffusion)

$$\partial_t q_{ij} + \frac{f^x_{i+\frac{1}{2},j} - f^x_{i-\frac{1}{2},j}}{\Delta x} + \frac{f^y_{i,j+\frac{1}{2}} - f^y_{i,j-\frac{1}{2}}}{\Delta y} = 0 \tag{2}$$

$$f^x_{i+\frac{1}{2},j} = \frac{1}{2} J^x (q_{i+1,j} + q_{ij}) - \frac{1}{2} D^x (q_{i+1,j} - q_{ij}) \tag{3}$$

$$D_x = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & 0 \\ a_3 & 0 & a_4 \end{pmatrix} \qquad D_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}$$

Actually, you don't need to be multi-d for vorticity preservation:

Directionally split Roe-type scheme (central + diffusion)

$$\partial_t q_{ij} + \frac{f^x_{i+\frac{1}{2},j} - f^x_{i-\frac{1}{2},j}}{\Delta x} + \frac{f^y_{i,j+\frac{1}{2}} - f^y_{i,j-\frac{1}{2}}}{\Delta y} = 0 \tag{2}$$

$$f^x_{i+\frac{1}{2},j} = \frac{1}{2} J^x (q_{i+1,j} + q_{ij}) - \frac{1}{2} D^x (q_{i+1,j} - q_{ij}) \tag{3}$$

$$D_x = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & 0 \\ a_3 & 0 & a_4 \end{pmatrix} \qquad D_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}$$

You can show that, $\boxed{\text{iff } a_1 = 0}$,

$$\underbrace{\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x}}_{\simeq \partial_x v} - \underbrace{\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}}_{\simeq \partial_y u} + \underbrace{\frac{a_3}{c^2} \left(\frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{2\Delta y} - \frac{v_{i+1,j} - 2v_{ij} + v_{i-1,j}}{2\Delta x} \right)}_{\simeq \mathcal{O}(\Delta x, \Delta y)} \tag{4}$$

is **stationary** (and a discretization of $\partial_x v - \partial_y u$)!

Actually, you don't need to be multi-d for vorticity preservation:

Directionally split Roe-type scheme (central + diffusion)

$$\partial_t q_{ij} + \frac{f^x_{i+\frac{1}{2},j} - f^x_{i-\frac{1}{2},j}}{\Delta x} + \frac{f^y_{i,j+\frac{1}{2}} - f^y_{i,j-\frac{1}{2}}}{\Delta y} = 0 \tag{2}$$

$$f^x_{i+\frac{1}{2},j} = \frac{1}{2} J^x (q_{i+1,j} + q_{ij}) - \frac{1}{2} D^x (q_{i+1,j} - q_{ij}) \tag{3}$$

$$D_x = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & 0 \\ a_3 & 0 & a_4 \end{pmatrix} \qquad D_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}$$

You can show that, $\boxed{\text{iff } a_1 = 0}$,

$$\underbrace{\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x}}_{\simeq \partial_x v} - \underbrace{\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}}_{\simeq \partial_y u} + \underbrace{\frac{a_3}{c^2}\left(\frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{2\Delta y} - \frac{v_{i+1,j} - 2v_{ij} + v_{i-1,j}}{2\Delta x}\right)}_{\simeq \mathcal{O}(\Delta x, \Delta y)} \tag{4}$$

is **stationary** (and a discretization of $\partial_x v - \partial_y u$)!

Good luck with trying to prove this directly.

A selection of tricky statements:

1. Scheme (3) is vorticity preserving with discrete vorticity (4).

A selection of tricky statements:

1. Scheme (3) is vorticity preserving with discrete vorticity (4).
2. There is no discrete stationary vorticity for the upwind scheme (1).

A selection of tricky statements:

1. Scheme (3) is vorticity preserving with discrete vorticity (4).
2. There is no discrete stationary vorticity for the upwind scheme (1).
3. Vorticity preserving schemes for acoustics are just the ones that behave well in the limit of low Mach number.

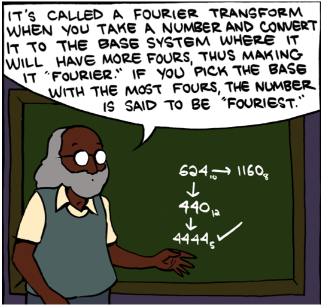
Solution: Use the Fourier transform!

Solution: Use the Fourier transform!

There is a Fourier transform for the **continuous** case, and also a **discrete** one!

Solution: Use the Fourier transform!

There is a Fourier transform for the **continuous** case, and also a **discrete** one!



