An IMEX FV Scheme for the Euler equations in the low Mach regime

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joint work with

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SHARK-FV 2019 Conference

May, 20 2019

Motivation/Problem description

- Aim: model flows close to the incompressible regime
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\begin{aligned}
\partial_t \rho + \nabla \cdot \mathbf{u} &= 0, \\
\partial_t \rho\mathbf{u} + \nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u}) + \frac{\nabla p}{M^2} &= 0, \\
\partial_t E + \nabla \cdot (\mathbf{u}(E + p)) &= 0, \\
E = \rho e + \frac{1}{2} M^2 \rho |\mathbf{u}|^2, p =& (\gamma - 1)\rho e.
\end{aligned}
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- Eigenvalues $u, u \pm \frac{c}{M}$
- Limit $M \to 0$: compressible Euler equations $\to$ incompressible Euler equations (Klainerman and Majda (1982), Klein (1995), Dellacherie (2010))

Numerical challenges

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- Numerical diffusion independent of M
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  - explicit: $\Delta t \leq M c_{cfl} \frac{\Delta x}{\max |Mu \pm a|}$
  - implicit: unconditionally stable wrt Δt, but solving non-linear implicit system
- Requirements
  - low computational costs
  - preserve physical properties as $\rho > 0$ and $e > 0$
  - mimic the transition from compressible to incompressible (AP property)

$$\begin{array}{ccc} w^M & \xrightarrow[\text{continuous}]{M \to 0} & w^0 \\ \Delta \to 0 \uparrow & & \uparrow \Delta \to 0 \\ w_\Delta^M & \xrightarrow[M \to 0]{\text{discrete}} & w_\Delta^0 \end{array}$$

Continuous limit equations

- w^0 solution of incompressible Euler equations

$$\begin{aligned} \rho &= \text{const.} \\ u_t + u \cdot \nabla u + \nabla \Pi &= 0 \\ \nabla \cdot u &= 0 \end{aligned}$$

- Start with well prepared data for $w = (\rho, \rho\mathbf{u}, E)^T$

$$\begin{aligned} \rho &= \rho_0 + \mathcal{O}(M), & \rho_0 &= \text{const.} \\ u &= u_0 + \mathcal{O}(M), & \nabla \cdot u_0 &= 0 \\ p &= p_0 + \mathcal{O}(M^2), & p_0 &= \text{const.} \end{aligned}$$

$$\Omega_{AP} = \left\{ w \in \mathbb{R}^{2+d} | \nabla \rho_0 = 0, \nabla p_0 = 0, \nabla \cdot u_0 = 0 \right\}$$

- Assume boundary conditions that preserve the scaling of p in time

Suliciu Relaxation

- Motivation:
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    $\Rightarrow$ all waves become contact discontinuities
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- Remove non-linearity by turning (1) into a relaxation equation with constant relaxation parameter a

$$\partial_t(\rho\pi) + \partial_x(\rho\pi u) + a^2 \partial_x u = \frac{\rho}{\varepsilon}(p(\rho, e) - \pi)$$

- Chapman-Enskog: Stable diffusive approximation of the original Euler equations for the subcharacteristic condition $a \geq \rho\sqrt{p_\rho(\rho, e)}$

Suliciu Relaxation Model[1]

$$\begin{aligned} \rho_t + \nabla \cdot \rho \mathbf{u} &= 0 \\ \rho \mathbf{u}_t + \nabla \cdot \rho \mathbf{u} \otimes \mathbf{u} + \nabla \frac{p}{M^2} &= 0 \\ E_t + \nabla \cdot \mathbf{u}(E + p) &= 0 \end{aligned}$$

[1] Berthon, Klingenberg, Zenk, 2018, Preprint

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$$\rho_t + \nabla \cdot \rho \mathbf{u} = 0$$
$$\rho \mathbf{u}_t + \nabla \cdot \rho \mathbf{u} \otimes \mathbf{u} + \nabla\left(p + \frac{1-M^2}{M^2} p\right) = 0$$
$$E_t + \nabla \cdot \mathbf{u}(E + M^2 p + (1-M^2)p) = 0$$

