Finite Volume schemes for kinetic problems

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An impressive list of Sharks!

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Outline

1. Motivation and set up
2. Kinetic theory
3. IMEX schemes with AP property
4. A local velocity grid approach
5. Mixtures of gases

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In this conference, we will be mainly occupied in

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eating nice food

In this conference, we will be mainly occupied in

drinking good wine

In this conference, we will be mainly occupied in

being very busy

...but mostly

our conversations will turn around

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our conversations will turn around CFD

CFD equations

We will mainly talk about how to numerically integrate equations which look like

$$\partial_t U + \nabla \cdot \boldsymbol{F}(U) = S(U),$$

where $U(\boldsymbol{x}, t) = [\rho, \rho\boldsymbol{u}, E]^T(\boldsymbol{x}, t)$, are density, momentum and total energy per unit volume, $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^3$, and

- S is the source term, and often will not be there;
- F is the flux function, which usually looks like this

$$F(U) = \begin{pmatrix} \rho\boldsymbol{u} \\ \rho\boldsymbol{u} \otimes \boldsymbol{u} + p\mathbb{I} \\ (E + p)\boldsymbol{u} \end{pmatrix}.$$

Now, the mysterious object is p, the pressure.
Where does it come from? What is it?

Equation of state

The pressure comes from the equation of state

$$p = p(\rho, e), \qquad e = (E - \tfrac{1}{2}\rho \boldsymbol{u}^2)\frac{1}{\rho}$$

in other words, pressure comes from thermodynamics. For a perfect gas in equilibrium, it is $p = \rho e(\gamma - 1)$, where γ is some parameter.

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- Kinetic theory addresses these (and several other) questions

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Kinetic equations

In kinetic theory, one studies the evolution of the distribution function $f(x, v, t)$ which gives the expected number $n(\boldsymbol{x}, t)$ of particles of mass m in a cell of volume $\mathrm{d}x\,\mathrm{d}v$ in phase space at time t. The evolution equation is

$$\partial_t f + v \cdot \nabla f = Q(f, f).$$

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Here $Q(f, f)$ is the collision kernel which accounts for the re-distribution of microscopic velocities through binary collisions.
Macroscopic quantities are recovered as expected values:

$$\begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho u^2 + \rho e \end{pmatrix} = m \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ \frac{1}{2} v^2 \end{pmatrix} f \,\mathrm{d}v = m < \phi(v) f >$$

Boltzmann

The key equation in kinetic theory is Boltzmann equation, which gives the evolution in time of the distribution function

$$\partial_t + v \cdot \nabla f = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \int_{S^+} B(|v - v_*|, \hat{n}) \left[f(v') f(v'_*) - f(v) f(v_*) \right] \, \mathrm{d}v_* \, \mathrm{d}\omega.$$

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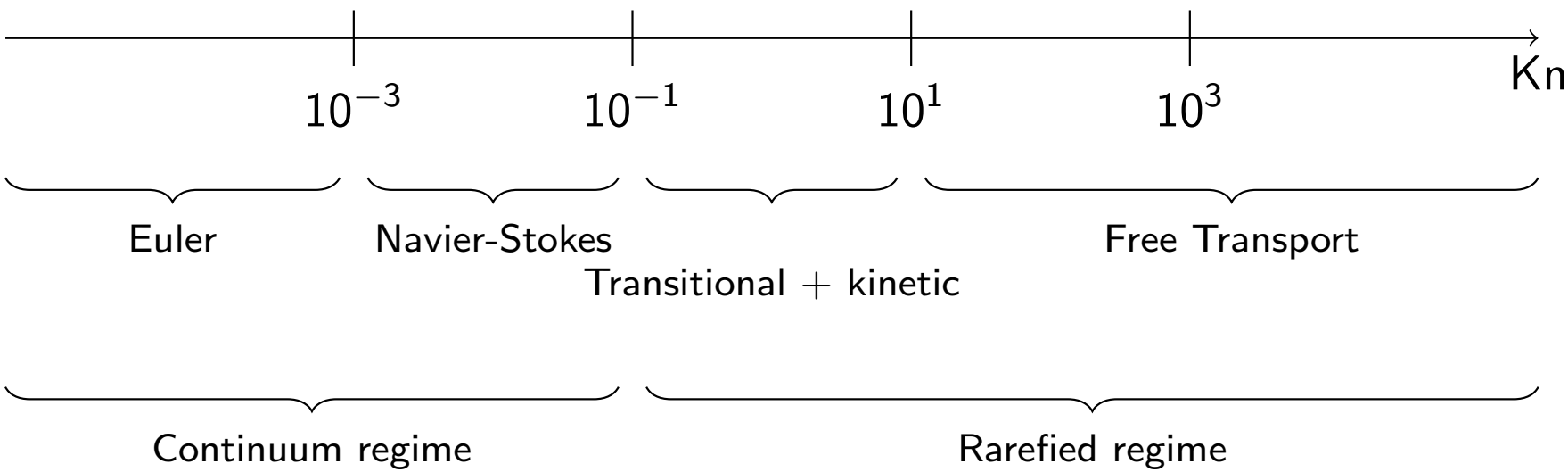
- (1) is the collision kernel, which contains the physics of the interaction.
- (2) is the gain term: particles that gain the velocity v because of an interaction.
- (3) is the loss term: particles that loose the velocity v because of an interaction.

From rarefied to continuum and back: the main scales

* λ the mean free path, L the representative physical length scale, ε collision time
* Knudsen number : $Kn = \dfrac{\lambda}{L} \simeq \varepsilon$



Kinetic vs CFD

Kinetic equations contain Fluid dynamics, which is recovered as $\varepsilon \to 0$. But there is more

- Non equilibrium effects: in case of strong shocks, the classic EOS does not hold.
- Natural modelling of transport coefficients in mixtures of fluids, or for complex molecules.
- Different scales, then Asymptotic preserving numerical methods, to ensure the possibility of under-resolving fast scales.
- Huge computational complexity: in 3D, 7 independent variables (3 in space, 3 in velocity plus time), so lots of fun with adaptive grids, high order methods.

Microscopic Conservation: identical particles

Conservation at the macroscopic level is due to the fact that collisions are supposed to be elastic: if v and v_* are the velocities of two identical particles before a collision occurs, while v' and v'_* denote the post collisional velocities,

- $v + v_* = v' + v'_*$ (conservation of momentum)
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At the macroscopic level, these principles yield

$$< Q(f,f)m >= 0 \qquad \text{mass conservation}$$

$$< Q(f,f)mv >= 0 \qquad \text{momentum conservation}$$

$$< Q(f,f)\tfrac{1}{2}m\, v^2 >= 0 \qquad \text{energy conservation}$$

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Thus, the macroscopic flux F is given by

$$F(x,t;f) = [< m\, v\, f >, < m\, v \otimes v\, f >, m < \tfrac{1}{2} v^2 v\, f >]^T$$

with pressure tensor $\mathbb{P} =< m\,(v-\boldsymbol{u}) \otimes (v-\boldsymbol{u})\, f >$.

Recovering Euler equations: equilibrium

* Equilibrium is obtained when $f = M_f$, where M_f is the Maxwellian distribution, s.t. $Q(M_f, M_f) = 0$

$$M_f[n, \boldsymbol{u}, T] = \frac{n}{(2\pi K/mT)^{3/2}} \exp\Big(- \frac{m|v - \boldsymbol{u}|^2}{2KT} \Big), \text{ with}$$

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* As $\varepsilon \to 0$, $f \to M_f$ and Euler equations are obtained. The EOS is given by the fact that the pressure tensor now becomes diagonal, with $p = \rho/mT$.
* Define the entropy as $< f \log f >$. The H-theorem ensures that entropy is dissipated driving the system to equilibrium.

Governing equations: BGK

The BGK model is an approximation of Boltzmann equation

$$\partial_t + v \cdot \nabla f = \int_{\mathbb{R}^d} \int_{S^+} B(|v - v_*|, \hat{n}) \left[f(v') f(v'_*) - f(v) f(v_*) \right] \, \mathrm{d}v_* \, \mathrm{d}\omega.$$

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In the BGK approximation, one supposes that

- The cross section of the interaction does not depend on the relative velocity of the interacting particles, $B(|v - v_*|, n) = B(n)$.
- The particles reach equilibrium after a single interaction, so the distribution of post interaction particles is Maxwellian.
- The equilibrium and the conservation principles of BGK and Boltzmann coincide.

Governing equations: ES-BGK

To recover also the NS equations for small Kn, the ES-BGK model is necessary

* $\partial_t f(\boldsymbol{x}, v, t) + v \cdot \nabla f(\boldsymbol{x}, v, t) = \frac{1}{\varepsilon}\Big(\mathcal{G}_f[\rho, \boldsymbol{u}, \mathcal{T}] - f(\boldsymbol{x}, v, t)\Big)$
* The equilibrium distribution is

$$\mathcal{G}_f[n, \boldsymbol{u}, T] = \frac{\rho}{\sqrt{2\pi \det \Theta}} e^{-\frac{1}{2}(v-\boldsymbol{u})^T \Theta^{-1}(v-\boldsymbol{u})}$$

* which is constructed using also the stress tensor

$$\Theta = (1-\nu)KT\mathbb{I} + \nu\mathbb{P}, \qquad \mathbb{P} = \frac{1}{n}\int_{\mathbb{R}^3} (v-\boldsymbol{u}) \otimes (v-\boldsymbol{u})^T dv$$

* As in BGK $\int_{\mathbb{R}^3} \phi(\xi) f dv = \int_{\mathbb{R}^3} \phi(v)\mathcal{G}_f d = \begin{pmatrix} \rho \\ \rho\boldsymbol{u} \\ E \end{pmatrix}$

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* Adaptive grids in velocity and physical space.
* Domain decomposition techniques.
* High order to contrast the curse of dimensionality.
* Robust modelling of more complex phenomena: mixtures and polyatomic gases.
* The entropy decay $H(f) =< f \log f >$ must also occur at the discrete level.