Continuous case:

Consider the following $n \times n$ hyperbolic system of PDEs

$$\partial_t q + J_x \partial_x q + J_y \partial_y q = 0 \tag{5}$$

$$\text{i.e. } \partial_t q + \mathbf{J} \cdot \nabla q = 0 \tag{6}$$

Continuous case:

Consider the following $n \times n$ hyperbolic system of PDEs

$$\partial_t q + J_x \partial_x q + J_y \partial_y q = 0 \tag{5}$$

$$\text{i.e. } \partial_t q + \mathbf{J} \cdot \nabla q = 0 \tag{6}$$

Insert a **Fourier mode**

$$q(t, \mathbf{x}) = \underbrace{\begin{pmatrix} \hat{u}(t, \mathbf{k}) \\ \hat{v}(t, \mathbf{k}) \\ \hat{p}(t, \mathbf{k}) \end{pmatrix}}_{=: \hat{q}(t, \mathbf{k})} \exp(\mathbb{i} \mathbf{k} \cdot \mathbf{x}) \tag{7}$$

Continuous case:

Consider the following $n \times n$ hyperbolic system of PDEs

$$\partial_t q + J_x \partial_x q + J_y \partial_y q = 0 \tag{5}$$

$$\text{i.e. } \partial_t q + \mathbf{J} \cdot \nabla q = 0 \tag{6}$$

Insert a **Fourier mode**

$$q(t, \mathbf{x}) = \underbrace{\begin{pmatrix} \hat{u}(t, \mathbf{k}) \\ \hat{v}(t, \mathbf{k}) \\ \hat{p}(t, \mathbf{k}) \end{pmatrix}}_{=:\hat{q}(t,\mathbf{k})} \exp(\mathbb{i}\mathbf{k} \cdot \mathbf{x}) \tag{7}$$

$$\partial_t \hat{q} + \mathbb{i} J_x k_x \hat{q} + \mathbb{i} J_y k_y \hat{q} = 0 \tag{8}$$

$$\text{i.e. } \partial_t \hat{q} + \mathbb{i}(\mathbf{J} \cdot \mathbf{k})\hat{q} = 0 \tag{9}$$

Derivatives become algebraic factors.

$$\boxed{\partial_t \hat{q} + \mathbb{i}(\mathbf{J} \cdot \mathbf{k})\hat{q} = 0} \tag{10}$$

Recall that **hyperbolicity** guarantees that $\mathbf{J} \cdot \mathbf{k}$ is real diagonalizable:

$$(\mathbf{J} \cdot \mathbf{k})e_j = \lambda_j e_j \qquad j = 1, \ldots, n \tag{11}$$

The solution to $\partial_t \hat{q} + \mathbb{i}(\mathbf{J} \cdot \mathbf{k})\hat{q} = 0$ can thus be constructed out of these eigenvectors:

$$\boxed{\hat{q}(t, \mathbf{k}) = \sum_{j=1}^{n} (\hat{q}_0 \cdot e_j) e_j \exp(-\mathbb{i}\lambda_j t)} \tag{12}$$

Every mode evolves with its own eigenvalue.

$$\boxed{\partial_t \hat{q} + \mathbb{i}(\mathbf{J} \cdot \mathbf{k})\hat{q} = 0} \tag{10}$$

Recall that **hyperbolicity** guarantees that $\mathbf{J} \cdot \mathbf{k}$ is real diagonalizable:

$$(\mathbf{J} \cdot \mathbf{k}) e_j = \lambda_j e_j \qquad j = 1, \dots, n \tag{11}$$

The solution to $\partial_t \hat{q} + \mathbb{i}(\mathbf{J} \cdot \mathbf{k})\hat{q} = 0$ can thus be constructed out of these eigenvectors:

$$\boxed{\hat{q}(t, \mathbf{k}) = \sum_{j=1}^{n} (\hat{q}_0 \cdot e_j) e_j \exp(-\mathbb{i}\lambda_j t)} \tag{12}$$

Every mode evolves with its own eigenvalue.

Definition (Nontrivial stationary states)

$\partial_t q + \mathbf{J} \cdot \nabla q = 0$ possesses *nontrivial stationary states* if $\det \mathbf{J} \cdot \mathbf{k} = 0 \ \forall \mathbf{k} \in \mathbb{R}^d$.

For example, linear advection has stationary states, but no nontrivial ones...

For linear acoustics in 2-d

$$\mathbf{J} \cdot \mathbf{k} = \begin{pmatrix} 0 & 0 & k_x \\ 0 & 0 & k_y \\ k_x c^2 & k_y c^2 & 0 \end{pmatrix} \tag{13}$$

has eigenvalues $(0, \pm c|\mathbf{k}|)$ such that

$$\hat{q}(t, \mathbf{k}) = (\hat{q}_0 \cdot e_1)e_1 + (\hat{q}_0 \cdot e_2)e_2 \cdot \exp\left(\mathbb{i}ct|\mathbf{k}|\right) + (\hat{q}_0 \cdot e_3)e_3 \cdot \exp\left(-\mathbb{i}ct|\mathbf{k}|\right) \tag{14}$$

with

$$e_1 = \begin{pmatrix} -k_y \\ k_x \\ 0 \end{pmatrix} \tag{15}$$

For linear acoustics in 2-d

$$\mathbf{J} \cdot \mathbf{k} = \begin{pmatrix} 0 & 0 & k_x \\ 0 & 0 & k_y \\ k_x c^2 & k_y c^2 & 0 \end{pmatrix} \quad (13)$$

has eigenvalues $(0, \pm c|\mathbf{k}|)$ such that

$$\hat{q}(t, \mathbf{k}) = (\hat{q}_0 \cdot e_1)e_1 + (\hat{q}_0 \cdot e_2)e_2 \cdot \exp(\mathbb{i}ct|\mathbf{k}|) + (\hat{q}_0 \cdot e_3)e_3 \cdot \exp(-\mathbb{i}ct|\mathbf{k}|) \quad (14)$$

with

$$e_1 = \begin{pmatrix} -k_y \\ k_x \\ 0 \end{pmatrix} \quad (15)$$

Solution is stationary, if initial data only contain modes parallel to e_1!

