SOLUTION PROPERTY PRESERVING METHOD FOR EULER EQUATIONS: A BVD_MOOD_APPROACH

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Outline

1. Introduction

2. General Framework

3. Solution Property Preserving Method

4. Numerical Results (1D and 2D cases)

5. Conclusion and Future Work

1. Introduction

❖ High order schemes developed for handling both smooth and discontinuous solution

- MUSCL scheme, PPM (2nd order scheme)
- WENO, DG scheme (3rd order scheme or higher-order)

∆ **One issue of high order scheme is generating non-physical negative density or pressure leads to blow-up the computation or code crash.**

❖ To prevent this issue:

- DG, finite volume/difference WENO flux limiter restricted the CFL number
- Flux Corrected Transport (FCT), cut-off limiter, bounded preserving, etc.
- Multi-dimensional Optimal Order Detection (MOOD)

1. Introduction

- **MOOD** is an *a posteriori* limiting process scheme:
  - Physical Admissible Detection (PAD)
  - Numerical Admissible Detection (NAD)

- **THINC** scheme using hyperbolic tangent function, mimics a jump-like solution and is employed to capture discontinuous solution

- **BVD algorithm** selecting the appropriate reconstructions rely on jump between reconstructed values at the cell boundary

A new BVD → $P_nT_m - BVD$ Multi-stage BVD

(Deng, *JCP*,2019)

Research Purpose

- **New Algorithm is proposed:**

1. How to get the high-accurate in smooth solutions?

    - High order polynomial based reconstruction

2. How to deal with discontinuous solutions?

    - Boundary Variation Diminishing (BVD) algorithm

3. How to preserve the positivity of physical properties of fluids?

    - Multi-dimensional Optimal Order Detection (MOOD)

➡ **Multi-stage BVD-MOOD Approach**

2. General Framework

➢ Finite Volume Method

The scalar hyperbolic conservation laws:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \qquad (1)$$

where U(x,t) is solution function and F(U) is the flux function. In the case of linear advection, F$(U) = \alpha U$ or $\alpha = F'(U)$, the characteristic speed.

A uniform discretization of the domain $\Omega = [x^L, x^R]$

$$x_i = x_0 + (i + \tfrac{1}{2})\Delta x \qquad \textit{For } i = 1,\ldots,N \qquad \textit{where } \Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$

The cell elements of control volumes

$$I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \qquad \textit{For } i = 1,\ldots,N$$

Introduce the cell average as volume-integrated average (VIA) as:

$$\overline{U}_i(t) = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} U(x,t)dx \qquad (2)$$

2. General Framework

For each cell I_i, the VIA $\overline{U}_i(t)$ is updated by

$$\frac{d\overline{U}_i}{dt} = -\frac{1}{\Delta x}\left(\tilde{F}_{i+\frac{1}{2}} - \tilde{F}_{i-\frac{1}{2}}\right), \quad (3)$$

where $\tilde{F}_{i+\frac{1}{2}}$ *and* $\tilde{F}_{i-\frac{1}{2}}$ *are numerical fluxes at cell boundaries*

Numerical fluxes computed by a Riemann Solver (HLLC in this work)

$$\tilde{F}_{i+\frac{1}{2}} = F^{Riemann}_{i+\frac{1}{2}}\left(U^L_{i+\frac{1}{2}}, U^R_{i+\frac{1}{2}}\right). \quad (4)$$

Particularly, the Riemann flux can be written into a canonical form

$$F^{Riemann}_{i+\frac{1}{2}}\left(U^L_{i+\frac{1}{2}}, U^R_{i+\frac{1}{2}}\right) = \underbrace{\frac{1}{2}\left(F\left(U^L_{i+\frac{1}{2}}\right) + F\left(U^R_{i+\frac{1}{2}}\right)\right)}_{\text{Central flux}} - \underbrace{\frac{\left|a_{i+\frac{1}{2}}\right|}{2}\left(U^R_{i+\frac{1}{2}} - U^L_{i+\frac{1}{2}}\right)}_{\text{Dissipation}} \quad (5)$$

- The spatial discretization reconstructed by piecewise polynomial reconstruction scheme and THINC schemes
- The time integration scheme is 4th-order Runge-Kutta (SSPRK)

3. Solution Property Preserving Method

Some properties of numerical solution should be preserved by the numerical scheme:

- High accuracy in regular zones → Accuracy on smooth profile.
- Free from spurious oscillation close to steep gradient → Non oscillatory behavior.
- Sharp capture of discontinuity → Accuracy on discontinuous profile.
- Robustness for extreme situations → Fail-safe behavior.

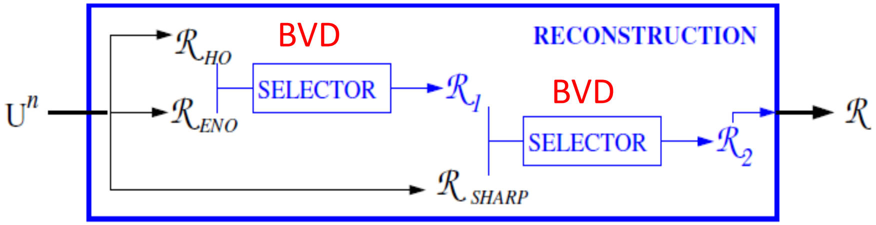


Illustration of the solution property preserving method

- $\mathcal{R}_{HO}$ is a linear 5th-order upwind scheme
- $\mathcal{R}_{ENO}$ and $\mathcal{R}_{SHARP}$ is THINC scheme with small beta and large beta value

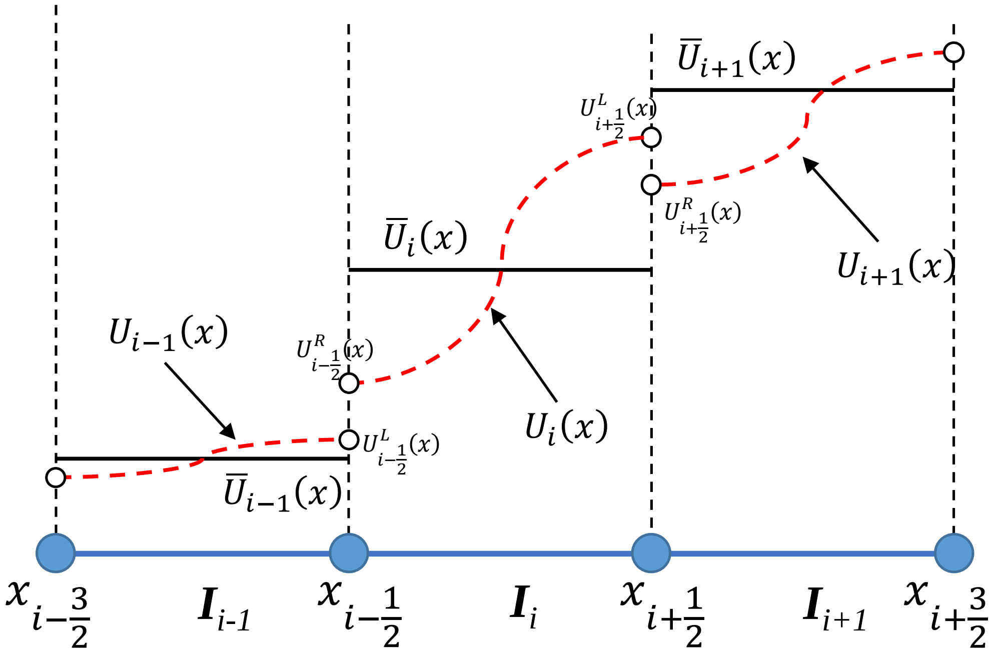
3. Solution Property Preserving Method

➢ THINC Method

THINC scheme uses the hyperbolic tangent function which is sigmoid function and is differentiable and monotone function.

- THINC reconstruction function is defined as

$$U_i(x) = U_{min} + \frac{U_{max}}{2}\left(1 + \gamma tanh\left(\beta\left(\frac{x - x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} - \tilde{x}_i\right)\right)\right), \quad (6)$$



where $U_{min} = min(\bar{U}_{i-1}, \bar{U}_{i+1})$,

$U_{max} = max(\bar{U}_{i-1}, \bar{U}_{i+1}) - U_{min}$

and $\gamma = sgn(\bar{U}_{i+1} - \bar{U}_{i-1})$.

β is used for controlling the jump thickness

$$\bar{U}_i(x) = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} U_i(x)dx$$

3. Solution Property Preserving Method

➢ THINC Method

- The value of cell I_i interface for left-side and right-side:

$$U_i^L(x_{i+\frac{1}{2}}) = U_{min} + \frac{U_{max}}{2}\left(1 + \gamma \frac{\tanh(\beta) + A}{1 + A\tanh(\beta)}\right),$$
$$U_i^R(x_{i-\frac{1}{2}}) = U_{min} + \frac{U_{max}}{2}(1 + \gamma A), \tag{7}$$

where $A = \frac{\frac{B}{\cosh(\beta)} - 1}{\tanh(\beta)}$, $B = exp(\gamma\beta(2C - 1))$ and $C = \frac{\bar{U}_i - \bar{U}_{min} + \epsilon}{\bar{U}_{max} + \epsilon}$ with $\epsilon = 10^{-20}$.