- Pressure splitting: $p = M^2 p + (1-M^2)p$

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\rho \mathbf{u}_t + \nabla \cdot \rho \mathbf{u} \otimes \mathbf{u} + \nabla\left(\pi + \frac{1 - M^2}{M^2} p\right) &= 0 \\
E_t + \nabla \cdot \mathbf{u}(E + M^2 \pi + (1 - M^2) p) &= 0 \\
\rho \pi_t + \nabla \cdot (\rho \pi \mathbf{u} + a^2 \mathbf{u}) &= \frac{\rho}{\varepsilon}(p - \pi)
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$$\rho \hat{\mathbf{u}}_t + \nabla \cdot \rho \mathbf{u} \otimes \hat{\mathbf{u}} + \frac{\nabla \psi}{M^2} = \frac{\rho}{\varepsilon}(\mathbf{u} - \hat{\mathbf{u}})$$
$$\rho \psi_t + \nabla \cdot (\rho \psi \mathbf{u} + a^2 \hat{\mathbf{u}}) = \frac{\rho}{\varepsilon}(p - \psi)$$

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\rho \hat{\mathbf{u}}_t + \nabla \cdot \rho \mathbf{u} \otimes \hat{\mathbf{u}} + \frac{\nabla \psi}{M^2} &= \frac{\rho}{\varepsilon}(\mathbf{u} - \hat{\mathbf{u}}) \\
\rho \psi_t + \nabla \cdot (\rho \psi \mathbf{u} + a^2 \hat{\mathbf{u}}) &= \frac{\rho}{\varepsilon}(p - \psi)
\end{aligned}
$$

- Pressure splitting: $p = M^2 p + (1 - M^2)p$
- Suliciu relaxation introducing π
- Ordered linear degenerate eigenvalues: u (2 times), $u \pm \frac{a}{\rho}, u \pm \frac{a}{\rho M}$
- Stable diffusive approximation of non-dimensional Euler equations with $a \geq \rho\sqrt{p_\rho(\rho, e)}$
- 3 scales: $\mathcal{O}(1), \mathcal{O}(M^{-2}), \mathcal{O}(\varepsilon^{-1})$

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Splitting

$$W_t + \nabla \cdot F(W) + \frac{1}{M^2}\nabla \cdot G(W) = \frac{1}{\varepsilon}R(W)$$

$$F(W) = \begin{pmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\otimes\mathbf{u} + \pi\mathbb{1} + \frac{1-M^2}{M^2}\psi\mathbb{1} \\ (E + M^2\pi + (1-M^2)\psi)\mathbf{u} \\ \rho\pi\mathbf{u} + a^2\mathbf{u} \\ \rho\mathbf{u}\otimes\hat{\mathbf{u}} \\ \rho\psi\mathbf{u} \end{pmatrix}, \quad G(W) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \psi \\ a^2M^2\hat{\mathbf{u}} \end{pmatrix},$$

$$R(W) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \rho(p-\pi) \\ \rho(\mathbf{u}-\hat{\mathbf{u}}) \\ \rho(p-\psi) \end{pmatrix}, \qquad W = \begin{pmatrix} \rho \\ \rho u \\ E \\ \rho\pi \\ \rho\hat{\mathbf{u}} \\ \rho\psi \end{pmatrix}$$

Time semi-discrete scheme

$$
\begin{aligned}
\text{Implicit:} \quad & W_t + \frac{1}{M^2}\nabla \cdot G(W) = 0,\\
\text{Explicit:} \quad & W_t + \nabla \cdot F(W) = 0,\\
\text{Projection:} \quad & W_t = \frac{1}{\varepsilon}R(W).
\end{aligned}
$$

- Start from equilibrium

$$W^{n,eq} := (\rho^n, \rho\mathbf{u}^n, E^n, \rho p^n, \rho\mathbf{u}^n, \rho p^n)^T$$

- Solve implicit equations

$$W^{(1)} - W^{n,eq} + \frac{\Delta t}{M^2}\nabla \cdot G(W^{(1)}) = 0$$

- Solve explicit equations

$$W^{(2)} - W^{(1)} + \Delta t \nabla \cdot F(W^{(1)}) = 0$$

- Project onto equilibrium

$$W^{n+1} = W^{(2),eq}$$

Implicit part

$$\frac{\rho^{(1)} - \rho^n}{\Delta t} = 0,$$
$$\frac{(\rho\hat{\mathbf{u}})^{(1)} - (\rho\hat{\mathbf{u}})^n}{\Delta t} + \frac{1}{M^2}\nabla\psi^{(1)} = 0,$$
$$\frac{(\rho\psi)^{(1)} - (\rho\psi)^n}{\Delta t} + a^2\nabla\cdot\hat{\mathbf{u}}^{(1)} = 0.$$