$$\hat{q}(t,\mathbf{k}) = (\hat{q}_0 \cdot e_1)e_1 + (\hat{q}_0 \cdot e_2)e_2 \cdot \exp\left(\mathbb{i}ct|\mathbf{k}|\right) + (\hat{q}_0 \cdot e_3)e_3 \cdot \exp\left(-\mathbb{i}ct|\mathbf{k}|\right) \tag{16}$$

$$e_1 = \begin{pmatrix} -k_y \\ k_x \\ 0 \end{pmatrix} \tag{17}$$

Define projector P_1 onto e_1

$$P_1 e_2 = P_1 e_3 = 0 \qquad\qquad P_1 e_1 = e_1 \tag{18}$$

Then **for all initial data**

$$P_1 \hat{q}(t,\mathbf{k}) = P_1 \hat{q}(0,\mathbf{k}) \tag{19}$$

i.e.

$$\partial_t (P_1 \hat{q}) = 0 \tag{20}$$

$$\hat{q}(t,\mathbf{k}) = (\hat{q}_0 \cdot e_1)e_1 + (\hat{q}_0 \cdot e_2)e_2 \cdot \exp\left(\mathbb{i}ct|\mathbf{k}|\right) + (\hat{q}_0 \cdot e_3)e_3 \cdot \exp\left(-\mathbb{i}ct|\mathbf{k}|\right) \tag{16}$$

$$e_1 = \begin{pmatrix} -k_y \\ k_x \\ 0 \end{pmatrix} \tag{17}$$

Define projector P_1 onto e_1

$$P_1 e_2 = P_1 e_3 = 0 \qquad\qquad P_1 e_1 = e_1 \tag{18}$$

Then **for all initial data**

$$P_1 \hat{q}(t,\mathbf{k}) = P_1 \hat{q}(0,\mathbf{k}) \tag{19}$$

i.e.

$$\partial_t (P_1 \hat{q}) = 0 \tag{20}$$

For acoustics iin 2-d: $P_1 \hat{q} = -k_y \hat{u} + k_x \hat{v} = \mathbb{F}[\nabla \times \mathbf{v}]$ → **stationary vorticity** (**involution**, or **constant of motion**).

Dafermos 1986; WB 2017; WB 2018, subm.

Existence of nontrivial stationary states is equivalent to the existence of involutions.

Discrete Fourier modes, e.g. in 2-d

$$q_{ij}(t) = \hat{q}(t, \mathbf{k}) \exp\left(\mathbb{i} k_x (i\Delta x) + \mathbb{i} k_y (j \Delta y)\right) \qquad \text{instead of} \qquad q(t, \mathbf{x}) = \hat{q}(t, \mathbf{k}) \exp(\mathbb{i} k_x x + \mathbb{i} k_y y)$$

Discrete Fourier modes, e.g. in 2-d

$$q_{ij}(t) = \hat{q}(t, \mathbf{k}) \exp\left(\mathbb{i} k_x (i\Delta x) + \mathbb{i} k_y (j\Delta y)\right) \qquad \text{instead of} \qquad q(t, \mathbf{x}) = \hat{q}(t, \mathbf{k}) \exp(\mathbb{i} k_x x + \mathbb{i} k_y y)$$

Consider a finite difference, e.g.

$$\frac{1}{2\Delta x} \left(q_{i+1,j} - q_{i-1,j}\right) \tag{21}$$

Inserting the Fourier mode

$$\exp\left(\mathbb{i} k_x (i\Delta x) + \mathbb{i} k_y (j\Delta y)\right) \cdot \frac{1}{2\Delta x}\Big(\exp(\mathbb{i} k_x \Delta x) - \exp(-\mathbb{i} k_x \Delta x)\Big)\hat{q} \tag{22}$$

Discrete Fourier modes, e.g. in 2-d

$$q_{ij}(t) = \hat{q}(t, \mathbf{k}) \exp\left(\mathbb{i} k_x (i\Delta x) + \mathbb{i} k_y (j\Delta y)\right) \qquad \text{instead of} \qquad q(t, \mathbf{x}) = \hat{q}(t, \mathbf{k}) \exp(\mathbb{i} k_x x + \mathbb{i} k_y y)$$

Consider a finite difference, e.g.

$$\frac{1}{2\Delta x} (q_{i+1,j} - q_{i-1,j}) \tag{21}$$

Inserting the Fourier mode

$$\exp\left(\mathbb{i} k_x (i\Delta x) + \mathbb{i} k_y (j\Delta y)\right) \cdot \frac{1}{2\Delta x} \Big(\exp(\mathbb{i} k_x \Delta x) - \exp(-\mathbb{i} k_x \Delta x) \Big) \hat{q} \tag{22}$$