- $\mathcal{R}_{ENO}$ is THINC reconstruction with $\beta \leq 1.2$
- $\mathcal{R}_{SHARP}$ is THINC reconstruction with $\beta \geq 1.6$

(Sun,*JCP*,2016 & Xiao,*JMF*,2005, & Deng,CF,2018)

3. Solution Property Preserving Method

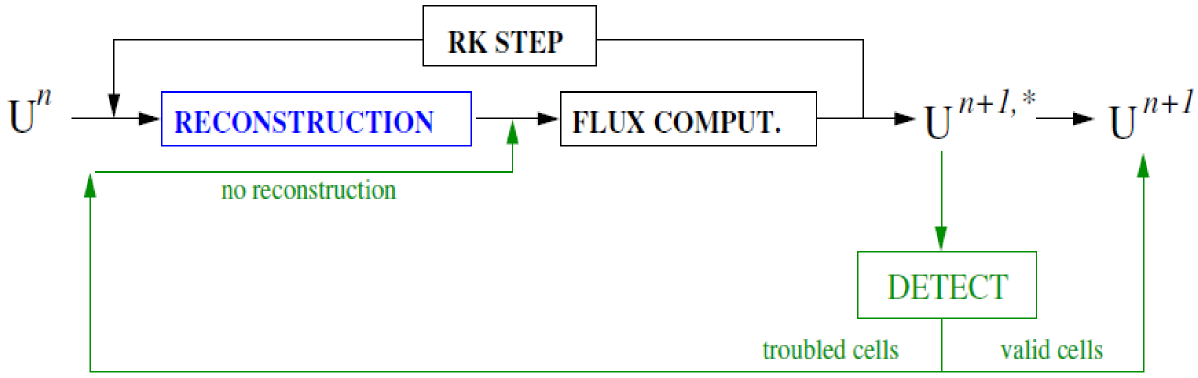
➢ An *a posteriori* MOOD procedure

- The detection criteria are split into a Physical Admissible Detection (PAD) and a Numerical Admissible Detection (NAD) (Clain,*JCP*,2011 & Diot,*JCP*,2012)

Detect: PAD$(U_i^{n+1,*})$, and NAD$(U_i^{n+1,*})$

○ **Physical Admissible Detection (PAD)**

$$\rho_i^* > 0 \quad \text{and} \quad p_i^* > 0 \qquad (8)$$



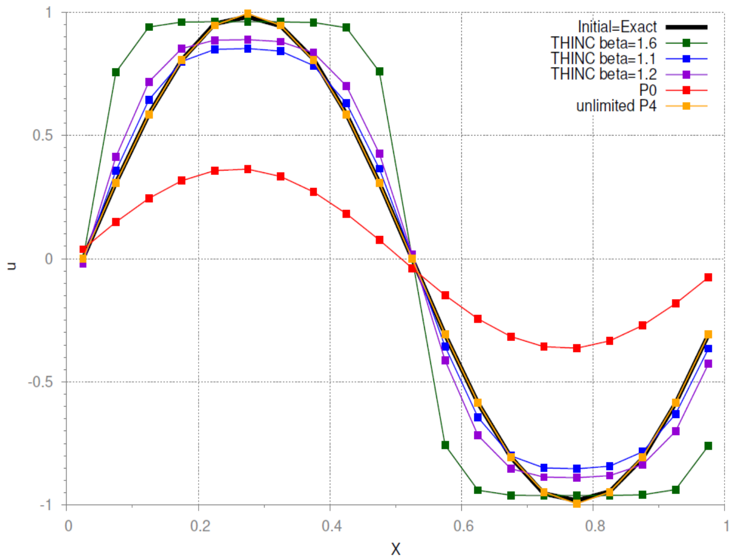
Sketch of the finite volume with a posteriori MOOD procedure

3. Solution Property Preserving Method

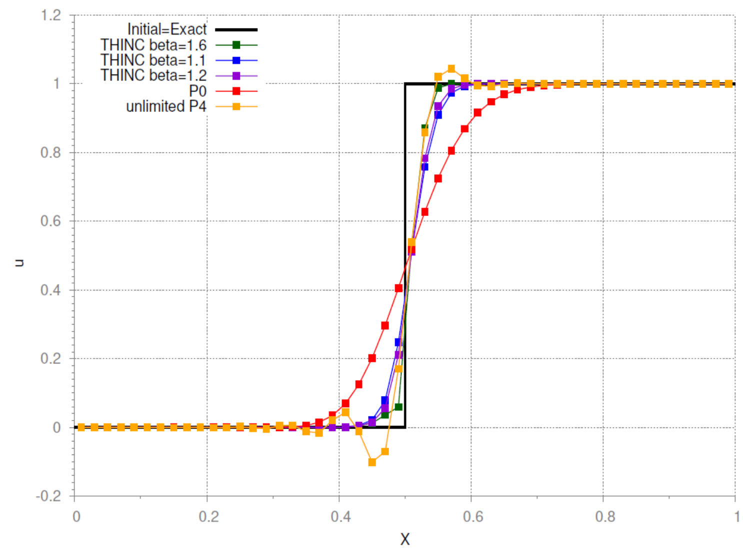
➢ Illustration of the behavior of the reconstruction $\mathcal{R}$

The behavior of FV schemes with different reconstructions

- $\mathcal{R}_{HO}$ is a linear 5th-order upwind scheme (P4)
- $\mathcal{R}_{ENO_2}$ and $\mathcal{R}_{ENO_1}$ are THINC schemes with $\beta = 1.2$ and $\beta = 1.1$
- $\mathcal{R}_{SHARP}$ is THINC scheme with $\beta = 1.6$
- $\mathcal{R}_{LO}$ is a piece-wise constant scheme (P0)



Smooth sine profile $U(x) = \sin(2\pi x)$



Discontinuous step profile $U(x) = \frac{1}{2}\left(1 + \frac{(x-1/4)}{|x-1/4|}\right)$

3. Solution Property Preserving Method

➢ Local selection of reconstruction operator: a 3-stage BVD algorithm

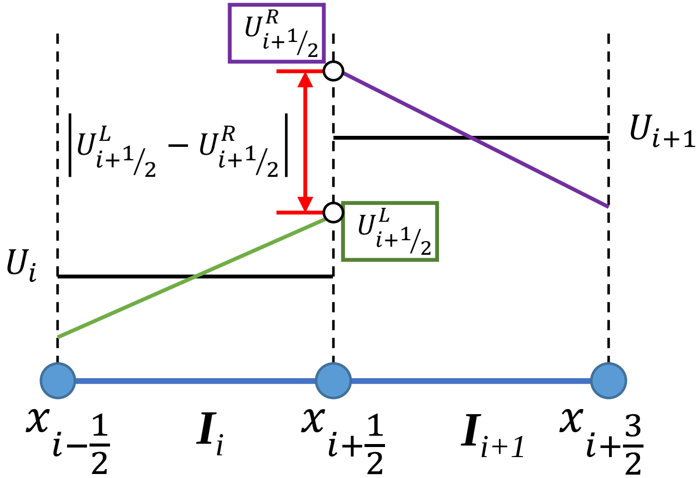
The selector relies on a 3-stage Boundary Variation Diminishing (BVD) algorithm.

- Total boundary variation (TBV) by the sum of the jumps of $\mathcal{R}$ at interfaces:

$$TBV_i^{\mathcal{R}} = \left|U_{i-\frac{1}{2}}^{L,\mathcal{R}} - U_{i-\frac{1}{2}}^{R,\mathcal{R}}\right| + \left|U_{i+\frac{1}{2}}^{L,\mathcal{R}} - U_{i+\frac{1}{2}}^{R,\mathcal{R}}\right| \geq 0. \quad (9)$$

where each term represents the amount of dissipation in the numerical flux in (5) for one edge of cell I_i.

BVD algorithm is to compare the $TBV_i^{\mathcal{R}_1}$ and $TBV_i^{\mathcal{R}_2}$ of the reconstructions $\mathcal{R}_1$ and $\mathcal{R}_2$ of the same date U, and selects the least dissipative one in cell I_i.



BVD principle

(Deng,CF2018 & Deng, *JCP*,2019)

3. Solution Property Preserving Method

➢ Local selection of reconstruction operator: a 3-stage BVD algorithm

A 3-stage BVD algorithm procedure is as following:

- Stage 1. Selection between $\mathcal{R}_{HO}$ and $\mathcal{R}_{ENO_2}$ → $\mathcal{R}_{ST_1}$

  For all cell i, if $TBV_i^{\mathcal{R}_{HO}} > TBV_i^{\mathcal{R}_{ENO_2}}$ then $(r_{i-1}, r_i, r_{i+1}) = ENO_2, else\ r_i = HO$.