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- AP scheme: we want a scheme for the kinetic system that, with a fixed Δt, becomes a consistent scheme for the relaxed system, when $\varepsilon \to 0$ (Jin 2012, Pareschi-Russo, 2011, Dimarco-Pareschi 2014, Dimarco-Loubère 2016, Degond, ...).

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- Moreover, the scheme should not require the solution of large non linear systems of eqs. (Filbet-Jin 2010).
- When ε is small, the scheme should reduce to a discretization of CNS.

Asymptotic preserving property [Filbet and Jin,2010]

* "An AP scheme preserves the discrete analogy of the Chapman-Enskog expansion, namely, it is a suitable scheme for the kinetic equation, yet, when holding the mesh size and time step fixed and letting the Knudsen number go to zero, the scheme becomes a suitable scheme for the limiting Euler equations"

* "implicit collision terms can be implemented explicitly, or at least more efficiently than using a Newton type solver for nonlinear algebraic systems"

See also [Jin and Levermore, 1991], [Coron and Perthame, 1991], [Jin, 1995], [Jin, Pareschi and Toscani, 1998], [Jin, 1999], [Dimarco Pareschi, 2014].

A Runge Kutta tutorial: IMEX

Suppose you are given a system of ODE's with slow and fast scales

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and you want to use an explicit scheme for the slow scale and an implicit scheme for the fast scale, to overcome stability constraints

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This construction uses two Butcher's tableaux (Asher, Carpenter, Pareschi, Russo, Boscarino, Dimarco...)

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- which are computed by:

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Solution of the implicit step

At each stage i, the equation for $f^{(i)}$ can be written as:

$$f^{(i)} = B^{(i)} + \frac{\Delta t}{\tau^{(i)}} a_{ii} \left(M^{(i)} - f^{(i)} \right)$$

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Note that $M^{(i)}$ and $\tau^{(i)}$ depend non linearly on $f^{(i)}$ through the moments of $f^{(i)}$.
We want to compute $f^{(i)}$ without solving non linear equations, even though the equation is highly non linear in $f^{(i)}$.
[Pieraccini, P. 2006]

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Hence we immediately find (explicitly!) the macroscopic variables $\rho^{(i)}$, $u^{(i)}$ and $T^{(i)}$ corresponding to $f^{(i)}$. With these the updated Maxwellian is obtained:

$$M^{(i)} = \frac{\rho^{(i)}}{(2\pi R T^{(i)})^{N/2}} \exp\left(-\frac{||v - u^{(i)}||^2}{2RT^{(i)}} \right)$$

Solution of the implicit step

With the knowledge of $M^{(i)}$, (and of $\tau^{(i)}$), the equation for $f^{(i)}$ is linear:

$$f^{(i)} = B^{(i)} + \frac{\Delta t}{\tau^{(i)}} a_{ii} \left(M^{(i)} - f^{(i)} \right)$$

and can be solved exactly, for each x, v. Thus, the scheme keeps both the gain and the loss terms at the same time level (which is crucial for mantaining the asymptotic properties), but

- the Maxwellian is computed at the new time level, without the restrictions due to an explicit discretization of the relaxation term.

Space discretization

We choose a uniform grid in space, and discretize the equation with a conservative finite difference scheme, based on a high order non-oscillatory reconstruction.

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4 A local velocity grid approach

5 Mixtures of gases

Discrete velocity grids (with F. Bernard and A. Iollo)

Kinetic computations are extremely time consuming, because of the large number of independent variables: 3 in space, 3 in velocity space plus time.

* The space of microscopic velocities is $\mathbb{R}^3$. Integrating a kinetic model, requires a discretization and a quadrature in velocity space.
* The number of grid nodes introduced in velocity space determines the number of PDE's of the form

$$\partial_t f_k(x,t) + v_k \cdot \nabla f_k = Q(f,f)|_{v_k}, \quad f_k(x,t) = f(x, v_k, t)$$

  to be solved. Thus it is essential to control the number of velocity grid nodes.
* Further, the maximum value of $|v_k|$ determines the CFL condition.
* Need to avoid interpolation between different grids

The discrete velocity approximation

For fixed Cartesian velocity grids, describing correctly the Maxwellian distribution function implies:

* $\min(v) \leq \min_{t,x}(u(t,x) - \alpha\sqrt{RT(t,x)})$
* $\max(v) \geq \max_{t,x}(u(t,x) + \alpha\sqrt{RT(t,x)})$
* $\Delta v \leq \min_{t,x} \sqrt{\frac{K}{m} T(t,x)}$

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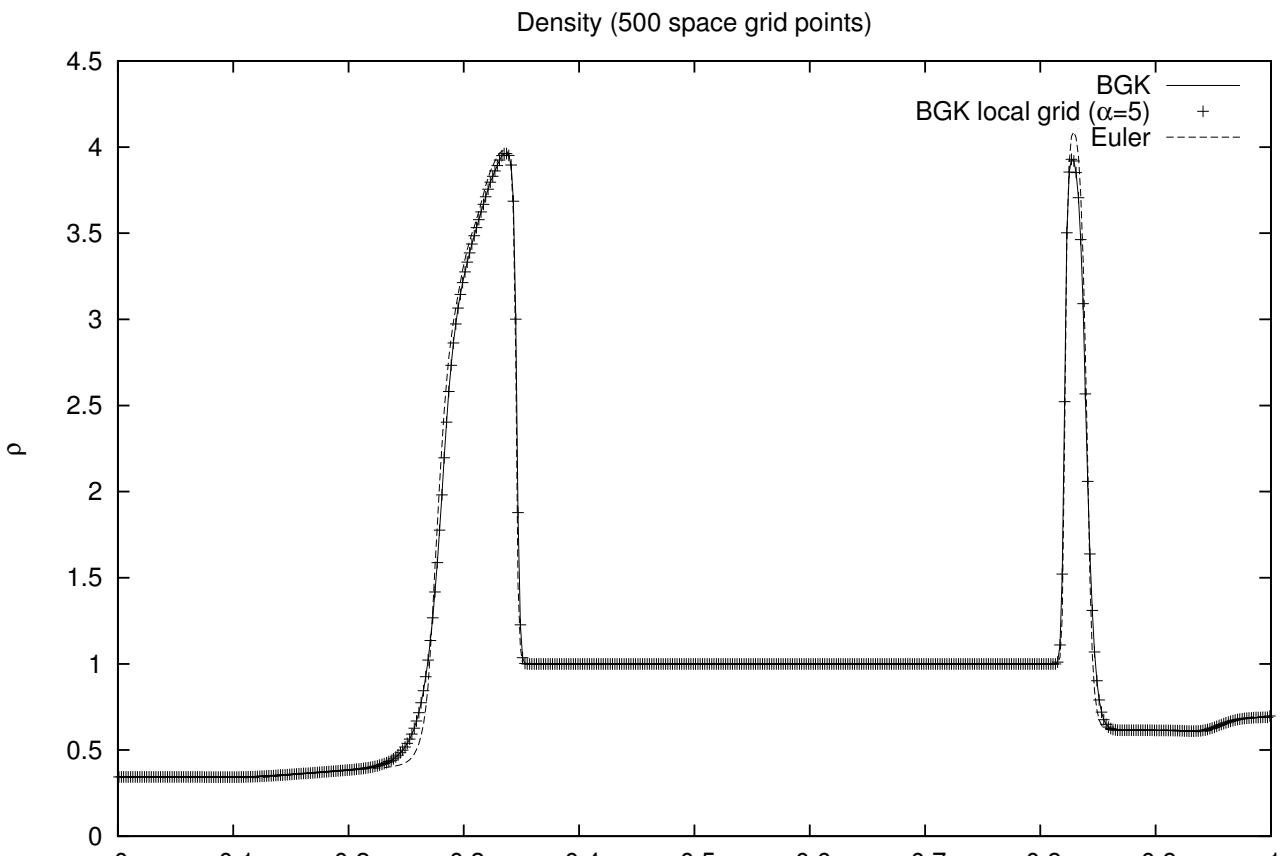
Solution : **Define dynamically local velocity grids based on local criteria.**

Two interacting blast waves ($Kn = 10^{-5}$)

$$\begin{cases} \rho = 1, & u = 0, & p = 1000 & \text{in } [0,0.1] \\ \rho = 1, & u = 0, & p = 0.01 & \text{in } [0.1,0.9] \\ \rho = 1, & u = 0, & p = 100 & \text{in } [0.9,1] \end{cases}$$

$\Longrightarrow v \in [-160, 160]$, 3200 grid points in 1D!