- Rewrite into a single equation, independent of $\hat{\mathbf{u}}^{(1)}$, $\tau^n = \frac{1}{\rho^n}$

$$\psi^{n+1} - \frac{\Delta t^2 a^2}{M^2}\tau^n\nabla\cdot\left(\tau^n\nabla\psi^{n+1}\right) = \psi^n - \Delta t a^2\tau^n\nabla\cdot u^n$$

- Space discretization: central differences
- Scaling of $\psi^{(1)}$ wrt M is important for
  - AP property
  - Positivity of ρ and e
  - Mach number independent numerical diffusion

AP property

Scaling of $\psi^{(1)}$

Let $w \in \Omega_{AP}$. Then the expansion of $\psi^{(1)}$ with respect to the Mach number M with p_0 constant, is given as

$$\psi^{(1)} = p_0 + M^2\psi_2 + \mathcal{O}(M^3).$$

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Discrete limit equations

$$\begin{cases} \rho_0^{n+1} - \rho_0^n + \Delta t \rho_0^n \nabla \cdot u_0^n & = 0 \\ \rho_0^{n+1} u_0^{n+1} - \rho_0^n u_0^n + \Delta t (\rho_0^n \nabla \cdot u_0^n \otimes u_0^n + \nabla \psi_2^{(1)}) & = 0 \\ e_0^{n+1} - e_0^n + \Delta t \nabla \cdot u_0^n (e_0^n + p_0^{(1)}) & = 0 \end{cases}$$

$$\overset{w^n \in \Omega_{AP}}{\Rightarrow} \begin{cases} \rho_0^{n+1} - \rho_0^n & = 0 \\ u_0^{n+1} - u_0^n + \Delta t \left(u_0^n \cdot \nabla u_0^n + \dfrac{\nabla \psi_2^{(1)}}{\rho_0} \right) & = 0 \\ p_0^{n+1} - p_0^n & = 0 \end{cases}$$

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Divergence free property

Let $w \in \Omega_{AP}$. Then we have $\nabla \cdot u_0^{n+1} = \mathcal{O}(\Delta t)$.

Explicit part

- Dimensional splitting to derive multi-dimensional scheme on a cartesian grid

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- Dimensional splitting to derive multi-dimensional scheme on a cartesian grid
- Equations in 1D

$$
\begin{aligned}
\partial_t \rho &+ \partial_x \rho u &= 0\\
\partial_t \rho u &+ \partial_x (\rho u^2 + \pi + \frac{1-M^2}{M^2}\psi) &= 0\\
\partial_t E &+ \partial_x (E + M^2\pi + (1-M^2)\psi) u &= 0\\
\partial_t \rho\pi &+ \partial_x (\rho\pi + a^2) u &= 0\\
\partial_t \rho\hat{u} &+ \partial_x \rho u \hat{u} &= 0\\
\partial_t \rho\psi &+ \partial_x \rho\psi u &= 0
\end{aligned}
$$

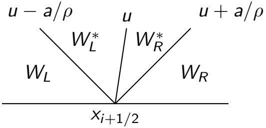
- Linearly degenerate system with ordered eigenvalues u (4 times), $u \pm \frac{a}{\rho}$
- CFL-restriction $\Delta t < \frac{1}{2} \frac{\Delta x}{\max |u \pm a/\rho|}$ independent of M
- Godunov-type Riemann solver $\Rightarrow$ solve Riemann Problem

Approx. Riemann solver according to Harten, Lax, Leer

- Solve Riemann problem at interfaces

$$W_{i+1/2}(x) = \begin{cases} W_L := W_i & \textit{if } x < x_{i+1/2} \\ W_R := W_{i+1} & \textit{if } x > x_{i+1/2}. \end{cases}$$

- structure of solution:



- Define Riemann solver

$$W_{\mathcal{RS}}\left(\frac{x - x_{i-1/2}}{t}; W_L, W_R\right) = \begin{cases} W_L & \frac{x_1}{t} < \lambda^-, \\ W_L^* & \lambda^- < \frac{x_1}{t} < \lambda^u, \\ W_R^* & \lambda^u < \frac{x_1}{t} < \lambda^+, \\ W_R & \lambda^+ < \frac{x_1}{t}, \end{cases}$$