  ⇨ $\mathcal{R}_{ST_1} = \{r_i, i = 1, \dots, N\}$

- Stage 2. Selection between $\mathcal{R}_{ST_1}$ and $\mathcal{R}_{ENO_1}$ → $\mathcal{R}_{ST_2}$

  For all cell i, if $TBV_i^{\mathcal{R}_{ST_1}} > TBV_i^{\mathcal{R}_{ENO_1}}$ then $(r_{i-1}, r_i, r_{i+1}) = ENO_1, else\ r_i = ST_1$.

  ⇨ $\mathcal{R}_{ST_2} = \{r_i, i = 1, \dots, N\}$

- Stage 3. Selection between $\mathcal{R}_{ST_2}$ and $\mathcal{R}_{SHARP}$ → $\mathcal{R}_{ST_3}$

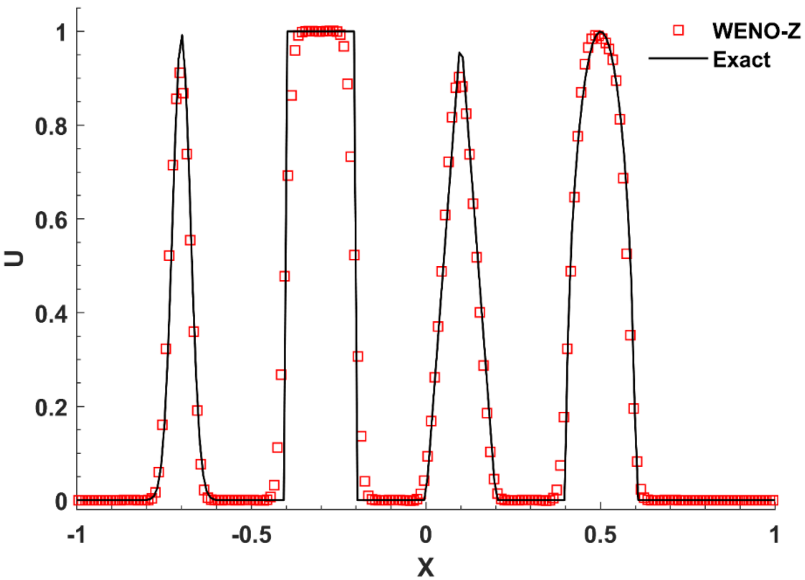
  For all cell i, if $TBV_i^{\mathcal{R}_{ST_2}} > TBV_i^{\mathcal{R}_{SHARP}}$ then $r_i = SHARP, else\ r_i = ST_2$.

  ⇨ $\mathcal{R}_{ST_3} = \{r_i, i = 1, \dots, N\}$

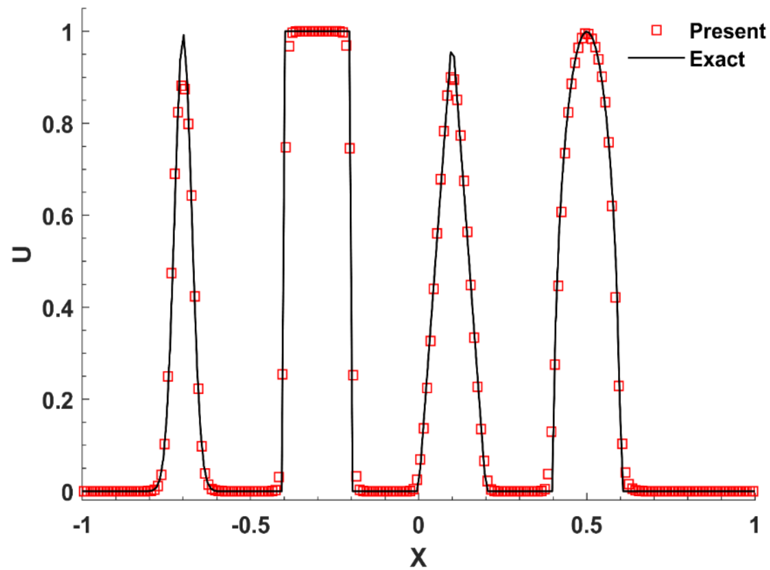
4. Numerical Experiments

✓ 1D Linear Advection Equation

WENO-Z



Multi-stage BVD-MOOD



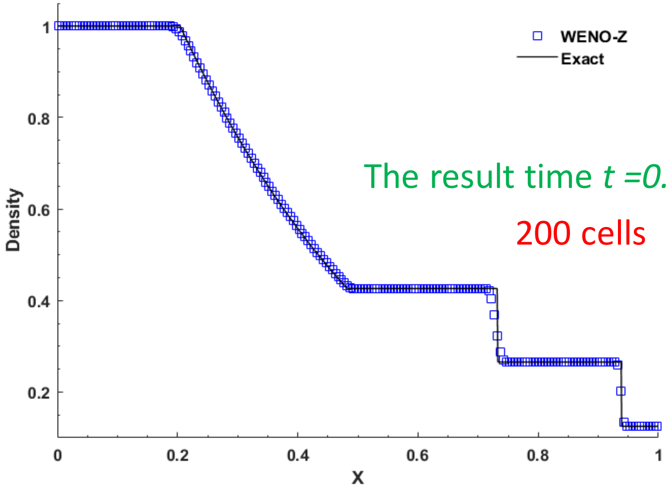
	Min Value
WENO-Z	-0.001131
Present	1.88e-18