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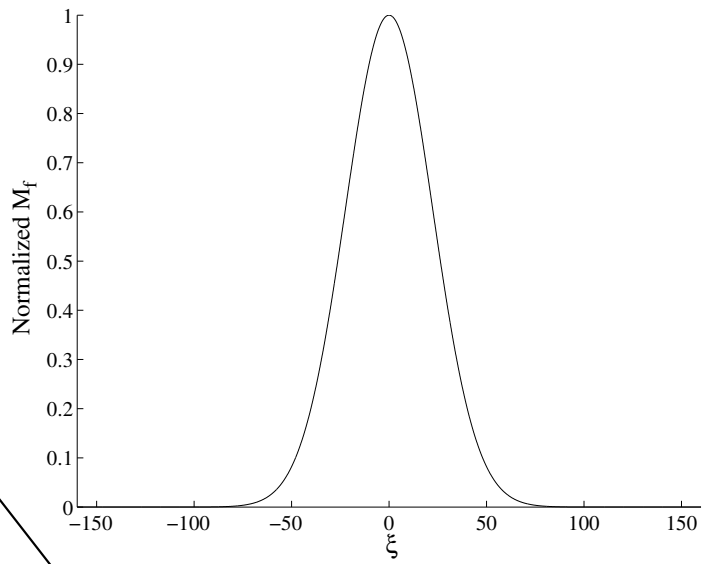

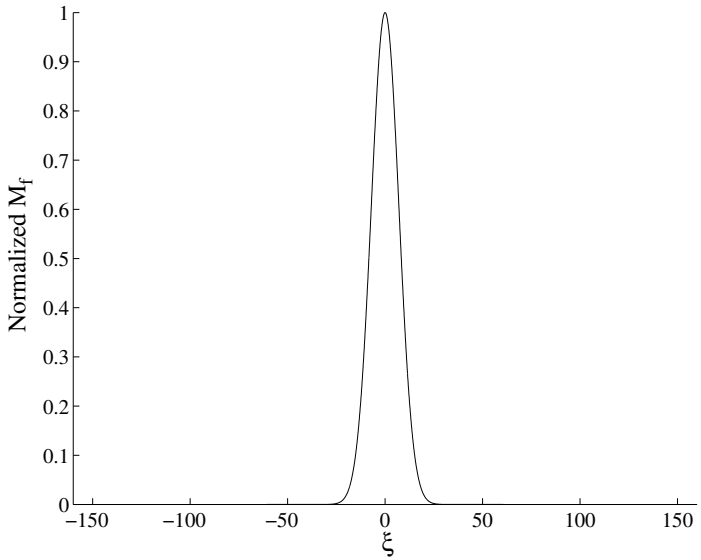

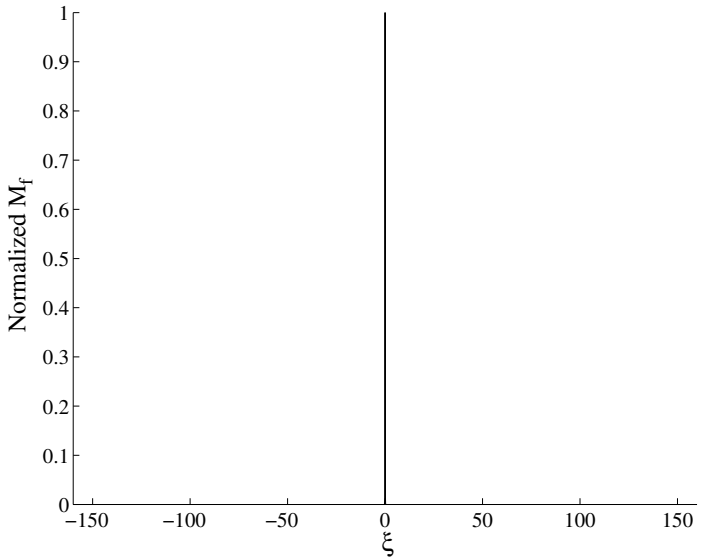

A local velocity grid approach in 1D

Idea: Build local velocity grid $[v_{min}(\boldsymbol{x}, t), v_{max}(\boldsymbol{x}, t)]$ based on a virtual global grid with amplitude Δ

* Define the local grid $\hat{v}$ at each time step

$$\hat{v}_{min}(t^n, x_i) = v_M \quad \text{such that} \quad M = \max_{k \in \mathbb{Z}}(k \ : \ v_k \leq u_i^n - \alpha\sqrt{T_i^n})$$

$$\hat{v}_{max}(t^n, x_i) = v_M \quad \text{such that} \quad M = \min_{k \in \mathbb{Z}}(k \ : \ v_k \geq u_i^n + \alpha\sqrt{T_i^n})$$

* Before the transport step, perform the union of the neighbouring velocity grids

$$v_{min}^i = \min(\hat{v}_{min}(x_{i-1}), \hat{v}_{min}(x_i), \hat{v}_{min}(x_{i+1}))$$
$$v_{max}^i = \max(\hat{v}_{max}(x_{i-1}), \hat{v}_{max}(x_i), \hat{v}_{max}(x_{i+1}))$$

* No interpolation is needed, because Δ is fixed.

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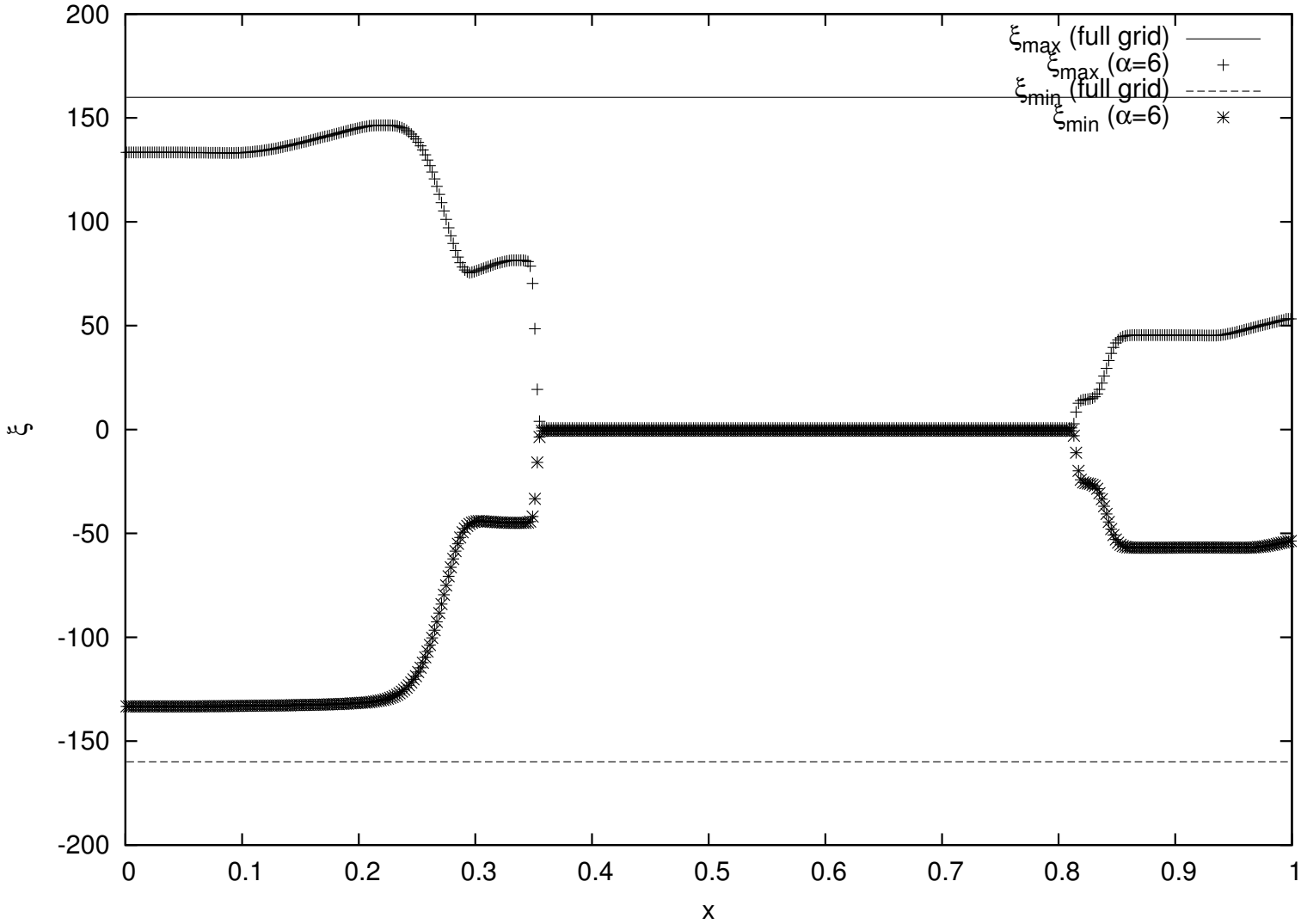


Bounds of the local velocity grid in x at the final time

Two interacting blast waves ($Kn = 10^{-5}$)