- Compute intermediate states $W_{L,R}^*$ using Riemann invariants

Flux at the interface

$$F_{i+1/2} = \begin{cases} F(W_L^{(1)}) & \lambda^- > 0 \\ F(W_L^{*,(1)}) & \lambda^u > 0 > \lambda^- \\ F(W_R^{*,(1)}) & \lambda^+ > 0 > \lambda^u \\ F(W_R^{(1)}) & \lambda^+ < 0 \end{cases}$$

$$W^{(2)} = W^{(1)} - \frac{\Delta t}{\Delta x}(F_{i+1/2} - F_{i-1/2})$$

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- Physical variables independent of $\hat{\mathbf{u}}$
  $\Rightarrow$ not necessary to compute $\hat{\mathbf{u}}$
- Update only the physical variables $\rho, \rho\mathbf{u}, E$
- $M = 1$: Riemann solver for compressible Euler equations
  $\Rightarrow$ fully explicit scheme

Positivity of density and internal energy

Let $w_i^{n,eq} \in \Omega_{AP} \cap \left(\Omega_{phy} = \left\{ w \in \mathbb{R}^{d+2} | \rho > 0, e > 0 \right\}\right)$, then for an relaxation parameter a sufficiently large and independent of M, we have $w_i^{n+1} \in \Omega_{phy}$.

- After implicit step: $w_i^{(1)} = w^{n,eq} \in \Omega_{phy}$ and $\psi_i^{(1)} = p_0 + \mathcal{O}(M^2)$
- With ordering of eigenvalues $u_L - \frac{a}{\rho_L} < u^*$:

$$\frac{1}{\rho_L^*} = \frac{1}{\rho_L} + \frac{u^* - u_L}{a} \geq \frac{1}{\rho_L} - \frac{1}{\rho_L} = 0.$$

- Inserting π_L^* into e_L^*

$$e_L^* = e_L + \frac{1}{8}(u_L - u_R)^2 + \frac{1}{2a^2}\left(-\pi_L^2 + \frac{1}{4}\left(\pi_L + \pi_R + \frac{1-M^2}{M^2}(\psi_R - \psi_L)\right)^2 + \frac{1}{2}\psi_L(1-M^2)\left(\pi_R - \pi_L + \frac{1-M^2}{M^2}(\psi_R - \psi_L)\right)\right) - \frac{1}{4a}(u_L - u_R)\left(\pi_L + \pi_R + \frac{1-M^2}{M^2}(\psi_R - \psi_L) + (1-M^2)\psi_L\right)$$

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- After implicit step: $w_i^{(1)} = w^{n,eq} \in \Omega_{phy}$ and $\psi_i^{(1)} = p_0 + \mathcal{O}(M^2)$
- $W_{\mathcal{RS}}^{(\rho,\rho u,E)} \in \Omega_{phy}$.
- Write solution as convex combination of Riemann Solvers

$$w_i^{n+1} = \frac{1}{\Delta x}\left(\int_{x_i-1/2}^{x_i} W_{\mathcal{RS}}^{(\rho,\rho u,E)}\left(\frac{x}{t^{n+1}}, W_{i-1}^{(1)}, W_i^{(1)}\right) dx + \int_{x_i}^{x_i+1/2} W_{\mathcal{RS}}^{(\rho,\rho u,E)}\left(\frac{x}{t^{n+1}}, W_i^{(1)}, W_{i+1}^{(1)}\right) dx\right)$$

Mach number independence of Diffusion

Scaling wrt Mach number

	$W^{n,eq}$	$W^{(1)}$	W^*
$\rho_{L,R}$	$\rho_0 + \mathcal{O}(M)$	$\rho_0 + \mathcal{O}(M)$	$\rho_0 + \mathcal{O}(\Delta x) + \mathcal{O}(M)$
$u_{L,R}$	$u_{0,L,R} + \mathcal{O}(M)$	$u_{0,L,R} + \mathcal{O}(M)$	$u_{0,L,R} + \mathcal{O}(\Delta x) + \mathcal{O}(M)$
$e_{L,R}$	$e_0 + \mathcal{O}(M)$	$e_0 + \mathcal{O}(M)$	$e_0 + \mathcal{O}(\Delta x) + \mathcal{O}(M)$
$\pi_{L,R}$	$p_0 + \mathcal{O}(M^2)$	$p_0 + \mathcal{O}(M^2)$	$p_0 + \mathcal{O}(\Delta x) + \mathcal{O}(M^2)$
$\psi_{L,R}$	$p_0 + \mathcal{O}(M^2)$	$p_0 + \mathcal{O}(M^2)$	$p_0 + \mathcal{O}(M^2)$