Obviously, the factor $\exp(\mathbb{i} k_x \Delta x) =: t_x$ acts as shift by one cell to the right. The above can be rewritten as

$$\exp\left(\mathbb{i} k_x (i\Delta x) + \mathbb{i} k_y (j\Delta y)\right) \cdot \frac{1}{2\Delta x} \left(t_x - t_x^{-1}\right) \hat{q} \tag{23}$$

Discrete Fourier modes, e.g. in 2-d

$$q_{ij}(t) = \hat{q}(t, \mathbf{k}) \exp\left(\mathbb{i} k_x (i\Delta x) + \mathbb{i} k_y (j \Delta y) \right) \qquad \text{instead of} \qquad q(t, \mathbf{x}) = \hat{q}(t, \mathbf{k}) \exp(\mathbb{i} k_x x + \mathbb{i} k_y y)$$

Consider a finite difference, e.g.

$$\frac{1}{2\Delta x} (q_{i+1,j} - q_{i-1,j}) \tag{21}$$

Inserting the Fourier mode

$$\exp\left(\mathbb{i} k_x (i\Delta x) + \mathbb{i} k_y (j \Delta y) \right) \cdot \frac{1}{2\Delta x} \Big(\exp(\mathbb{i} k_x \Delta x) - \exp(-\mathbb{i} k_x \Delta x) \Big) \hat{q} \tag{22}$$

Obviously, the factor $\exp(\mathbb{i} k_x \Delta x) =: t_x$ acts as shift by one cell to the right. The above can be rewritten as

$$\exp\left(\mathbb{i} k_x (i\Delta x) + \mathbb{i} k_y (j \Delta y) \right) \cdot \frac{1}{2\Delta x} \left(t_x - t_x^{-1} \right) \hat{q} \tag{23}$$

There is a bijection between linear finite difference formulae and Laurent polynomials in t_x, t_y. E.g.

$$\frac{1}{8\Delta x} \Big((q_{i+1,j+1} + 2q_{i+1,j} + q_{i+1,j-1}) - (q_{i-1,j+1} + 2q_{i-1,j} + q_{i-1,j-1}) \Big) \tag{24}$$

$$\simeq \frac{1}{8\Delta x} \Big((t_x t_y + 2t_x + t_x t_y^{-1}) - (t_x^{-1} t_y + 2t_x^{-1} + t_x^{-1} t_y^{-1}) \Big) \tag{25}$$

$$= \frac{(t_x + 1)(t_x - 1)}{2 t_x \Delta x} \cdot \frac{(t_y + 1)^2}{4 t_y} \hat{q} \tag{26}$$

$\frac{t_x - 1}{\Delta x}$	discretization of ∂_x
$\frac{t_x + 1}{2} \frac{t_x - 1}{\Delta x}$	central difference $q_{i+1} - q_{i-1}$
$\frac{(t_x - 1)^2}{\Delta x^2}$	second derivative $\frac{q_{i+1} - 2q_i + q_{i-1}}{\Delta x^2}$

Lemma

The Fourier transform of a finite difference that approximates ∂_x^n contains precisely n factors $t_x - 1$.

Lemma (Normalization)

Consider the Fourier transform $\frac{1}{t_x^k} \prod_{j=1}^{2k} (t_x - s_j)$ with $s_j = 1$ for $j = 1, 2, \ldots, n$ and otherwise $s_j \neq 1$. Then, as $\Delta x \to 0$, it approximates $A \Delta x^n \partial_x^n$ with $A = \prod_{j=1, s_j \neq 1}^{2k} (1 - s_j)$.

Example:

Consider a **central scheme** for linear acoustics (**unstable with explicit time integration**!)

$$\partial_t q + J_x \partial_x q + J_y \partial_y q = 0 \tag{27}$$

Example:

Consider a **central scheme** for linear acoustics (**unstable with explicit time integration**!)

$$\partial_t q + J_x \partial_x q + J_y \partial_y q = 0 \tag{27}$$

$$\partial_t q + J_x \frac{q_{i+1,j} - q_{i-1,j}}{2\Delta x} + J_y \frac{q_{i,j+1} - q_{i,j-1}}{2\Delta y} = 0 \tag{28}$$

Example:

Consider a **central scheme** for linear acoustics (**unstable with explicit time integration**!)

$$\partial_t q + J_x \partial_x q + J_y \partial_y q = 0 \tag{27}$$

$$\partial_t q + J_x \frac{q_{i+1,j} - q_{i-1,j}}{2\Delta x} + J_y \frac{q_{i,j+1} - q_{i,j-1}}{2\Delta y} = 0 \tag{28}$$

$$\partial_t \hat{q} + \underbrace{\left(J_x \frac{(t_x + 1)(t_x - 1)}{2 t_x \Delta x} + J_y \frac{(t_y + 1)(t_y - 1)}{2 t_y \Delta y} \right)}_{=:\mathcal{E}} \hat{q} = 0 \tag{29}$$

Example:

Consider a **central scheme** for linear acoustics (**unstable with explicit time integration**!)

$$\partial_t q + J_x \partial_x q + J_y \partial_y q = 0 \tag{27}$$

$$\partial_t q + J_x \frac{q_{i+1,j} - q_{i-1,j}}{2\Delta x} + J_y \frac{q_{i,j+1} - q_{i,j-1}}{2\Delta y} = 0 \tag{28}$$

$$\partial_t \hat{q} + \underbrace{\left(J_x \frac{(t_x+1)(t_x-1)}{2t_x \Delta x} + J_y \frac{(t_y+1)(t_y-1)}{2t_y \Delta y} \right)}_{=:\mathcal{E}} \hat{q} = 0 \tag{29}$$

In the discrete, the evolution matrix $\mathcal{E}$ plays the role of $\mathbb{i}(\mathbf{J} \cdot \mathbf{k})$.