Comparison between local and global velocity grids

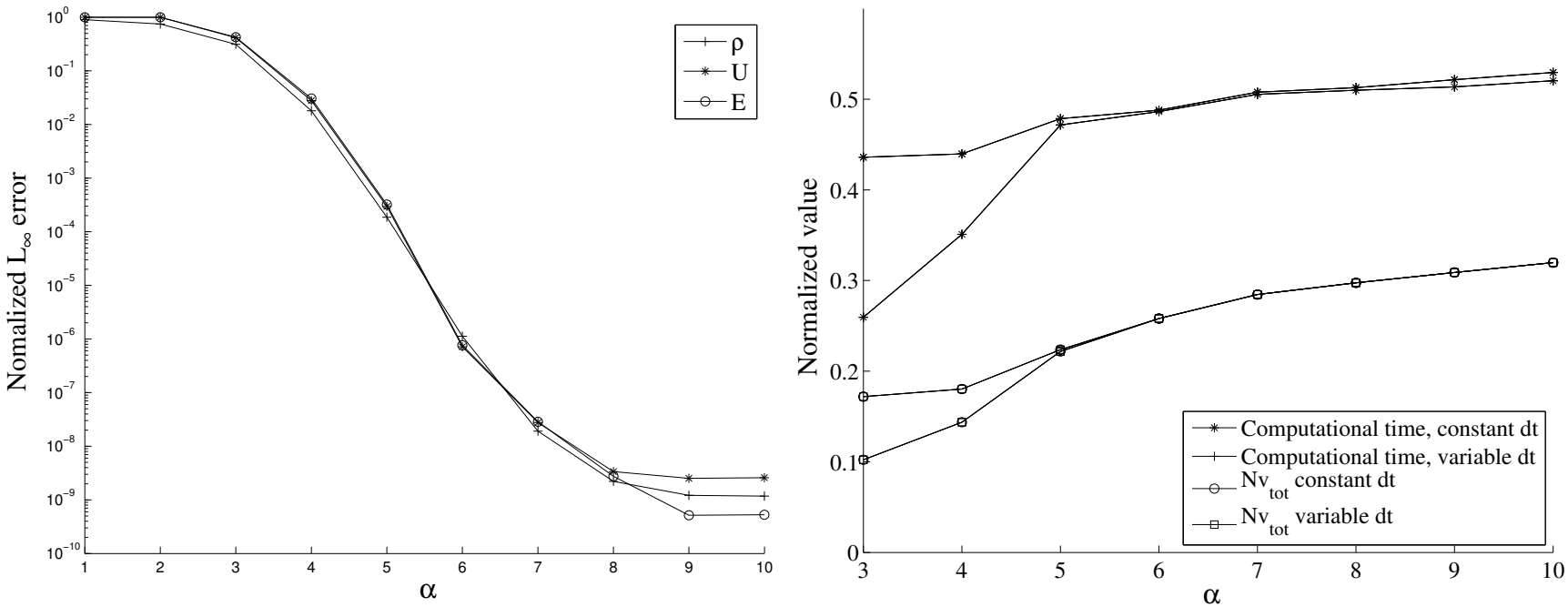


Figure: Error L_∞ norm (left) for density, velocity and energy. Computational time and total number of velocity grid points used (right)

A local velocity grid approach in 1D

"What about the rarefied regime? The distribution function can be far from being Maxwellian"

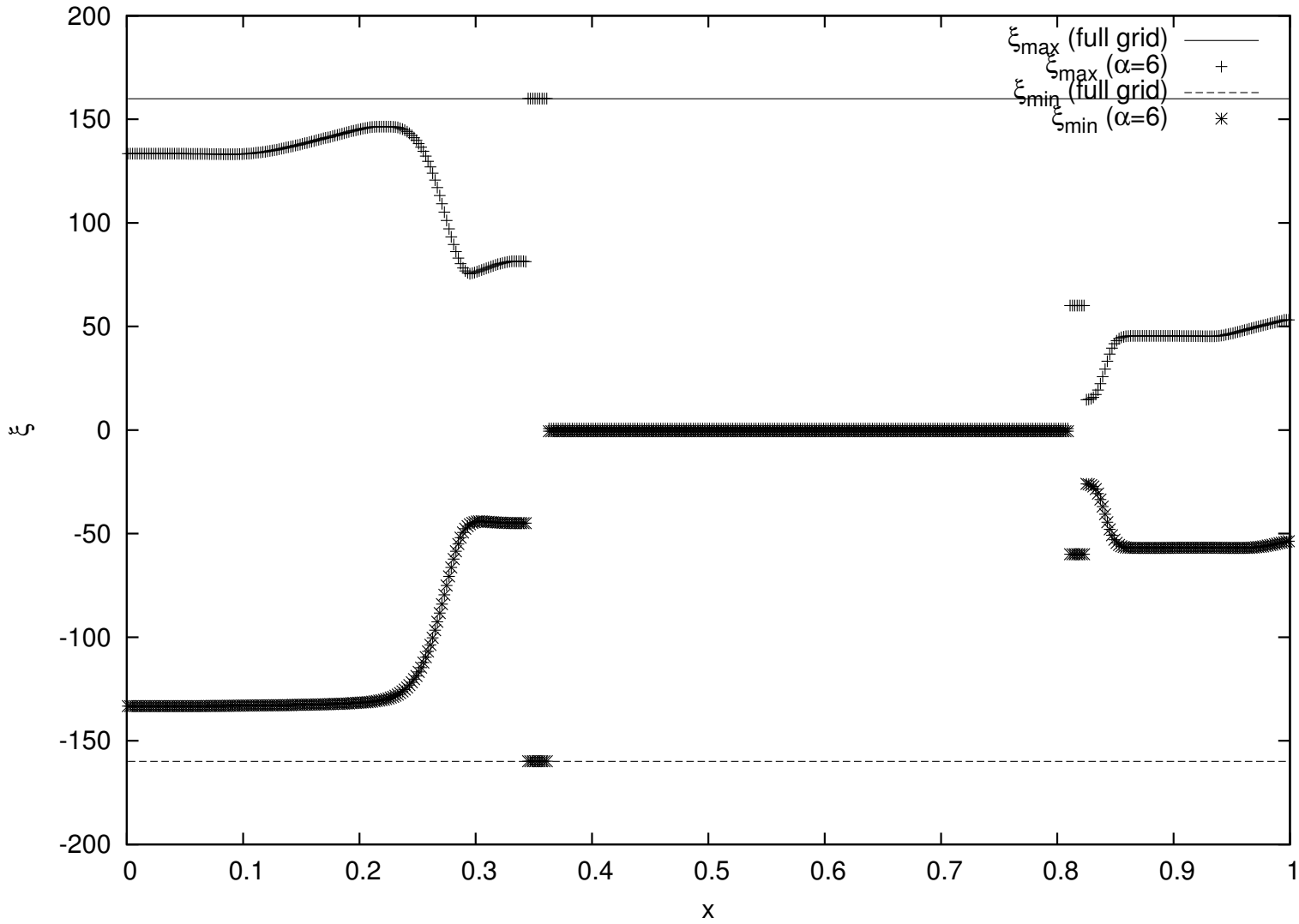
Set a tolerance for the error in mass conservation. Accept the new grid if

$$\frac{|\int_{\mathcal{G}_v^n} f^n dv - \int_{\mathcal{G}_v^{n-1}} f^n dv|}{\int_{\mathcal{G}_v^{n-1}} f^n dv} < tol$$

with $\mathcal{G}_v^{n-1}$ the local velocity grid at $t = t^{n-1}$ and $\mathcal{G}_v^n$ the local velocity grid defined with α.

If the condition is false, $\mathcal{G}_v^n = \mathcal{G}_v^{n-1}$ (usually the tolerance is set to 10^{-5}).

Keeping track of non Maxwellian effects ($Kn = 10^{-5}$)



Bounds of the local velocity grid in x at the final time

Two interacting blast waves ($Kn = 10^{-2}$)

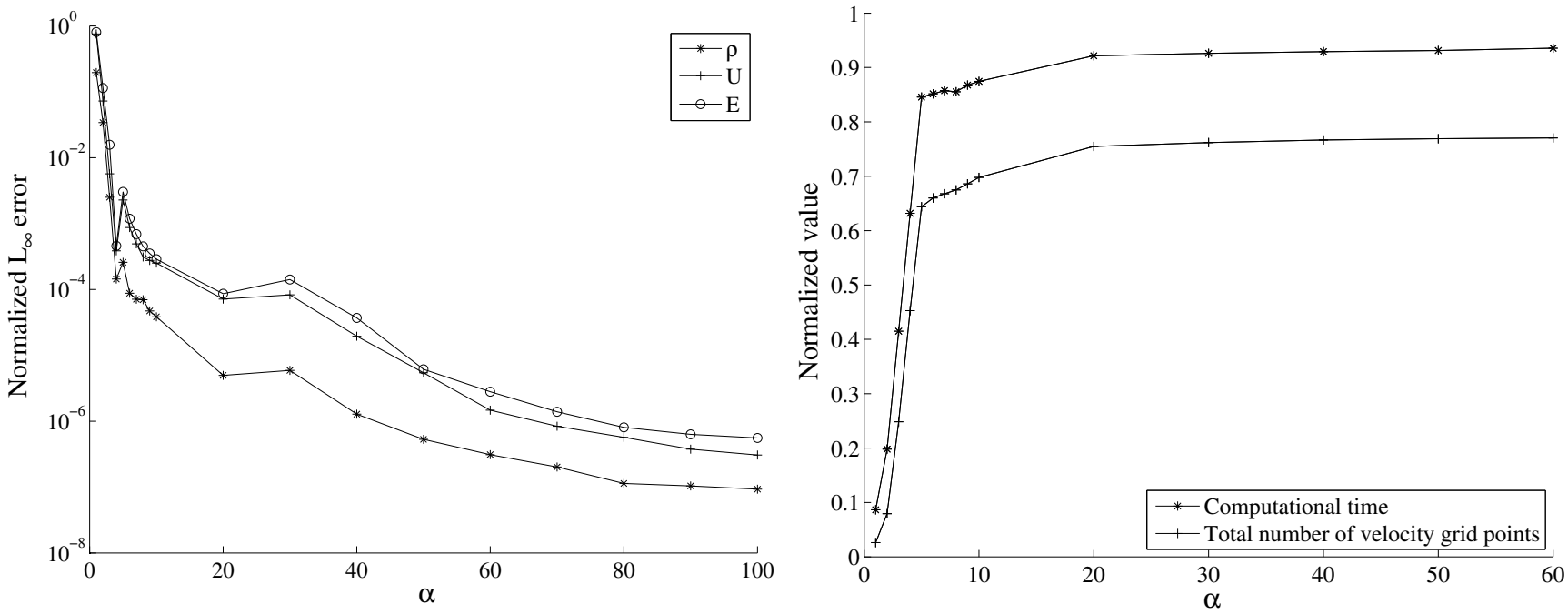


Figure: Error L_∞ norm (left) for density, velocity and energy. Computational time and total number of velocity grid points used (right)