Diffusion form

$$D(w_R - w_L) = \frac{1}{2}(f(W_L) + f(W_R)) - F^{(\rho,\rho u,E)}(W_{\mathcal{RS}})$$

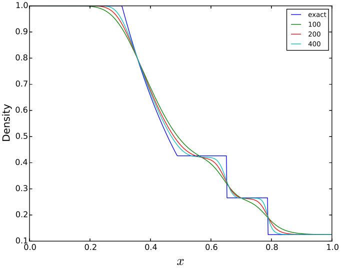
$$F^{(\rho,\rho u,E)}(W^*_{L,R}) = \begin{pmatrix} (\rho u)^* \\ (\rho u^2 + \pi + \frac{1-M^2}{M^2}\psi)^* \\ (u(E + M^2\pi + (1-M^2)\psi))^* \end{pmatrix} = \begin{pmatrix} \rho_0 u_{0,L,R} + \mathcal{O}(1) \\ \rho_0 u^2_{0,L,R} + \frac{p_0}{M^2} + \mathcal{O}(1) \\ u_{0,L,R}(E_0 + p_0) + \mathcal{O}(1) \end{pmatrix}$$

Numerical test cases

SOD shock tube

$$\rho_L = 1, \qquad \rho_R = 0.125,$$
$$u_L = 0, \qquad u_R = 0,$$
$$p_L = 1, \qquad p_R = 0.1,$$

compressible regime $M = 1$



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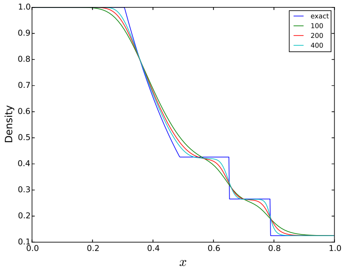
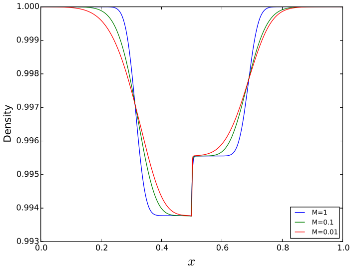


Mach number dependent shock

$$\rho_L = 1, \quad \rho_R = 1,$$
$$u_L = 0, \quad u_R = 0.008,$$
$$p_L = 0.4, \quad p_R = 0.399.$$

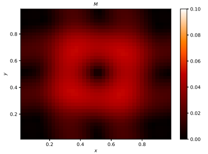
Scaling:
$\rho_{ref} = 1,\ u_{ref} = 1,\ p_{ref} = 1/M^2$



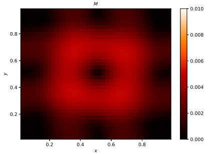
2D - Gresho test case

Setting:
Mesh 40x40

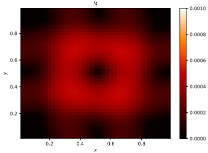
$M = 10^{-1}$



$M = 10^{-2}$

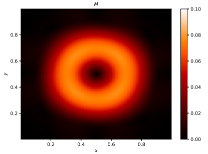


$M = 10^{-3}$

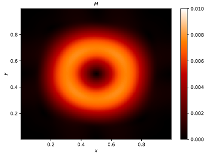


Setting:
Mesh 200x200

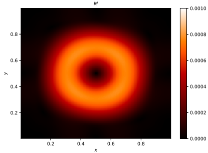
$M = 10^{-1}$



$M = 10^{-2}$

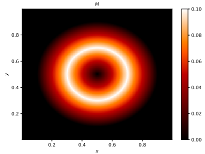


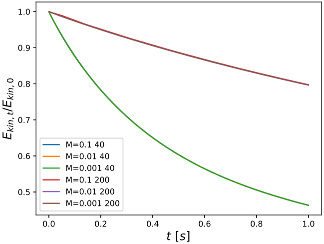
$M = 10^{-3}$



Initial configuration

$M = 10^{-1}$





Second order extension

Space

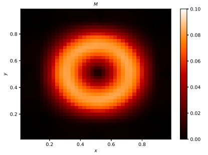
- Construction of piecewise linear functions in $(\rho, \rho\mathbf{u}, E, \psi)$
- Interface values given by $w^{\mp}_{i\pm 1/2} = w_i \pm \sigma_i \frac{\Delta x_i}{2}$
- Ensure **positivity** of ρ, p: Use minmod limiter for slope reconstruction