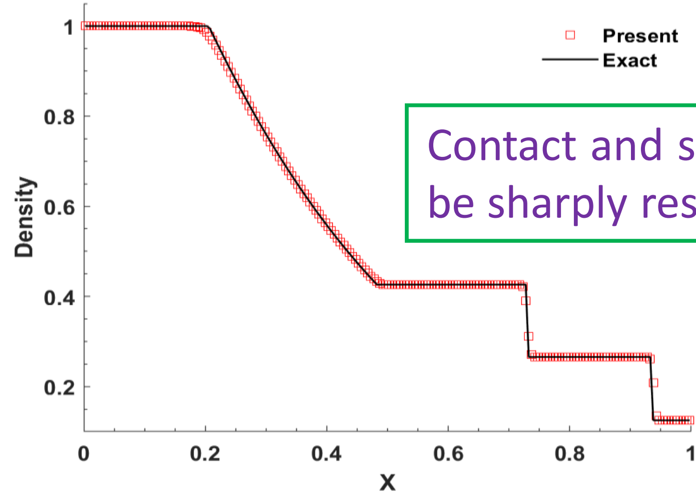
The result time $t=2$

4. Numerical Experiments

✓ 1D Euler Equations

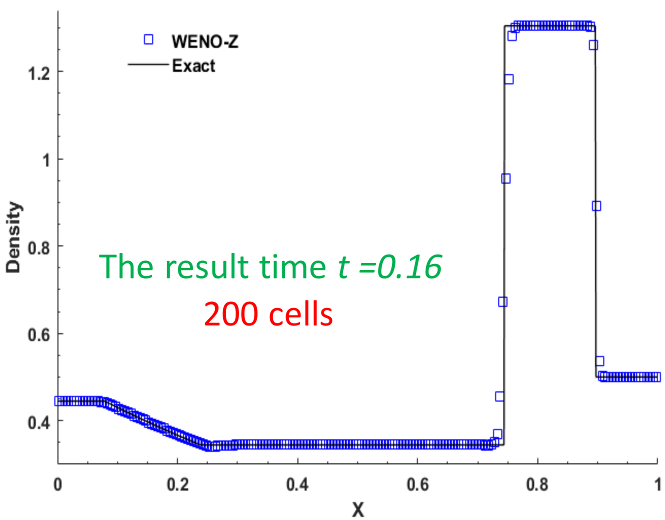
- SOD and Lax Shock tube problem

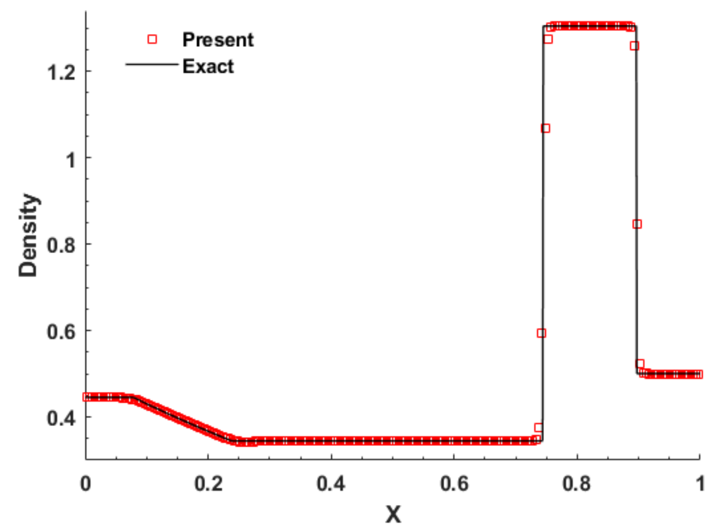




Contact and shock can be sharply resolved

WENO-Z **Present**



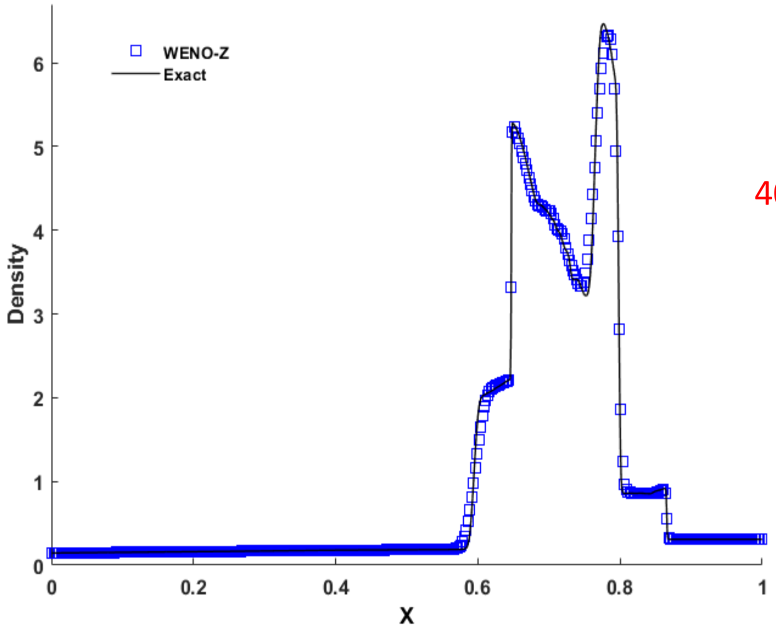


4. Numerical Experiments

✓ 1D Euler Equations

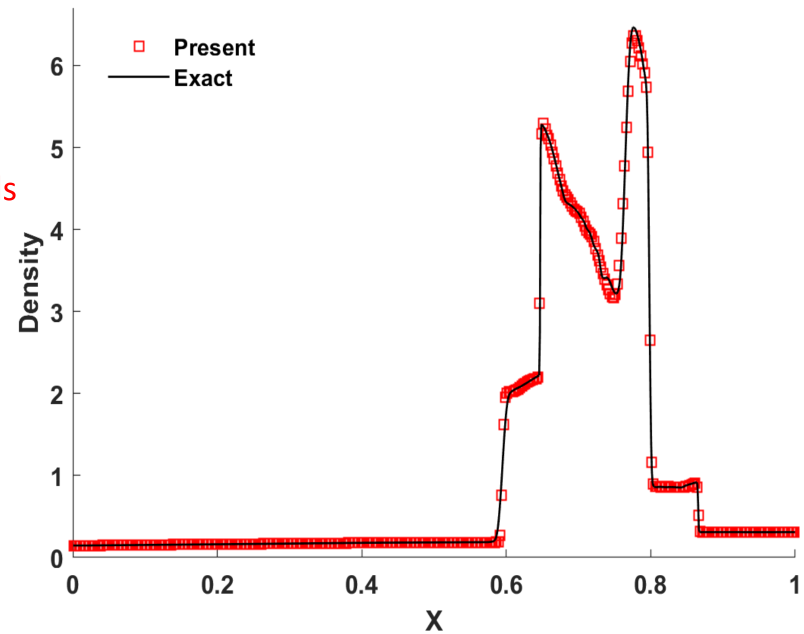
- *Collela-Woodward* Blast-wave

The result time *t =0.038*



WENO-Z

400 cells

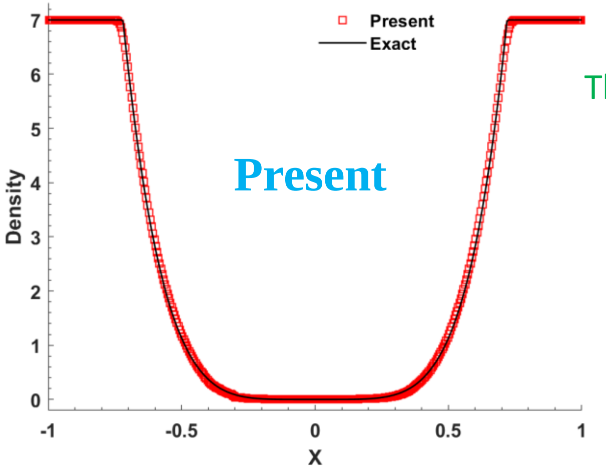


Present

4. Numerical Experiments

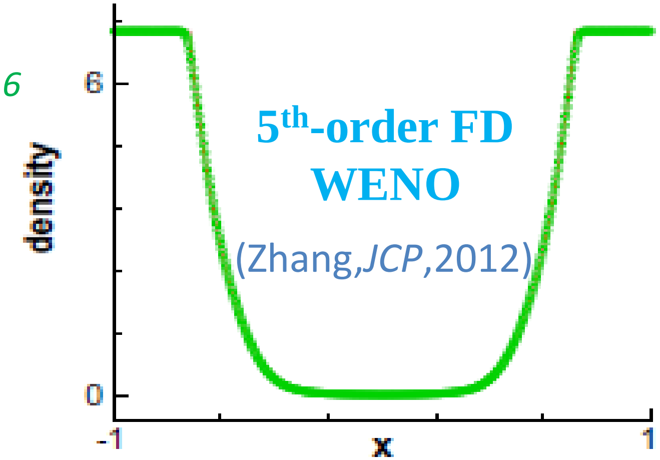
✓ 1D Euler Equations

- *Double rarefaction and Le Blanc problem*

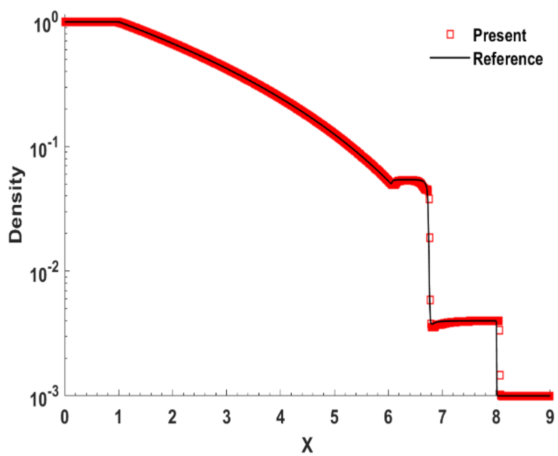


The result time $t = 0.6$

400 cells



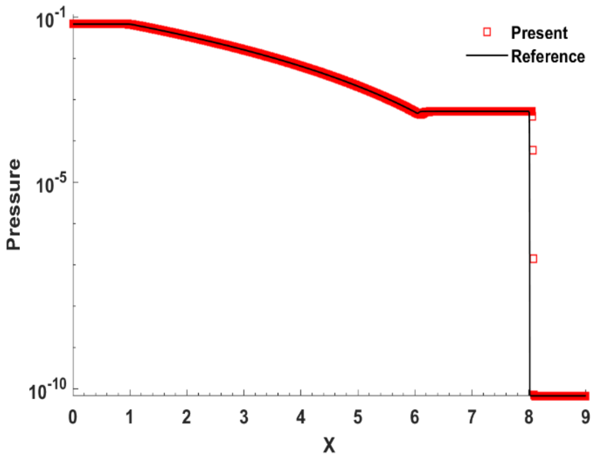
Positivity-preserving techniques