Computational time

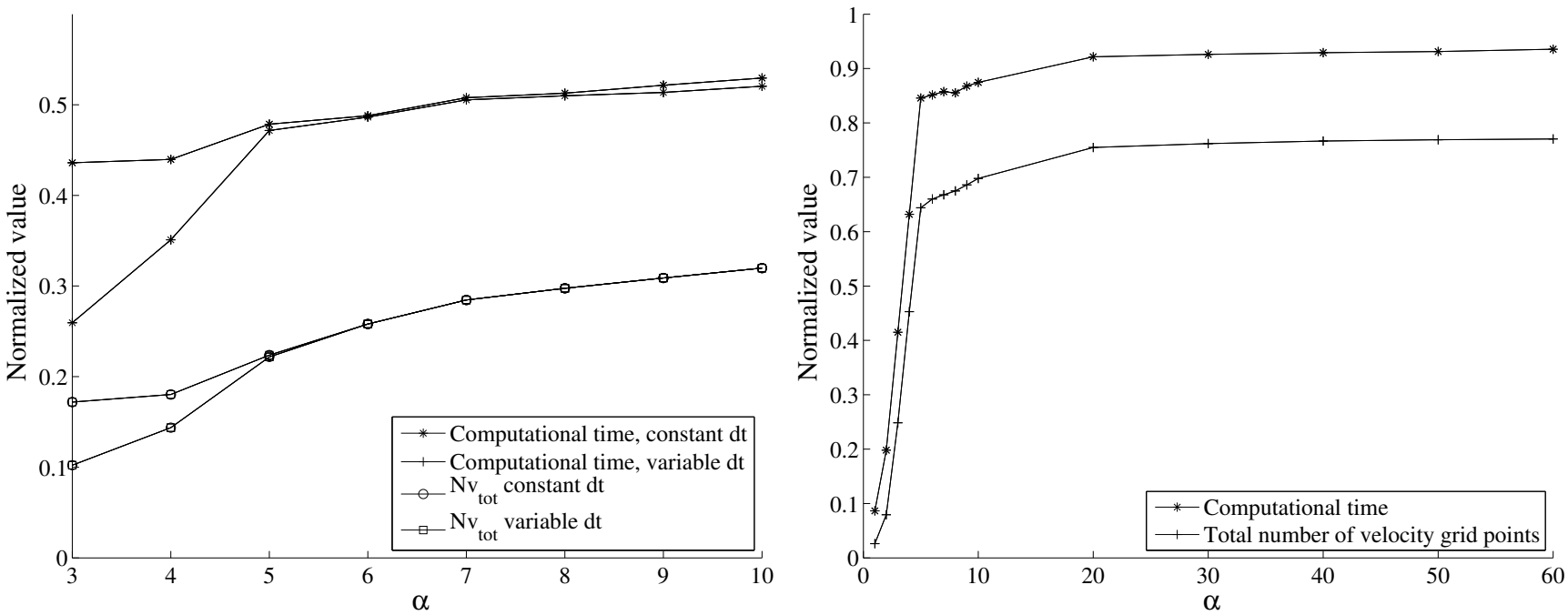
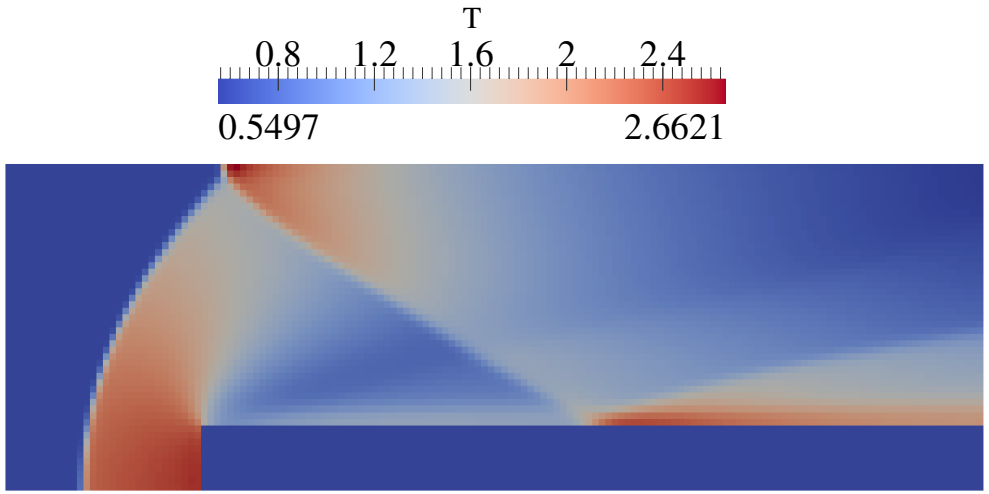
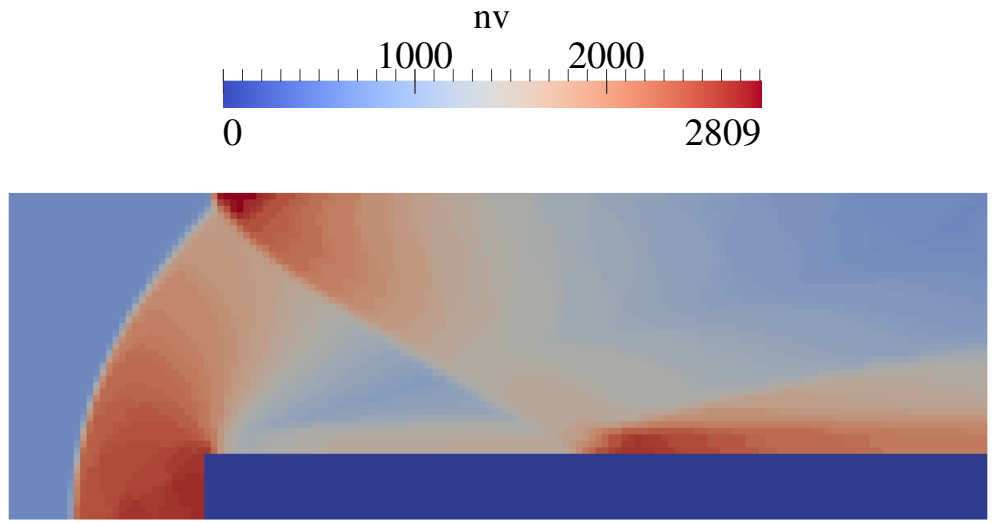


Figure: Gain in computational time and in the total number of velocity grid points used for $Kn = 10^{-5}$ (left) and $Kn = 10^{-2}$ (right), with respect to the global grid

A Mach 3 wind tunnel with a step ($Kn = 10^{-5}$)





A Mach 3 wind tunnel with a step ($Kn = 10^{-5}$ and $Kn = 10^{-2}$)