Example (cont.):

$$\partial_t \hat{q}(t) + \mathcal{E}\hat{q} = 0 \tag{30}$$

Observe that, as before, with $\mathcal{E} e_j = \lambda_j e_j$

$$\hat{q}(t) = \sum_{j=1}^{n} (q_0 \cdot e_j) e_j \exp(-\mathbb{i}\lambda_j t) \tag{31}$$

For example, for the acoustic equations the evolution of the discrete mode is

$$\hat{q}(t) = (\hat{q}_0 \cdot e_1)e_1 + (\hat{q}_0 \cdot e_2)e_2 \exp\left(\mathbb{i}ct\alpha\right) + (\hat{q}_0 \cdot e_3)e_3 \exp\left(-\mathbb{i}ct\alpha\right) \tag{32}$$

with

$$\alpha = \sqrt{\frac{\sin^2(k_y \Delta y)}{\Delta y^2} + \frac{\sin^2(k_x \Delta x)}{\Delta x^2}} \xrightarrow{\Delta x, \Delta y \to 0} |\mathbf{k}| \tag{33}$$

$$e_1 = \begin{pmatrix} -\frac{t_y^2 - 1}{t_y \Delta y} \\ \frac{t_x^2 - 1}{t_x \Delta x} \\ 0 \end{pmatrix} \simeq \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} = 0 \text{ and } p = \text{const} \tag{34}$$

$$\partial_t \hat{q} + \mathcal{E} \hat{q} = 0 \tag{35}$$

The projector onto the zero-eigenvector of $\mathcal{E}$ yields a **numerical constant of motion**, or **discrete involution** (discretization of vorticity in the case of acoustic equations).

For stability under explicit time integration, one needs to add numerical diffusion:

$$f^x_{i+\frac{1}{2},j} = \frac{1}{2} J^x (q_{i+1,j} + q_{ij}) - \frac{1}{2} D^x (q_{i+1,j} - q_{ij}) \tag{36}$$

For stability under explicit time integration, one needs to add numerical diffusion:

$$f^x_{i+\frac{1}{2},j} = \frac{1}{2} J^x (q_{i+1,j} + q_{ij}) - \frac{1}{2} D^x (q_{i+1,j} - q_{ij}) \tag{36}$$

$$\partial_t q + J_x \frac{q_{i+1,j} - q_{i-1,j}}{2\Delta x} + J_y \frac{q_{i,j+1} - q_{i,j-1}}{2\Delta y} \tag{37}$$

$$- D_x \frac{q_{i+1,j} - 2q_{ij} + q_{i-1,j}}{2\Delta x} - D_y \frac{q_{i,j+1} - 2q_{ij} + q_{i,j-1}}{2\Delta y} = 0 \tag{38}$$

For stability under explicit time integration, one needs to add numerical diffusion:

$$f^x_{i+\frac{1}{2},j} = \frac{1}{2} J^x (q_{i+1,j} + q_{ij}) - \frac{1}{2} D^x (q_{i+1,j} - q_{ij}) \tag{36}$$

$$\partial_t q + J_x \frac{q_{i+1,j} - q_{i-1,j}}{2\Delta x} + J_y \frac{q_{i,j+1} - q_{i,j-1}}{2\Delta y} \tag{37}$$

$$- D_x \frac{q_{i+1,j} - 2q_{ij} + q_{i-1,j}}{2\Delta x} - D_y \frac{q_{i,j+1} - 2q_{ij} + q_{i,j-1}}{2\Delta y} = 0 \tag{38}$$

$$\partial_t \hat{q} + \mathcal{E} \hat{q} = 0 \tag{39}$$

Evolution matrix:

$$\mathcal{E} = J_x \frac{t_x^2 - 1}{2\Delta x t_x} + J_y \frac{t_y^2 - 1}{2\Delta y t_y} - D_x \frac{(t_x - 1)^2}{2\Delta x t_x} - D_y \frac{(t_y - 1)^2}{2\Delta y t_y} \tag{40}$$

What are its eigenvalues?