Present

The result time $t = 6$

800 cells

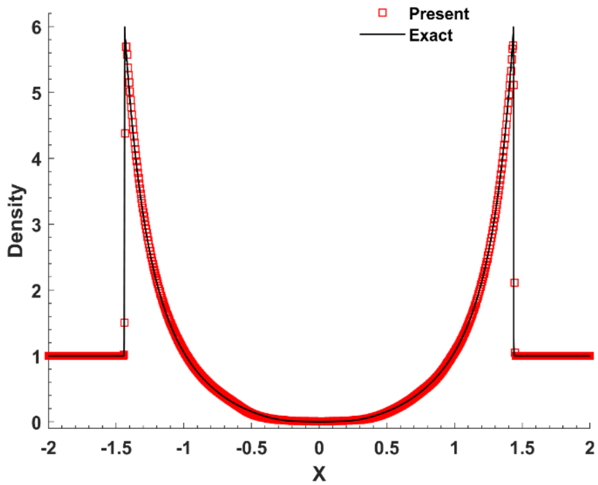


4. Numerical Experiments

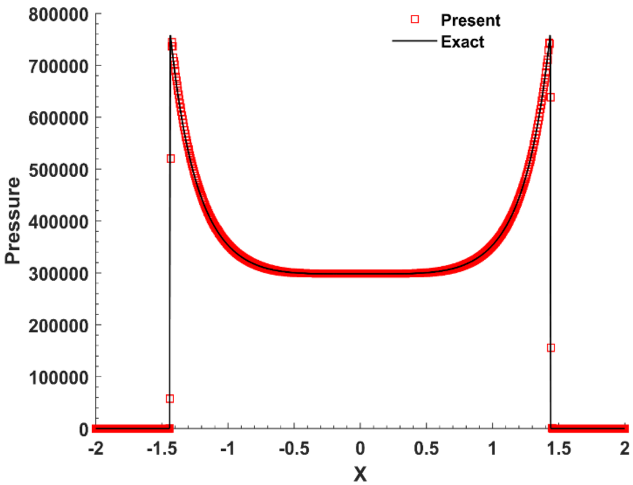
✓ 1D Euler Equations

- *Sedov Blast-waves*

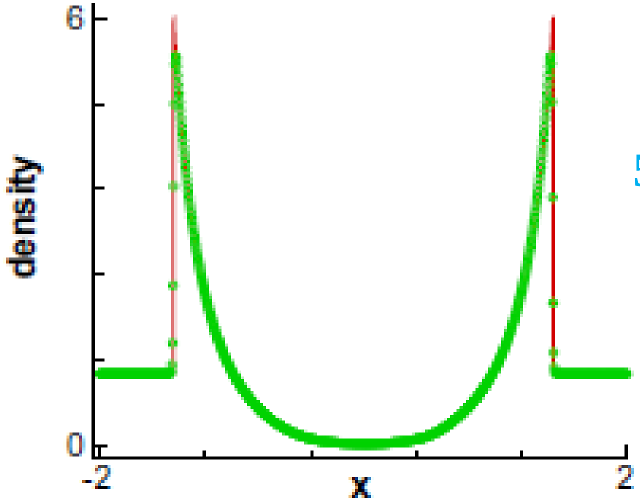
The result time *t =0.001*



Present

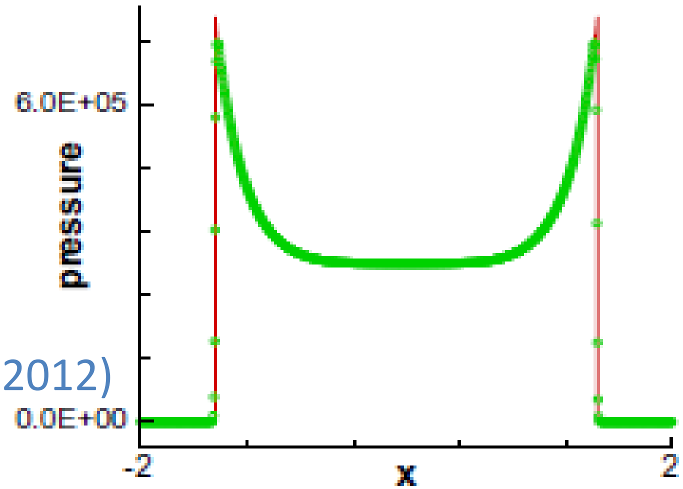


Positivity-preserving techniques



5th-order FD WENO

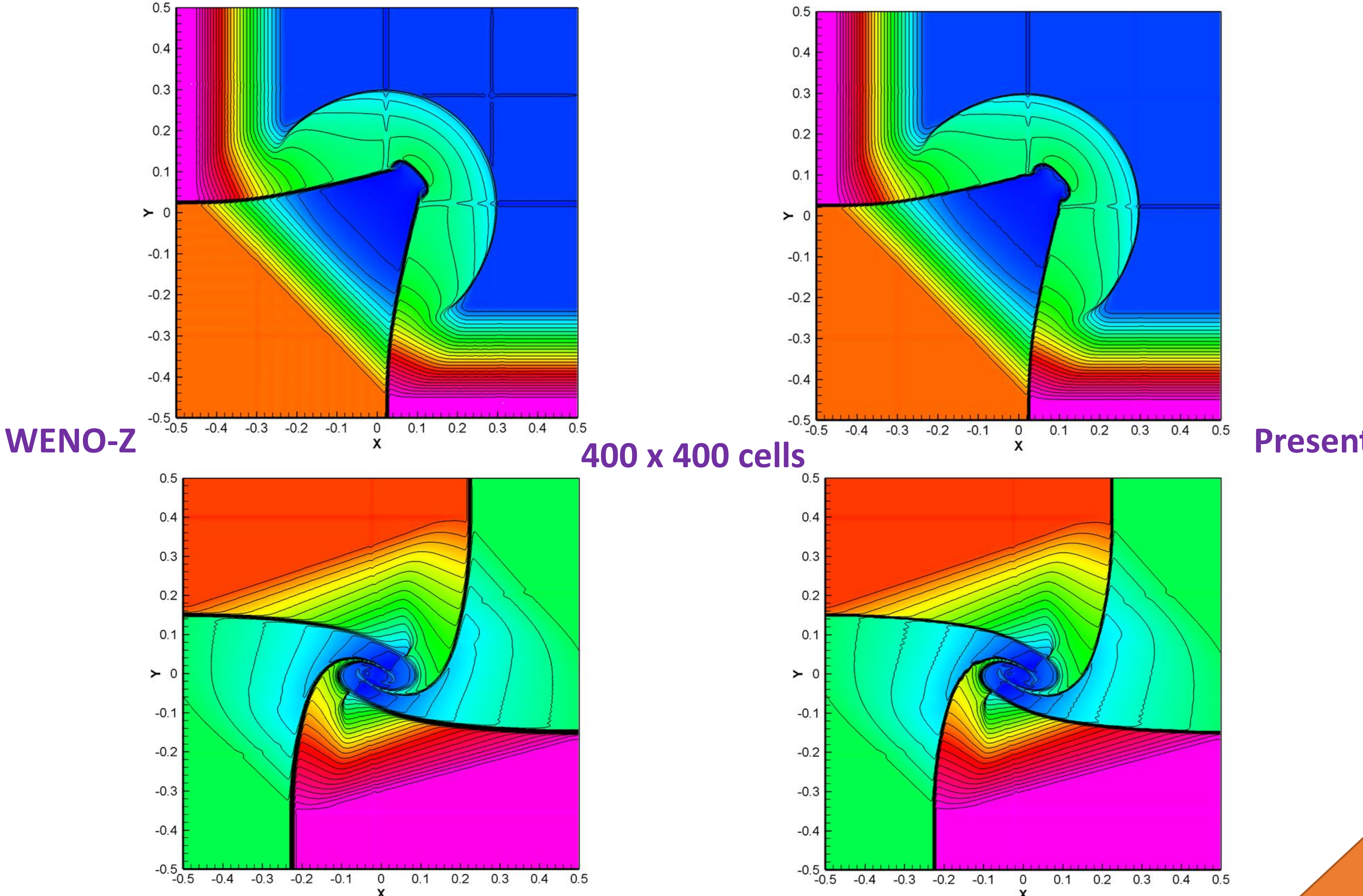
(Zhang,*JCP*,2012)



4. Numerical Experiments

✓ 2D Euler Equations

- *Riemann Problem*



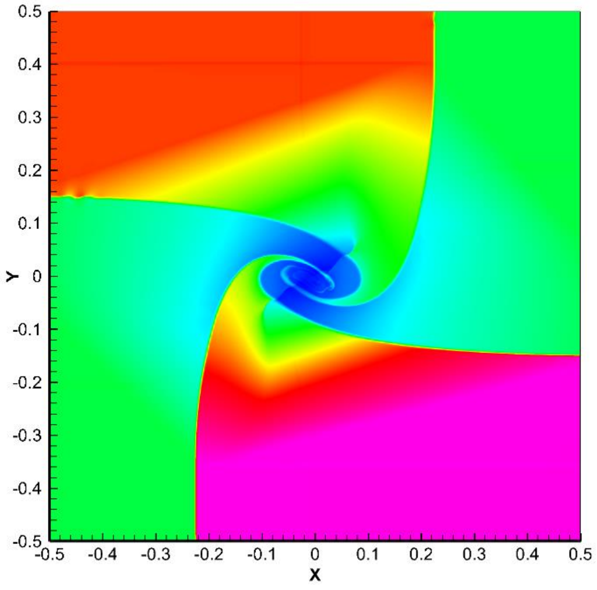
4. Numerical Experiments

✓ **2D Euler Equations**

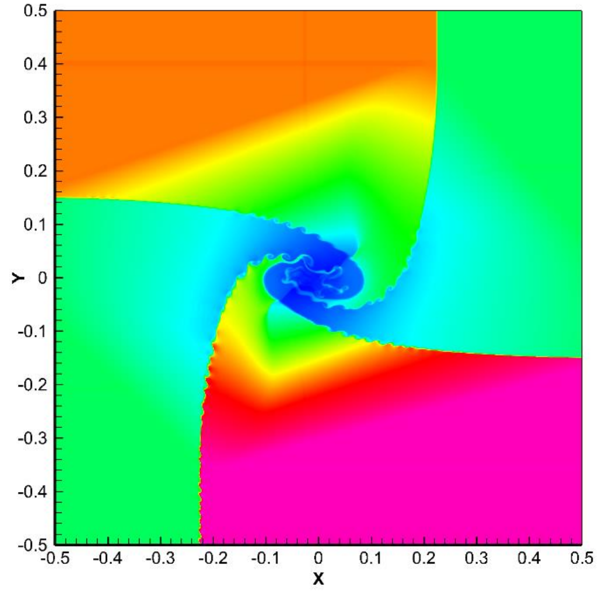
- *Performance of the present scheme*

The result time *t =0.3*

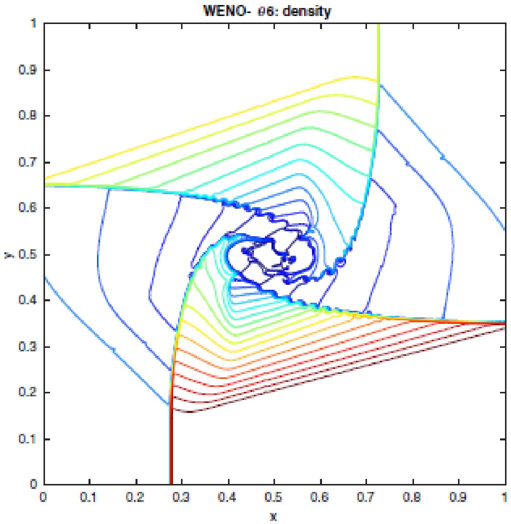
WENO-Z



Present



900 x 900 cells



6th-order WENO-θ

1200 x 1200 cells

(Jung, *Adv Comput Math*, 2017)