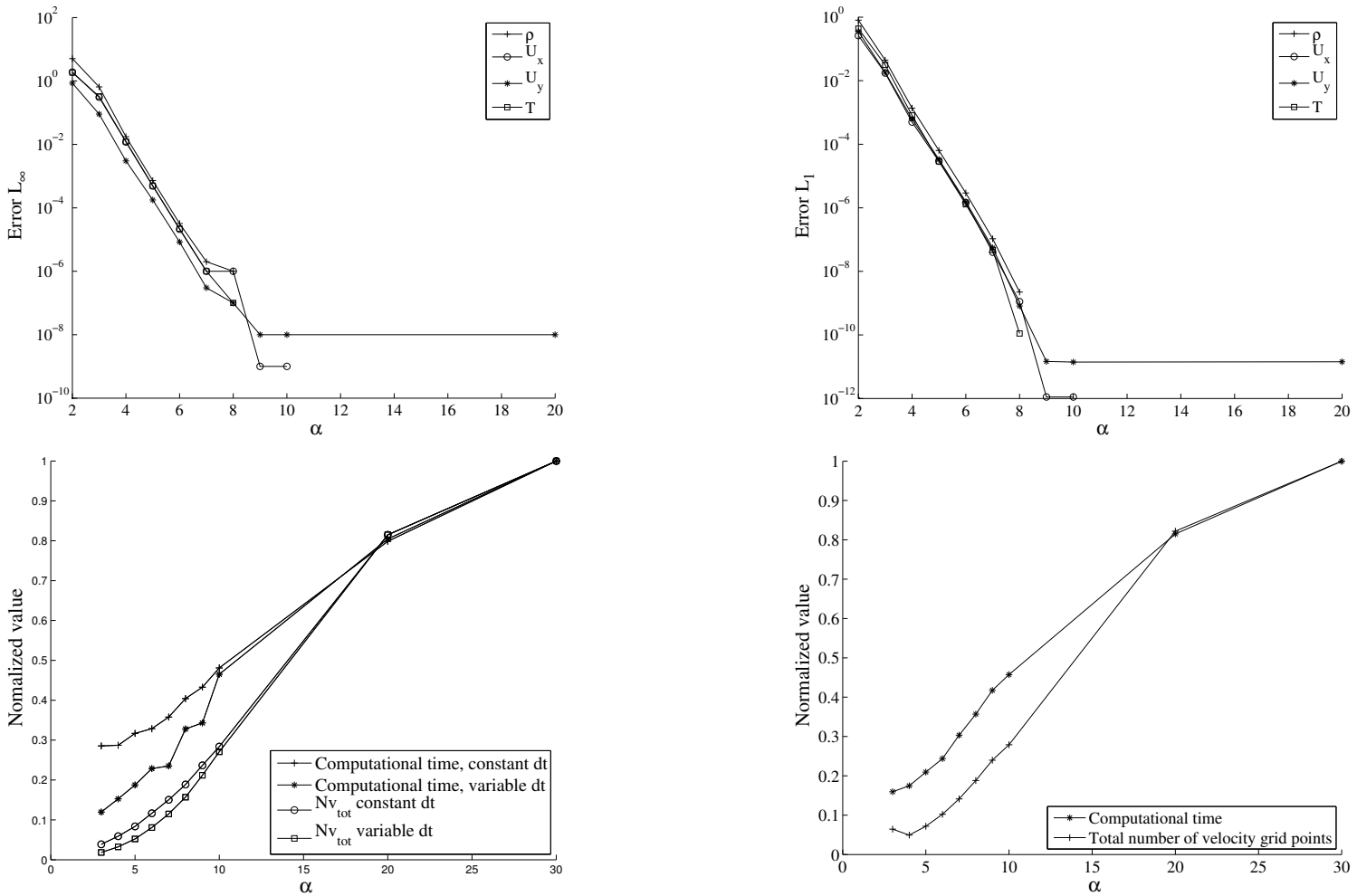
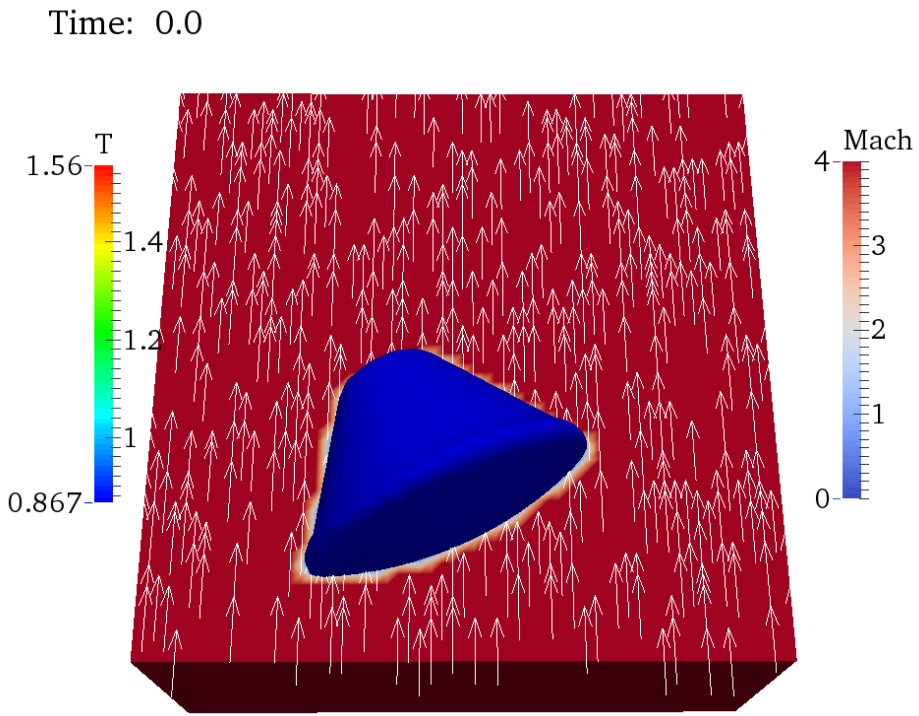


Figure: Error in L_∞ and L^1 norm for density, velocity and temperature and

3D: falling capsule



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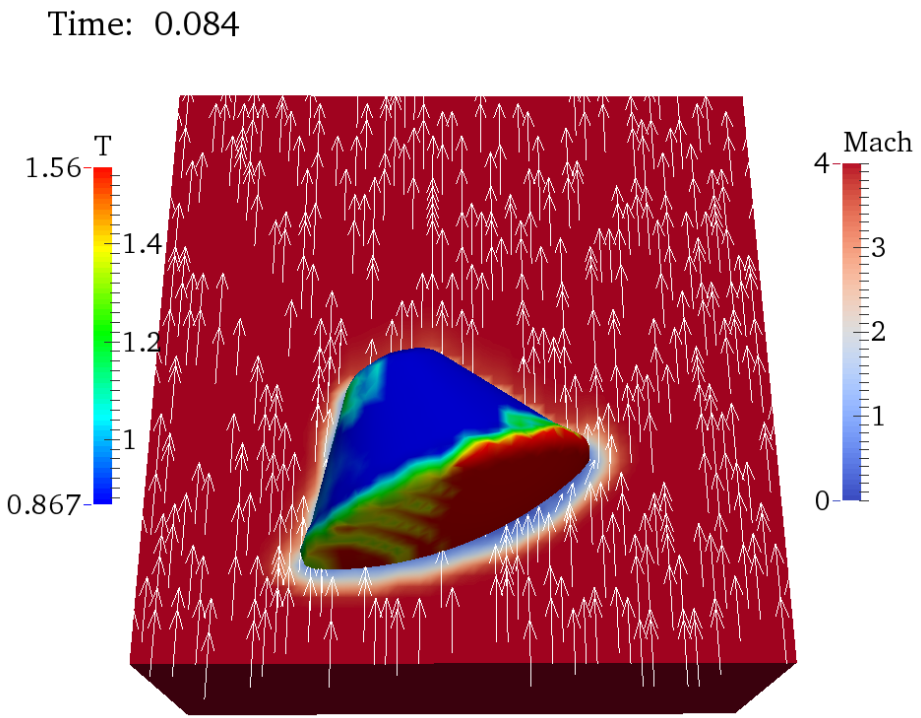

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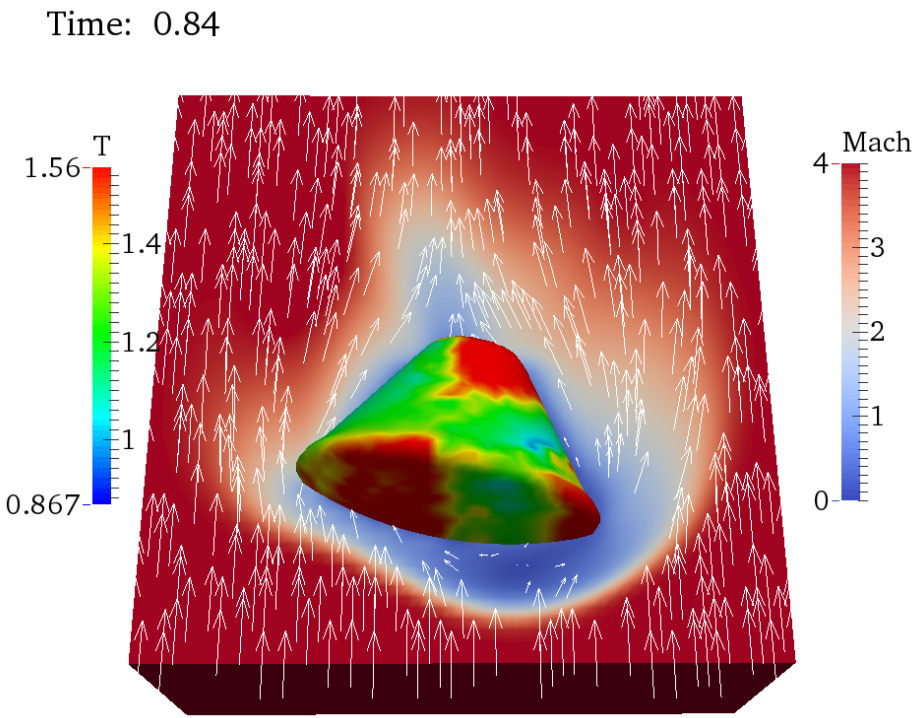

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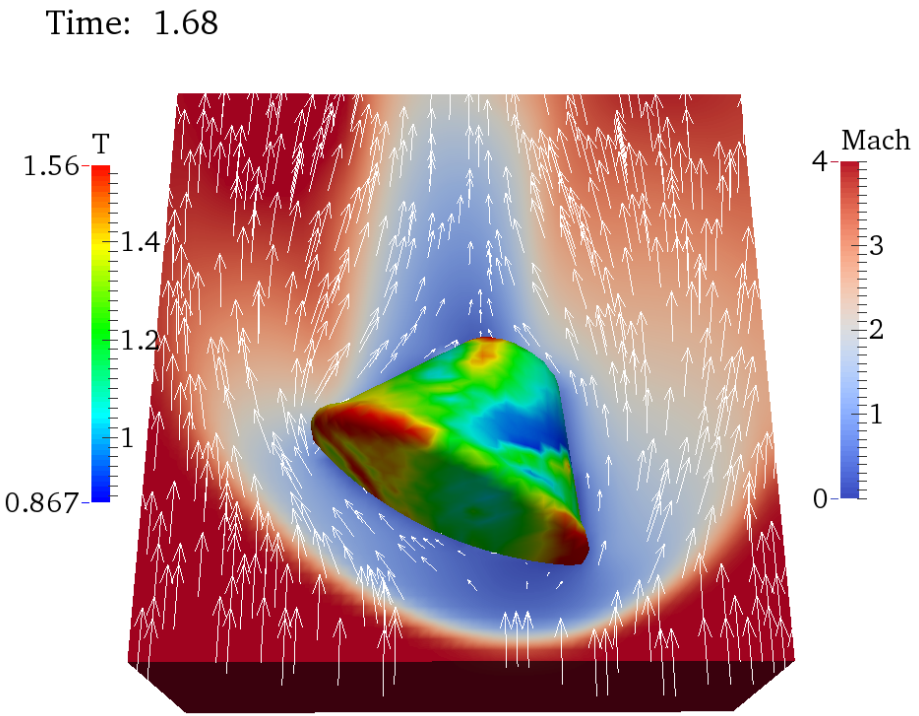