For stability under explicit time integration, one needs to add numerical diffusion:

$$f^x_{i+\frac{1}{2},j} = \frac{1}{2} J^x (q_{i+1,j} + q_{ij}) - \frac{1}{2} D^x (q_{i+1,j} - q_{ij}) \tag{36}$$

$$\partial_t q + J_x \frac{q_{i+1,j} - q_{i-1,j}}{2\Delta x} + J_y \frac{q_{i,j+1} - q_{i,j-1}}{2\Delta y} \tag{37}$$

$$- D_x \frac{q_{i+1,j} - 2q_{ij} + q_{i-1,j}}{2\Delta x} - D_y \frac{q_{i,j+1} - 2q_{ij} + q_{i,j-1}}{2\Delta y} = 0 \tag{38}$$

$$\partial_t \hat{q} + \mathcal{E} \hat{q} = 0 \tag{39}$$

Evolution matrix:

$$\mathcal{E} = J_x \frac{t_x^2 - 1}{2\Delta x t_x} + J_y \frac{t_y^2 - 1}{2\Delta y t_y} - D_x \frac{(t_x - 1)^2}{2\Delta x t_x} - D_y \frac{(t_y - 1)^2}{2\Delta y t_y} \tag{40}$$

What are its eigenvalues? Actually, for the **upwind scheme** ($D_x = |J_x|$, $D_y = |J_y|$)

$$\det \mathcal{E} \neq 0$$

All modes instationary! (von Neumann stability tells you what happens with them)

Definition

Stationarity preserving scheme: $\dim \ker \mathcal{E} = \dim \ker(\mathbf{J} \cdot \mathbf{k}) \; \forall \mathbf{k}$.

I. e. the scheme's stationary states discretize all the analytic stationary states.

Theorem

The upwind scheme is not stationarity preserving.

WB 2017; WB 2018, subm.

Consider the initial example

$$D_x = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & 0 \\ a_3 & 0 & a_4 \end{pmatrix} \qquad D_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}$$

Consider the initial example

$$D_x = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & 0 \\ a_3 & 0 & a_4 \end{pmatrix} \qquad D_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}$$

You can show that, $\boxed{\text{iff } a_1 = 0}$, then $\det \mathcal{E} = 0$, and the numerical scheme becomes stationarity preserving.

Consider the initial example

$$D_x = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & 0 \\ a_3 & 0 & a_4 \end{pmatrix} \qquad D_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}$$

You can show that, $\boxed{\text{iff } a_1 = 0}$, then $\det \mathcal{E} = 0$, and the numerical scheme becomes stationarity preserving.
The (right) eigenvector corresponding to eigenvalue zero is

$$e_1 = \begin{pmatrix} \frac{a_3(t_y - 2 + t_y^{-1})}{2\Delta y} - \frac{c^2(t_y - t_y^{-1})}{2\Delta y} \\ -\frac{a_3(t_x - 2 + t_x^{-1})}{2\Delta x} + \frac{c^2(t_x - t_x^{-1})}{2\Delta x} \\ 0 \end{pmatrix} \tag{41}$$

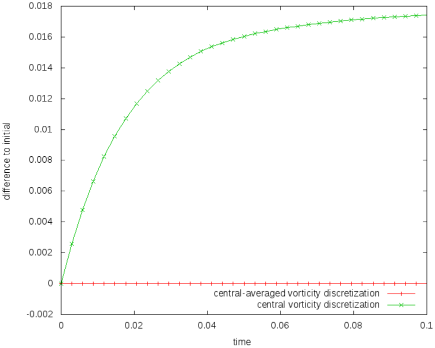
and therefore the vorticity discretization

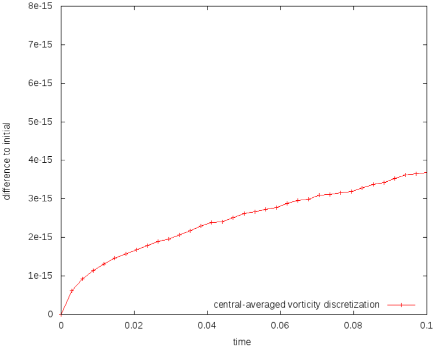
$$\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} + \frac{a_3}{c^2}\left(\frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{2\Delta y} - \frac{v_{i+1,j} - 2v_{ij} + v_{i-1,j}}{2\Delta x}\right) \tag{42}$$

is stationary (as claimed in the introduction).

So far:

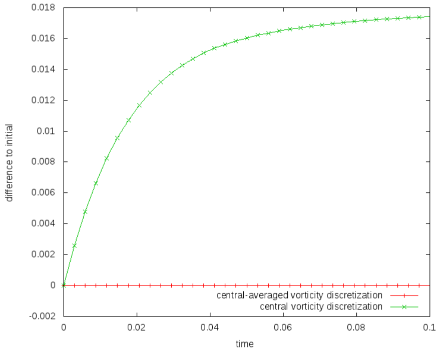
- Classification of all **vorticity preserving** schemes for linear acoustics
- Vorticity preservation is equivalent to the existence of nontrivial stationary states (**stationarity preserving**)

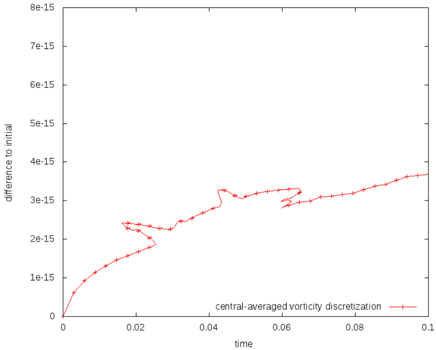




So far:

- Classification of all **vorticity preserving** schemes for linear acoustics
- Vorticity preservation is equivalent to the existence of nontrivial stationary states (**stationarity preserving**)