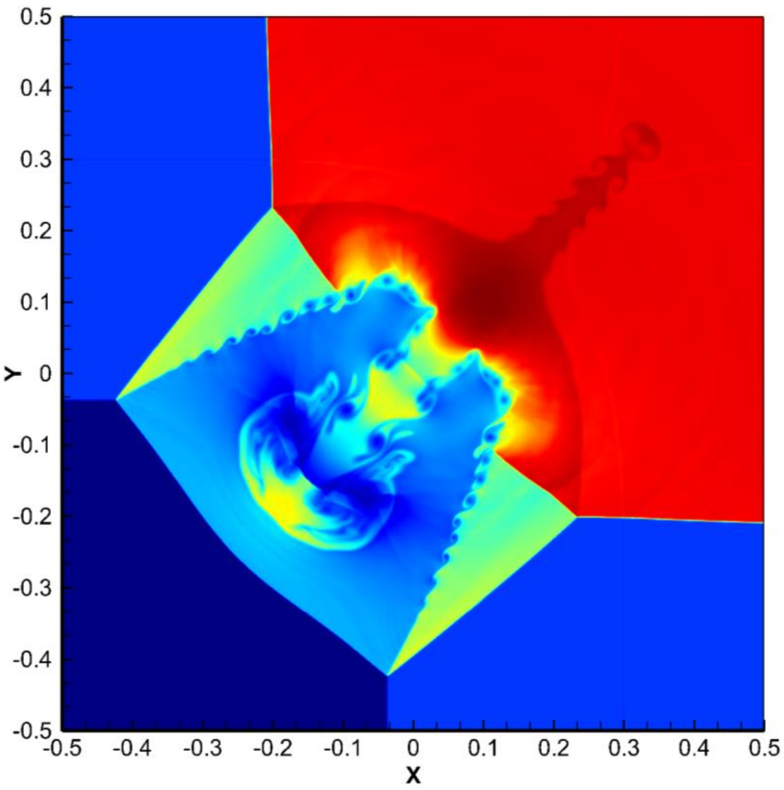
4. Numerical Experiments

✓ 2D Euler Equations

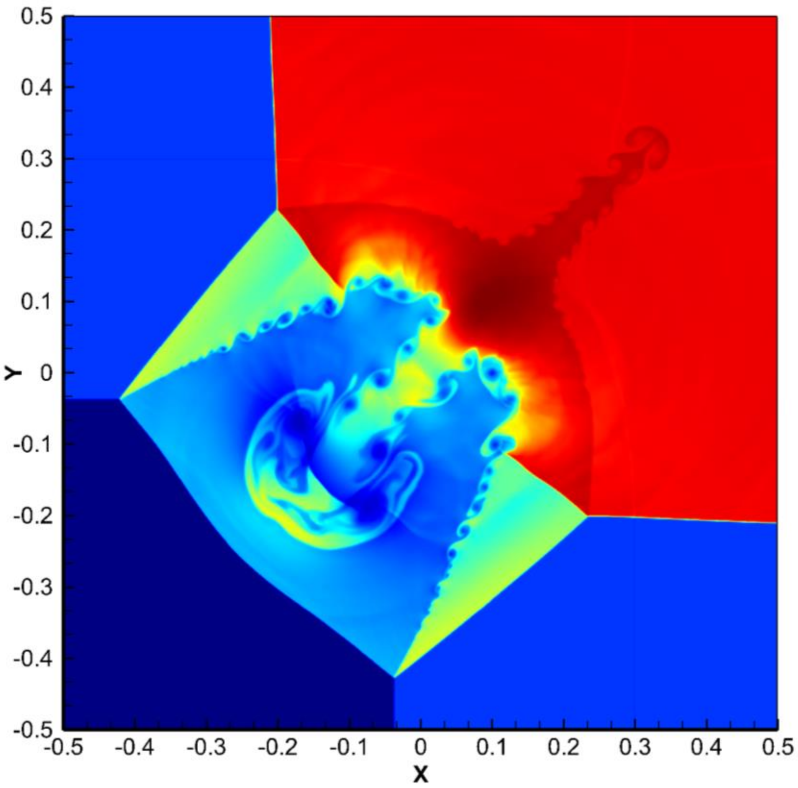
- *Performance of the present scheme*

The result time *t =0.8*

WENO-Z



Present



600 x 600 cells

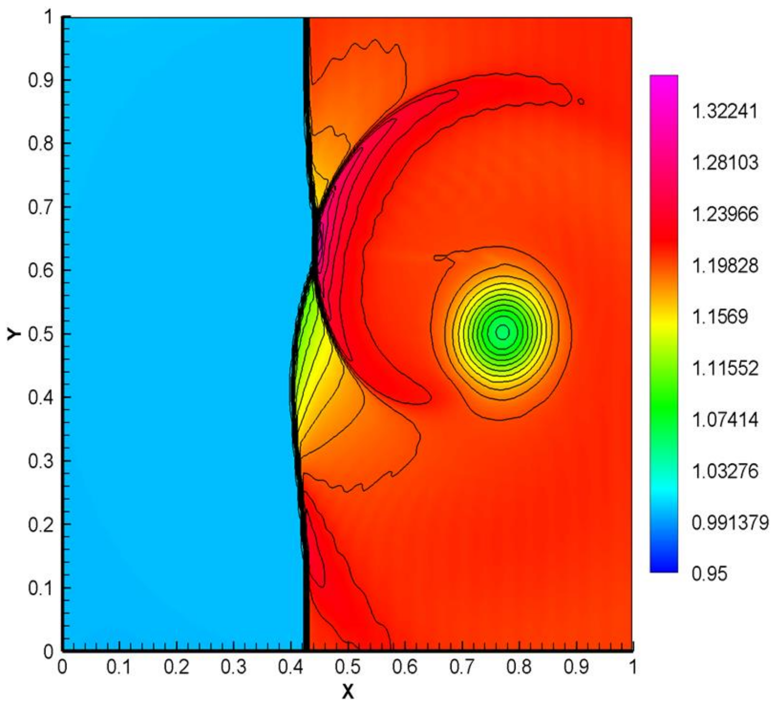
4. Numerical Experiments

- ✓ **2D Euler Equations**
  - *Shock-Vortex Interaction problem*

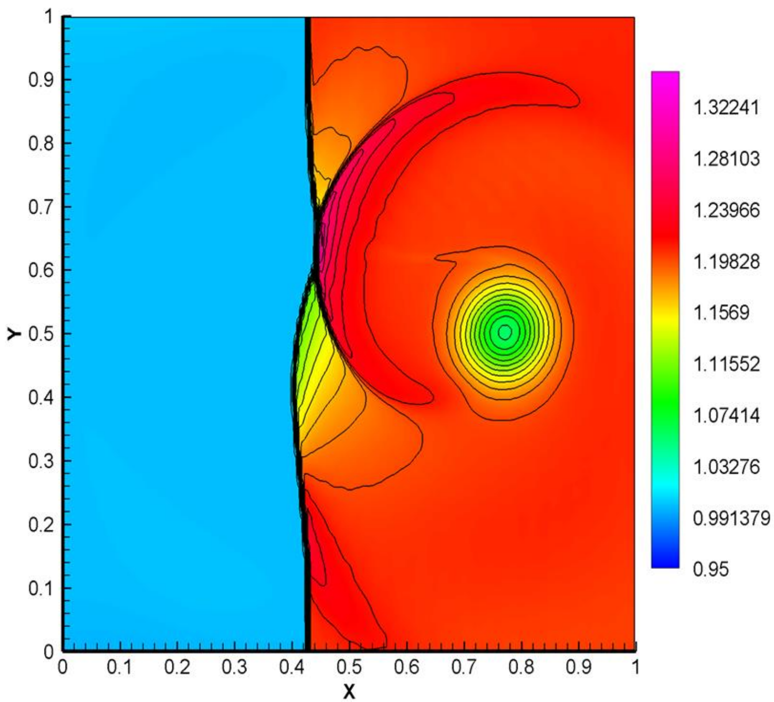
The result time *t =0.5*

200 x 200 cells

WENO-Z



Present

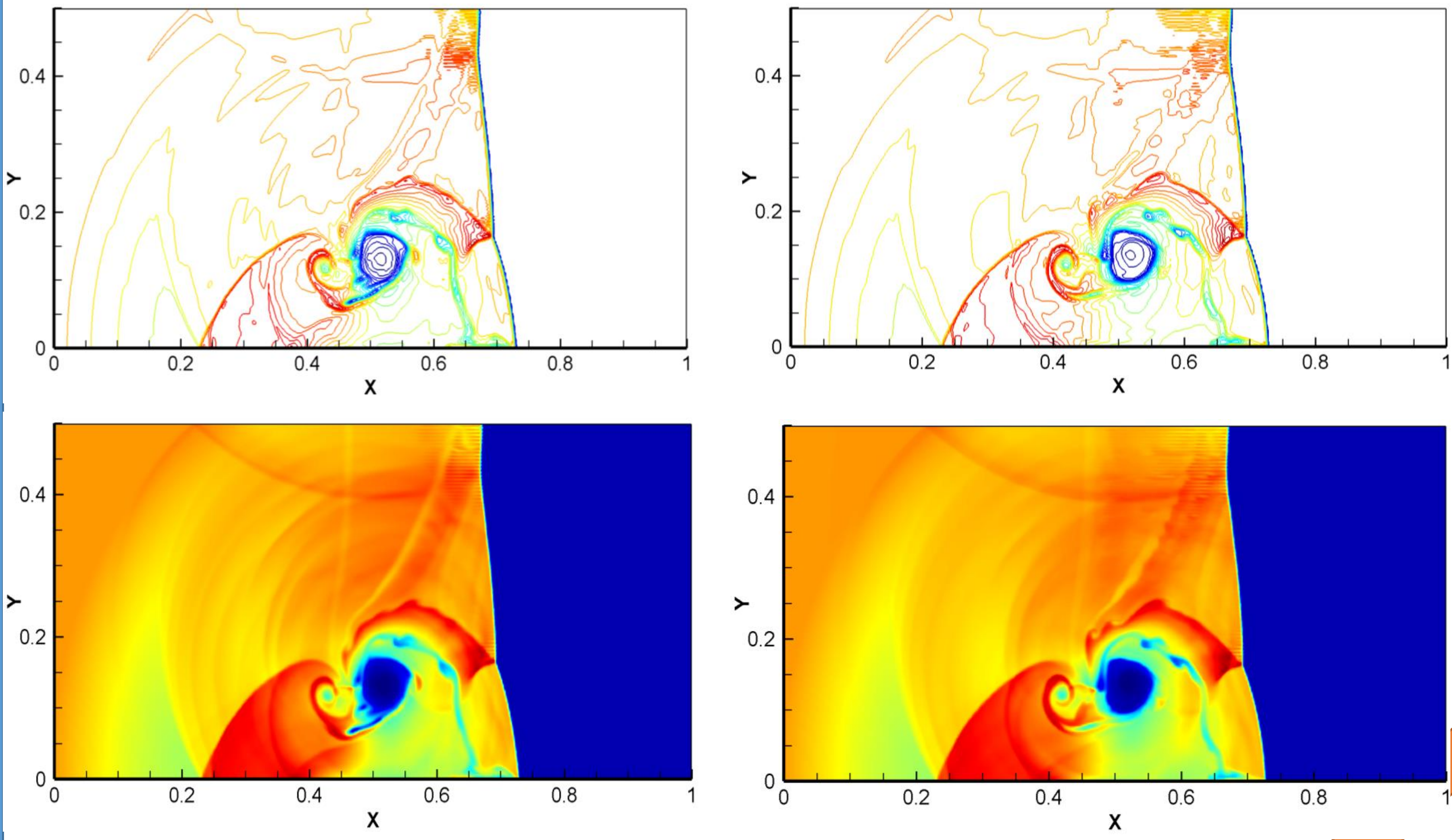


4. Numerical Experiments

- ✓ **2D Euler Equations**
  - *Shock-Bubble Interaction problem*

The result time *t =0.15*

WENO-Z **400 x 200 cells** **Present**

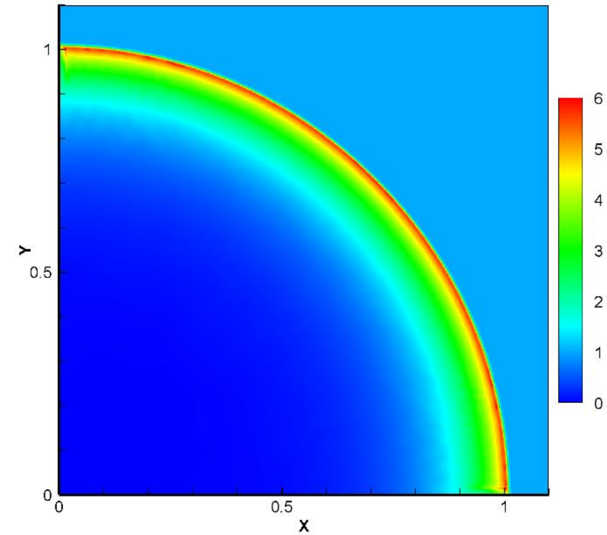