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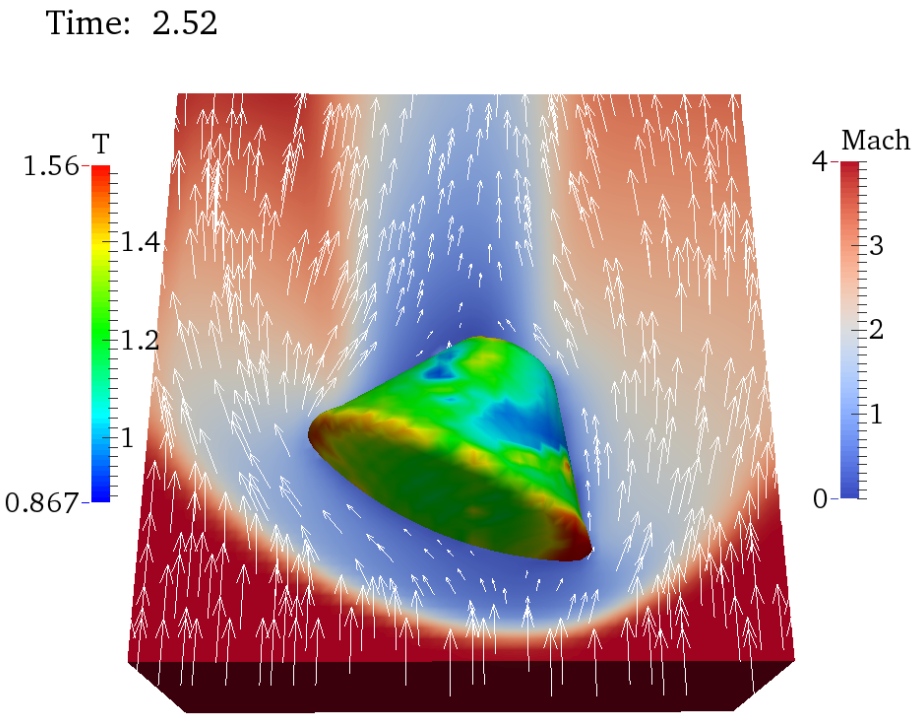

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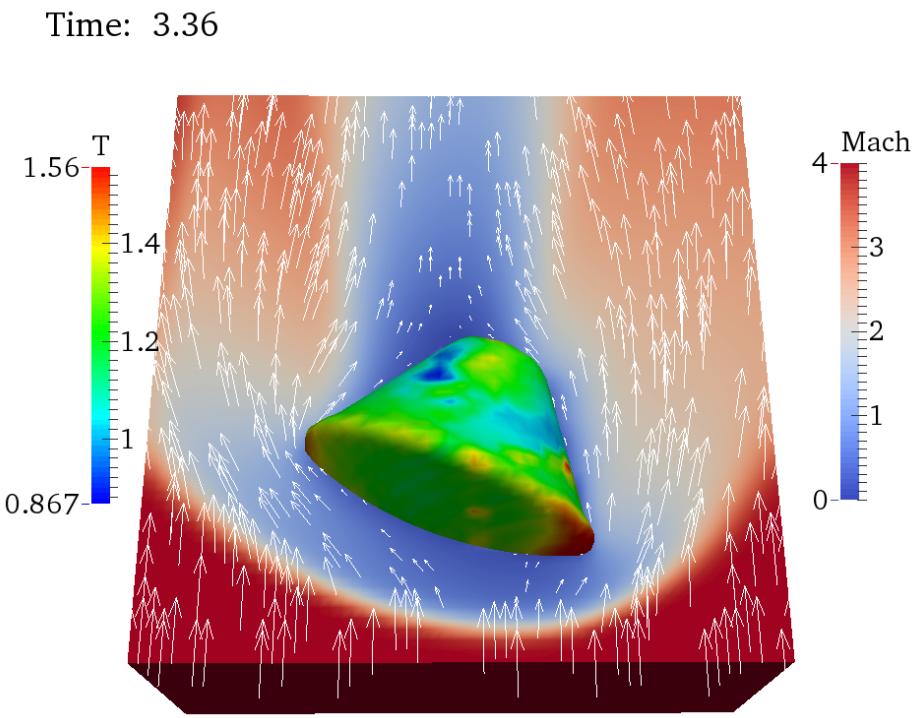

Thank to Florian Bernard

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We need more realistic models

Outline

1 Motivation and set up

2 Kinetic theory

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4 A local velocity grid approach

5 Mixtures of gases

Motivation

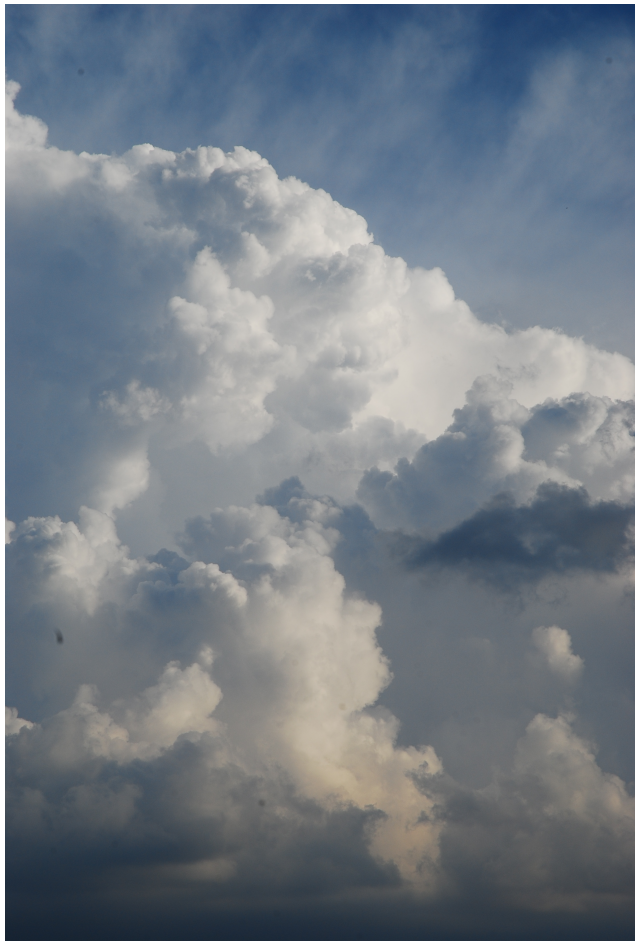
Fluid dynamics usually concentrates on the treatment of the evolution of a single gas.

Motivation

However, many phenomena actually involve mixtures of different fluids

Motivation

Just think of metheorology, where you need at least to deal with a mixture of air and vapour



Fluid dynamics of mixtures

The evolution of a mixture of two gases is modelled in several ways

- One can think that the two gases have different densities, speeds and temperatures. What then will be the exchange terms between the different species?
- Or one can think of a common speed and temperature, evolving only the volume fraction occupied by each of the different gases. But what would be the common equation of state?

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Kinetic theory applied to mixtures can help to model these phenomena.

Motivation

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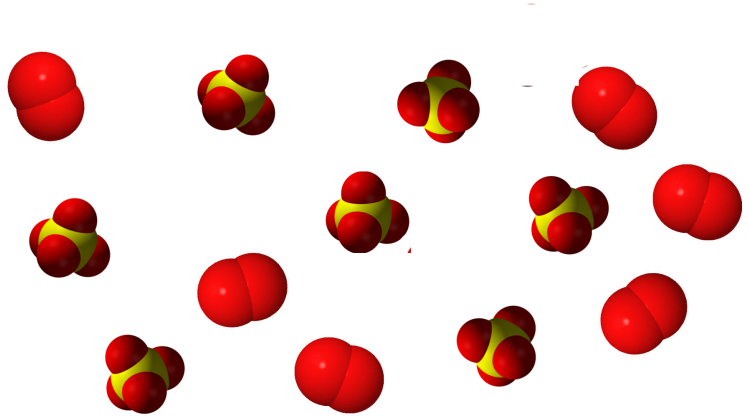
Then we must account for interactions of one species with itself, $Q(f_1, f_1)$, and interactions of one species with the other one, $Q(f_1, f_2)$. Thus, we obtain two equations, with two interaction terms each.