Time

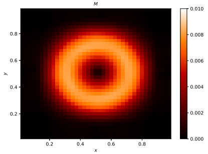
$$\begin{aligned}
\overline{W}^{(1)} - \quad & W^{n,eq} & + \frac{\Delta t}{M^2} \nabla \cdot G(\overline{W}^{(1)}) & = 0 \\
\overline{W}^{(2)} - \quad & \overline{W}^{(1)} & + \Delta t \nabla \cdot F(\overline{W}^{(1)}) & = 0 \\
\overline{\overline{W}}^{(1)} - \quad & \overline{W}^{(2),eq} & + \frac{\Delta t}{M^2} \nabla \cdot G(\overline{\overline{W}}^{(1)}) & = 0 \\
\overline{\overline{W}}^{(2)} - \quad & \overline{\overline{W}}^{(1)} & + \Delta t \nabla \cdot F(\overline{\overline{W}}^{(1)}) & = 0 \\
W^{n+1} = \quad & \frac{1}{2} W^{n,eq} + \frac{1}{2} \overline{\overline{W}}^{(2),eq} & &
\end{aligned}$$

Setting:
Mesh 40x40

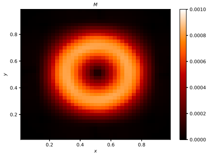
$M = 10^{-1}$



$M = 10^{-2}$

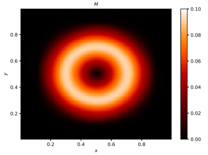


$M = 10^{-3}$

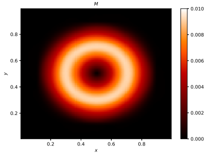


Setting:
Mesh 100x100

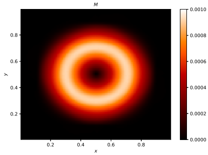
$M = 10^{-1}$



$M = 10^{-2}$

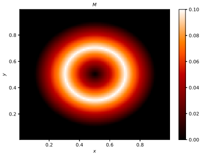


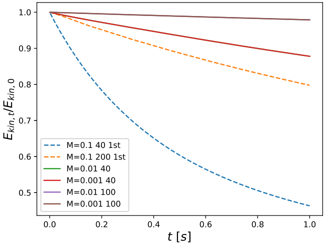
$M = 10^{-3}$



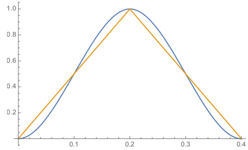
Initial configuration

$M = 10^{-1}$

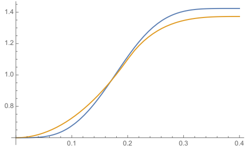




Convergence of second order scheme



Velocity



Pressure

original Gresho profiles
smooth Gresho profiles

N	ρ		u_1		u_2		p	
20	5.190E-004	—	2.017E-003	—	2.017E-003	—	5.530E-004	
40	1.164E-004	2.156	5.626E-004	1.843	5.626E-004	1.843	1.240E-004	2.156
60	4.505E-005	2.341	2.833E-004	1.692	2.833E-004	1.692	4.801E-005	2.341
80	2.305E-005	2.328	1.726E-004	1.721	1.726E-004	1.721	2.456E-005	2.329

L^1-error and convergence rates for $M = 0.1$ at $T = 0.016s$.

N	ρ		u_1		u_2		p	
20	1.050E-003	—	1.000E-002	—	1.000E-002	—	1.109E-003	—
40	1.731E-004	2.601	2.714E-003	1.882	2.714E-003	1.882	1.827E-004	2.601
60	5.566E-005	2.798	1.272E-003	1.868	1.272E-003	1.868	5.875E-005	2.798
80	2.443E-005	2.862	7.388E-004	1.890	7.388E-004	1.890	2.578E-005	2.862

L^1-error and convergence rates for $M = 0.001$ at $T = 0.079s$.

Summary

- Second order IMEX FV scheme for low Mach applications
- Mach number independent time step
- Asymptotic preserving towards incompressible Euler equations
- Preservation of $\rho > 0$ and $e > 0$

Outlook

- Open question: Entropy stability?
- Work in progress: Well balanced IMEX FV scheme for Euler equations with Gravity for low Mach applications

Thank you for your attention! :)