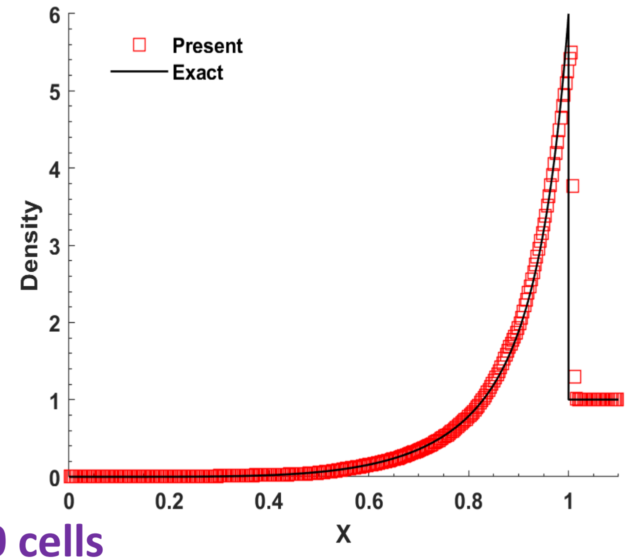
4. Numerical Experiments

✓ **2D Euler Equations**

- **Sedov Blast-waves (near vacuum region problem)**

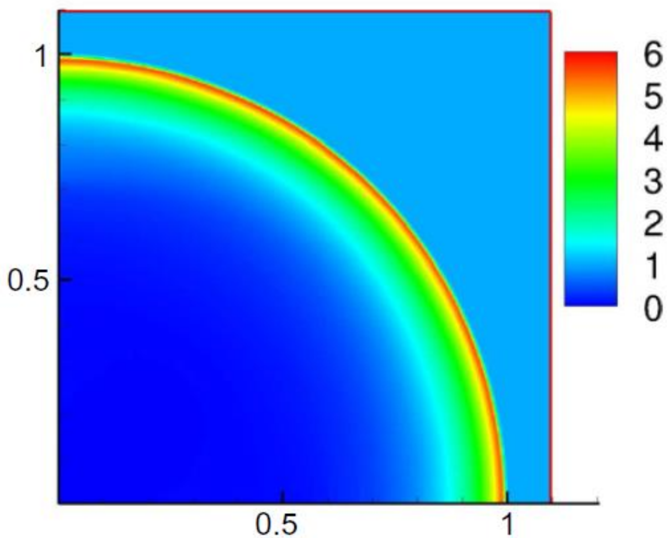
The result time $t = 1$

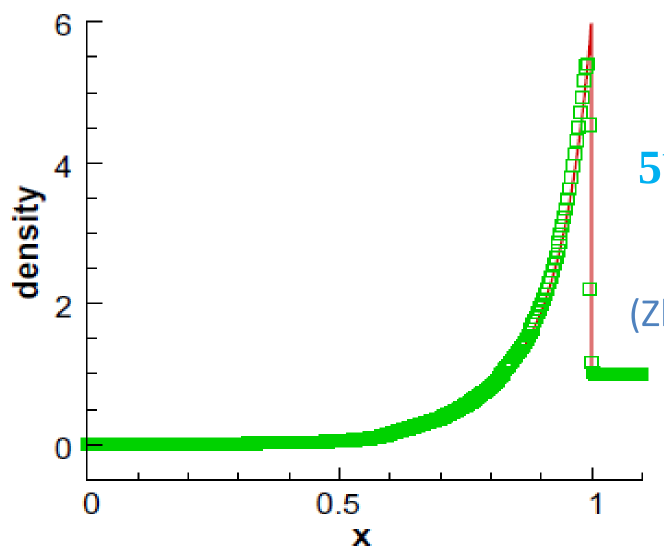




Present

320 x 320 cells





5^{th}-order FD WENO

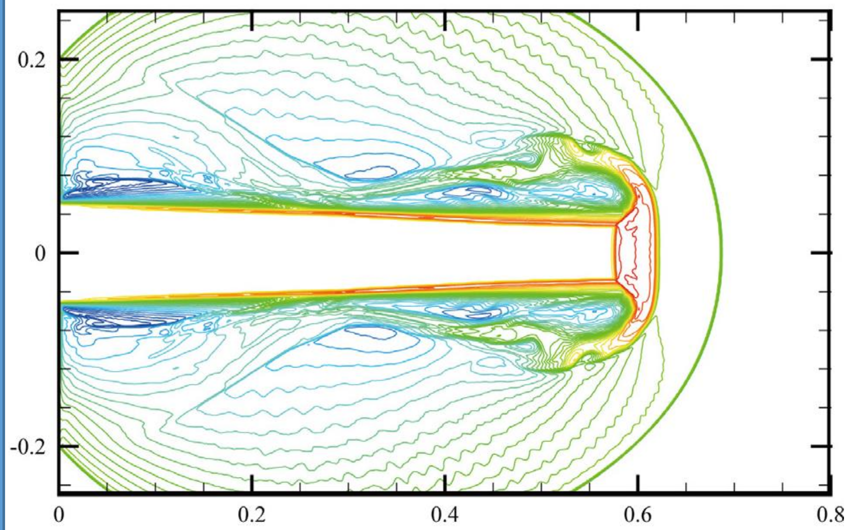
(Zhang,*JCP*,2012)

4. Numerical Experiments

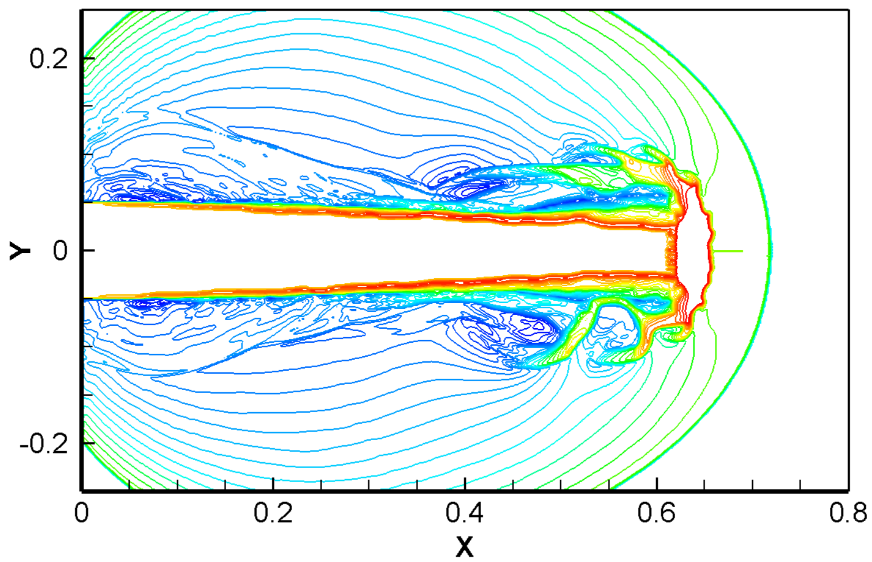
✓ **2D Euler Equations**

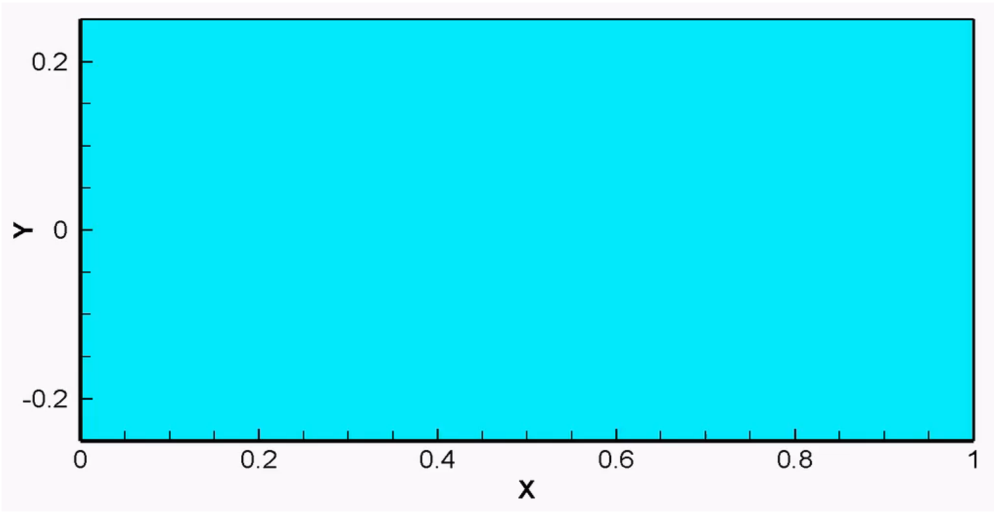
- **High Mach number astrophysical jets (2000)**

ACWNCS6-CU (800x400 cells)



Present (640x320 cells)





5. Conclusion

- The new approach differs from classical polynomial-based and relies on several types of reconstructions:

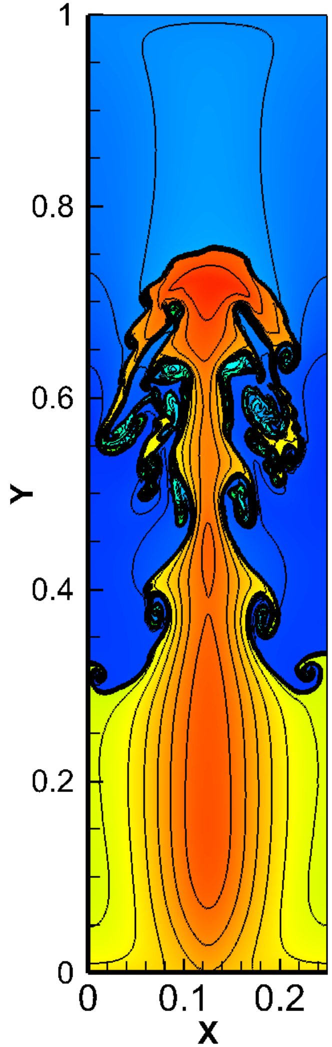
  1. 5th-order upwind scheme for smooth solution
  2. Viscous non-linear THINC functions to add dissipation
  3. Sharp non-linear ones to handle discontinuity and steep gradients
  4. No reconstruction at all extreme situations

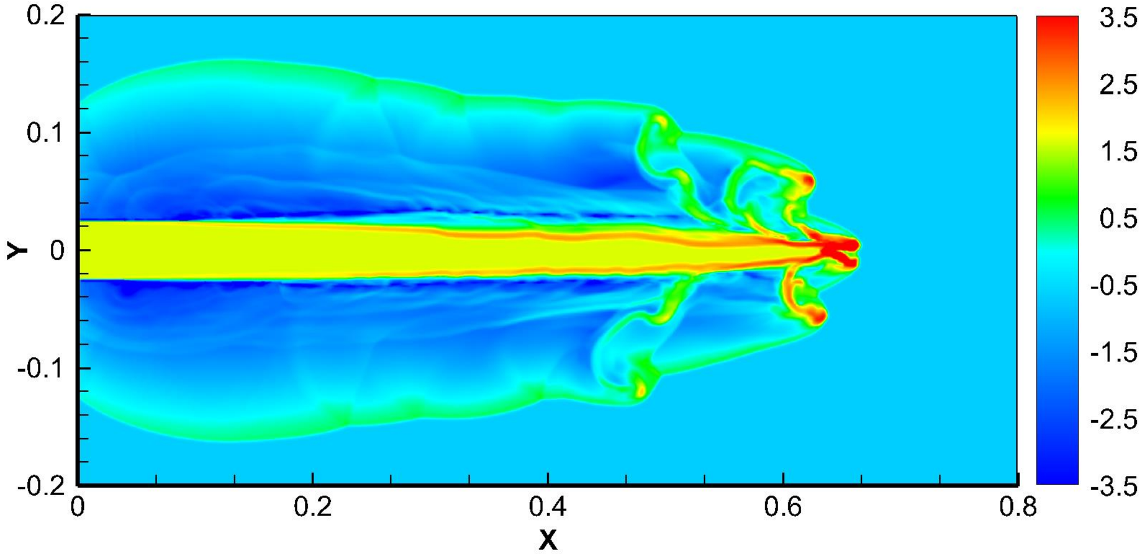
- This approach is to choose these reconstructions based on the BVD strategy.

✓ The present scheme can sharply capture both contact discontinuities and shocks (**THINC**) and is extremely robust to positivity issues (an *a posteriori* treatment).

Future Work

- Apply the scheme to complex PDEs with source terms and Reaction Euler equations





Thank you very much
for your attention and advices