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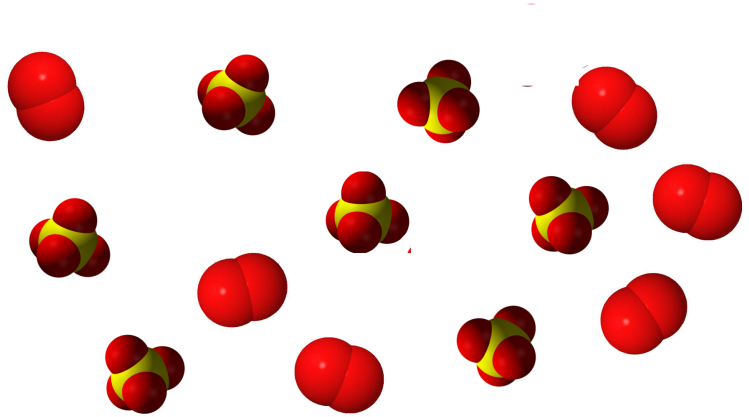

$$\partial_t f_1 + v \cdot \nabla f_1 = Q_{11}(f_1, f_1) + Q_{12}(f_1, f_2)$$
$$\partial_t f_2 + v \cdot \nabla f_2 = Q_{22}(f_2, f_2) + Q_{21}(f_2, f_1).$$

BGK for mixtures (with M. Pirner and C. Klingenberg, 2017)

A natural extension of the BGK model for mixtures can be written as

$$\partial_t f_1 + v \cdot \nabla f_1 = \frac{n_1}{\tau_{11}}(M_1 - f_1) + \frac{n_2}{\tau_{12}}(M_{12} - f_1)$$
$$\partial_t f_2 + v \cdot \nabla f_2 = \frac{n_2}{\tau_{22}}(M_2 - f_2) + \frac{n_1}{\tau_{21}}(M_{21} - f_2)$$

where the first terms account for the interactions of a species with itself, while the second terms account for cross-interactions, and τ_{ij} are the collision times of the different types of collisions, normalized with a typical value for n.

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- Models based on two collision terms, with differences in the exchange terms: Gross, 1956, Hamel, 1965, Asinari, 2008.

The Maxwellians

The internal Maxwellians are given by

$$M_i(t,x,v) = \frac{n_i(x,t)}{\sqrt{2\pi \frac{T_i(x,t)}{m_i}}^d} \exp\left(-\frac{(v-u_i(x,t))^2}{2T_i(x,t)/m_i}\right).$$

So $< \phi(v)\ M_i > - < \phi(v)\ f_i >= 0$ for $\phi(v) = [1, v, v^2]^T$, to ensure conservation of mass, momentum and energy when no interspecies collisions occur.

- In this fashion, collisions between particles of the same kind do not result in momentum or energy exchanges.

The Maxwellians, 2

The cross-species Maxwellians are given by

$$M_{ij}(t,x,v) = \frac{n_i(x,t)}{\sqrt{2\pi \frac{T_{ij}(x,t)}{m_i}}^d} \exp\left(-\frac{(v-u_{ij}(x,t))^2}{2T_{ij}(x,t)/m_i}\right),$$

where u_{ij} and T_{ij} are mixture velocities and temperatures.

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$$u_{12} = \delta u_1 + (1-\delta)u_2, \qquad 0 \le \delta \le 1,$$
$$T_{12} = \alpha T_1 + (1-\alpha)T_2 + \gamma|u_1 - u_2|^2 \qquad 0 \le \alpha \le 1, \gamma \ge 0.$$

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Then, u_{21} and T_{21} are completely determined by imposing global conservation of total momentum and energy. And,
if $u_1 = u_2 = U \implies u_{12} = u_{21} = U$,
while if $T_1 = T_2 = T$ and $u_1 = u_2 \implies T_{12} = T_{21} = T$.

The mixture velocity

The momentum loss (or gain) of each of the two gases is

$$q_{12} = \frac{m_1 n_1 n_2}{\tau_{12}}(u_{12} - u_1); \qquad q_{21} = \frac{m_2 n_1 n_2}{\tau_{21}}(u_{21} - u_2)$$

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- Momentum conservation implies $q_{21} + q_{12} = 0$.
- So, if you give $u_{12} = u_{12}(u_1, u_2)$, then you have $u_{21} = u_{21}(u_1, u_2)$.

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- For instance, if the mixture velocity is chosen as $u_{12} = \delta u_1 + (1 - \delta)u_2$, with say $\delta = m_1/(m_1 + m_2)$, you find

$$u_{21} = \tilde{\delta} u_1 + (1 - \tilde{\delta})u_2, \qquad \tilde{\delta} = \frac{\tau_{21}}{m_2}\frac{m_1}{\tau_{12}}(1 - \delta)$$

The mixture temperature

We choose the mixture temperature as resulting from a convex combination of the single species temperatures, and a dissipation of kinetic energy

$$T_{12} = \alpha T_1 + (1-\alpha) T_2 + \gamma ||u_1 - u_2||^2 \qquad 0 \leq \alpha \leq 1, \gamma \geq 0.$$

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$$E_{12} = \frac{n_1 n_2}{\tau_{12}} \left(\frac{m_1}{2}(u_{12}^2 - u_1^2) + \frac{d}{2}(T_{12} - T_1) \right)$$
$$E_{21} = \frac{n_1 n_2}{\tau_{21}} \left(\frac{m_2}{2}(u_{21}^2 - u_2^2) + \frac{d}{2}(T_{21} - T_2) \right)$$

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where α and γ may depend on the mass of the particles, or on other microscopic characteristics. Then the second mixture temperature is determined uniquely imposing $E_{12} + E_{21} = 0$. So

$$T_{21} = \mathcal{T}(\delta, \alpha, \gamma; \quad u_1, u_2, T_1, T_2, \dots)$$

where $\mathcal{T}$ is a complicated (but computable!) function.

Properties of the model

We can prove several properties for this model (Klingenberg, Pirner, P., KRM, 2017), in the space homogeneous case. Suppose that the initial data satisfy $f_1(t = 0, x, v) \geq 0$ and $f_2(t = 0, x, v) \geq 0$.

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$$\mathcal{H} =< f_1 \log f_1 + f_2 \log f_2 > \qquad \Longrightarrow \qquad \partial_t \mathcal{H} \leq 0$$

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- At equilibrium, $u_1 = u_2 = u_{21} = u_{12}$ and $T_1 = T_2 = T_{21} = T_{12}$.

References

- Models with a single collision term, which accounts for both types of interactions (a species with itself and with the other species): Andries et al., 2002, Brull et al., 2012., Groppi et al.

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- Then Bobylev, Bisi et al (KRM 2018) actually computed the transfer coefficients α, δ, γ, using the full Boltzmann equation. So now, the BGK and the Boltzmann mixture models have the same momentum and energy exchanges.
- Recently, Pirner with Klingenberg proved existence and uniqueness results for this model, (J.Diff. Eqs. 2018)

IMEX Runge Kutta schemes for kinetic eqs

The IMEX splitting shown before can be applied also in this setting.

$$f_i^k = f_i^n - \Delta t \sum_{\ell=1}^{k-1} \tilde{a}_{k\ell} v \cdot \nabla f_i^\ell + \Delta t \sum_{\ell=1}^{k} a_{k\ell} \left[\frac{n_i^\ell}{\tau_{ii}^\ell}(M_i^\ell - f_i^\ell) + \frac{n_j^\ell}{\tau_{ij}^\ell}(M_{ij}^\ell - f_i^\ell) \right]$$

$$f_i^{n+1} = f_i^n - \Delta t \sum_{\ell=1}^{\nu} \tilde{b}_\ell v \cdot \nabla f_i^\ell + \Delta t \sum_{\ell=1}^{\nu} b_\ell \left[\frac{n_i^\ell}{\tau_{ii}^\ell}(M_i^\ell - f_i^\ell) + \frac{n_j^\ell}{\tau_{ij}^\ell}(M_{ij}^\ell - f_i^\ell) \right]$$

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At each stage, one needs to solve

$$f_i^k = (\dots) + \Delta t\, a_{kk} \left[\frac{n_i^k}{\tau_{ii}^k}(M_i^k - f_i^k) + \frac{n_j^k}{\tau_{ij}^k}(M_{ij}^k - f_i^k) \right].$$

Thus the system is linear (and diagonal), once M_i^k, M_{ij}^k and the macroscopic quantities τ, n are known.

Predicting the Maxwellians

The Maxwellians can be predicted explicitely, as in Pieraccini, P., 2006, computing moments of the kinetic equations, to find

$$n_i^k = n_i^n - \Delta t \sum_{\ell=1}^{k-1} \tilde{a}_{k\ell} \nabla < v \; f_i^\ell >$$

$$u_i^k = u_i^n - \Delta t \sum_{\ell=1}^{k-1} \tilde{a}_{k\ell} \nabla < v \otimes v \; f_i^\ell > + \Delta t \sum_{\ell=1}^{k} a_{k\ell} \left[\frac{n_j^\ell}{\tau_{ij}^\ell} (u_{ij}^\ell - u_i^\ell) \right]$$

$$T_i^k = T_i^n - \frac{1}{3} \Delta t \sum_{\ell=1}^{k-1} \tilde{a}_{k\ell} \nabla < m_i (v - u_i)^2 \; f_i^\ell > +$$

$$\Delta t \sum_{\ell=1}^{k} a_{k\ell} \left[\frac{n_j^\ell}{\tau_{ij}^\ell} (T_{ij}^\ell - T_i^\ell) \right]$$

Predicting the Maxwellians, 2

Single out unknown quantities at each stage,

$$n_i^k = (\dots)^{k-1}$$

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Since u_{ij} and T_{ij} are known functions of u_i's and T_i's, one obtains a coupled algebraic system in the four unknowns $u_i^k, i = 1, 2$ and $T_i^k, i = 1, 2$. If the τ_{ij} and the convex combination weights δ, ... are constants, or depend only on n_i, the system is linear.

AP property

We have seen that it is possible to compute all macroscopic parameters, at the new time level, without the need to solve large non linear systems of eqs. Once, the macro parameters are known, all Maxwellians can be easily computed, and finally $f_i(x, v), i = 1, 2$ is found on each grid node in velocity and space, solving scalar linear eqs.

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- The AP property rests on the shoulders of the IMEX scheme chosen.
- The literature here is quite large: DiMarco, Pareschi, ACTA 2014, Pareschi-Russo, 2005, Boscarino et al., SISC 2013, Degond . . . , Jin . . .

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Computations are needed!!!

Obrigado!