Low Mach number limit

Take $\epsilon > 0$, and choose $M_{\text{loc}} = \dfrac{|\mathbf{v}|}{\sqrt{\gamma p/\varrho}} \in \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. Then this corresponds to solving

$$\begin{aligned} & & \partial_t \varrho + \mathbf{v} \cdot \nabla \varrho + \varrho \nabla \cdot \mathbf{v} &= 0 \\ \partial_t \mathbf{v} + \frac{\nabla p}{\epsilon^2} &= 0 & \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\nabla p}{\varrho \epsilon^2} &= 0 \\ \partial_t p + c^2 \nabla \cdot \mathbf{v} &= 0 & \partial_t p + \mathbf{v} \cdot \nabla p + \varrho c^2 \nabla \cdot \mathbf{v} &= 0 \end{aligned}$$

Acoustics “inherits” a low Mach number limit.

Take $\epsilon > 0$, and choose $M_{\text{loc}} = \frac{|\mathbf{v}|}{\sqrt{\gamma p/\varrho}} \in \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. Then this corresponds to solving

$$\begin{aligned} \partial_t \mathbf{v} + \frac{\nabla p}{\epsilon^2} &= 0 \\ \partial_t p + c^2 \nabla \cdot \mathbf{v} &= 0 \end{aligned} \qquad \begin{aligned} \partial_t \varrho + \mathbf{v} \cdot \nabla \varrho + \varrho \nabla \cdot \mathbf{v} &= 0 \\ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\nabla p}{\varrho \epsilon^2} &= 0 \\ \partial_t p + \mathbf{v} \cdot \nabla p + \varrho c^2 \nabla \cdot \mathbf{v} &= 0 \end{aligned}$$

Acoustics "inherits" a low Mach number limit.

$$\mathbf{J} \cdot \mathbf{k} = \begin{pmatrix} 0 & 0 & k_x/\epsilon^2 \\ 0 & 0 & k_y/\epsilon^2 \\ k_x c^2 & k_y c^2 & 0 \end{pmatrix} \tag{43}$$

has eigenvalues $\left(0, \pm \frac{c|\mathbf{k}|}{\epsilon}\right)$ such that

$$\hat{q}(t, \mathbf{k}) = (\hat{q}_0 \cdot e_1)e_1 + (\hat{q}_0 \cdot e_2)e_2 \exp\left(\mathbb{i}\frac{ct}{\epsilon}|\mathbf{k}|\right) + (\hat{q}_0 \cdot e_3)e_3 \exp\left(-\mathbb{i}\frac{ct}{\epsilon}|\mathbf{k}|\right) \tag{44}$$

Formal asymptotic analysis:

$$\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}\epsilon + \mathbf{v}^{(2)}\epsilon^2 + \dots \tag{45}$$

$$p = p^{(0)} + p^{(1)}\epsilon + p^{(2)}\epsilon^2 + \dots \tag{46}$$

$$\partial_t \mathbf{v} + \frac{\nabla p}{\epsilon^2} = 0 \tag{47}$$

$$\partial_t p + c^2 \nabla \cdot \mathbf{v} = 0 \tag{48}$$

Formal asymptotic analysis:

$$\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}\epsilon + \mathbf{v}^{(2)}\epsilon^2 + \dots \tag{45}$$

$$p = p^{(0)} + p^{(1)}\epsilon + p^{(2)}\epsilon^2 + \dots \tag{46}$$

$$\partial_t \mathbf{v} + \frac{\nabla p}{\epsilon^2} = 0 \tag{47}$$

$$\partial_t p + c^2 \nabla \cdot \mathbf{v} = 0 \tag{48}$$

Limit equations:

$$\nabla p^{(0)} = 0 \qquad \nabla \cdot \mathbf{v}^{(0)} = 0 \tag{49}$$

$$\nabla p^{(1)} = 0 \tag{50}$$

Take $\epsilon > 0$, and choose $M_{\text{loc}} = \dfrac{|\mathbf{v}|}{\sqrt{\gamma p/\varrho}} \in \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. Then this corresponds to solving

$$\partial_t \varrho + \mathbf{v} \cdot \nabla \varrho + \varrho \nabla \cdot \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \frac{\nabla p}{\epsilon^2} = 0 \qquad\qquad \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\nabla p}{\varrho \epsilon^2} = 0$$

$$\partial_t p + c^2 \nabla \cdot \mathbf{v} = 0 \qquad\qquad \partial_t p + \mathbf{v} \cdot \nabla p + \varrho c^2 \nabla \cdot \mathbf{v} = 0$$

Acoustics “inherits” a low Mach number limit.

$$\mathbf{J} \cdot \mathbf{k} = \begin{pmatrix} 0 & 0 & k_x/\epsilon^2 \\ 0 & 0 & k_y/\epsilon^2 \\ k_x c^2 & k_y c^2 & 0 \end{pmatrix} \tag{51}$$

has eigenvalues $\left(0, \pm \frac{c|\mathbf{k}|}{\epsilon}\right)$ such that

$$\hat{q}(t, \mathbf{k}) = (\hat{q}_0 \cdot e_1)e_1 + (\hat{q}_0 \cdot e_2)e_2 \exp\left(\mathbb{i}\frac{ct}{\epsilon}|\mathbf{k}|\right) + (\hat{q}_0 \cdot e_3)e_3 \exp\left(-\mathbb{i}\frac{ct}{\epsilon}|\mathbf{k}|\right) \tag{52}$$

Take $\epsilon > 0$, and choose $M_{\text{loc}} = \frac{|\mathbf{v}|}{\sqrt{\gamma p/\varrho}} \in \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. Then this corresponds to solving

$$\begin{aligned} & & \partial_t \varrho + \mathbf{v}\cdot\nabla\varrho + \varrho\nabla\cdot\mathbf{v} &= 0 \\ \partial_t \mathbf{v} + \frac{\nabla p}{\epsilon^2} &= 0 & \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\nabla p}{\varrho\epsilon^2} &= 0 \\ \partial_t p + c^2 \nabla\cdot\mathbf{v} &= 0 & \partial_t p + \mathbf{v}\cdot\nabla p + \varrho c^2 \nabla\cdot\mathbf{v} &= 0 \end{aligned}$$

Acoustics "inherits" a low Mach number limit.

$$\mathbf{J}\cdot\mathbf{k} = \begin{pmatrix} 0 & 0 & k_x/\epsilon^2 \\ 0 & 0 & k_y/\epsilon^2 \\ k_x c^2 & k_y c^2 & 0 \end{pmatrix} \tag{51}$$

has eigenvalues $\left(0, \pm\frac{c|\mathbf{k}|}{\epsilon}\right)$ such that

$$\hat{q}(t,\mathbf{k}) = (\hat{q}_0\cdot e_1)e_1 + (\hat{q}_0\cdot e_2)e_2 \exp\left(\mathbb{i}\frac{ct}{\epsilon}|\mathbf{k}|\right) + (\hat{q}_0\cdot e_3)e_3 \exp\left(-\mathbb{i}\frac{ct}{\epsilon}|\mathbf{k}|\right) \tag{52}$$

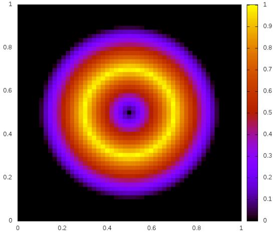
This means that

$$\boxed{\lim_{\epsilon\to 0,\, t \text{ fixed}} q = \lim_{\epsilon \text{ fixed},\, t\to\infty} q} \tag{53}$$

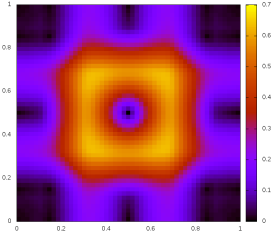
Low Mach number limit is the same as the long time limit.

E.g. for the upwind/Roe scheme

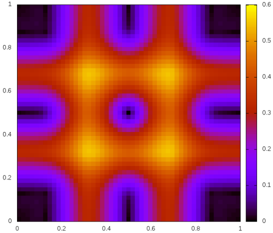
$t = 0$ and exact

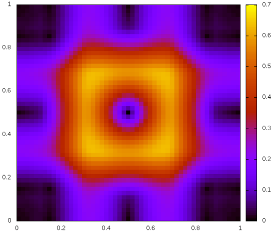


$\epsilon = 1, t = 2$

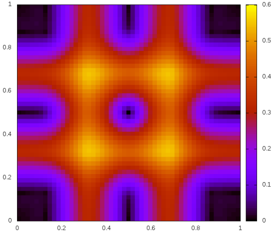


$\epsilon = 1, t = 5$





$\epsilon = \frac{1}{2}, t = 1$



$\epsilon = \frac{1}{5}, t = 1$

What happens to the (numerical) solution for long times?

What happens to the (numerical) solution for long times?

- Instationary modes are decaying by von Neumann stability
- Stationary modes remain

Correct limit in the discrete: only if all nontrivial stationary states are captured!

What happens to the (numerical) solution for long times?

- Instationary modes are decaying by von Neumann stability
- Stationary modes remain

Correct limit in the discrete: only if all nontrivial stationary states are captured!

Vorticity preserving $\Leftrightarrow$ **Stationarity preserving** $\Leftrightarrow$ **Low Mach compliant**

WB 2017; WB 2018, subm.

Summary:

- acoustics is an atomic system of equations in multi-d
- classification of all vorticity preserving schemes
- new understanding of the performance of schemes in the low Mach number limit
- complete understanding of the behaviour of linear schemes for acoustics

Vorticity preserving $\Leftrightarrow$ **Stationarity preserving** $\Leftrightarrow$ **Low Mach compliant**

Summary:

- acoustics is an atomic system of equations in multi-d
- classification of all vorticity preserving schemes
- new understanding of the performance of schemes in the low Mach number limit
- complete understanding of the behaviour of linear schemes for acoustics

Vorticity preserving $\Leftrightarrow$ **Stationarity preserving** $\Leftrightarrow$ **Low Mach compliant**

- you can treat similarly all kinds of linear systems (e.g. taking into account source terms)
- truly multi-dimensional discretizations (uniqueness)
- stationarity preserving limiters for higher order schemes?
- there exist nonlinear versions of stationarity preservation / vorticity "preservation": relation to low Mach number?

Challenge:

Generalization to non-Cartesian grids?