Entropy stable nodal discontinuous Galerkin spectral element method for the resistive MHD equations

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Motivation

- Approximate the solution of the resistive MHD equations with high-order discontinuous Galerkin (DG) method
- Broad range of applications in space or astrophysics
- Possible to have other auxiliary conserved quantities not built into the PDE
- Entropy of the system is one such quantity
- Important as entropy helps separate possible flow states from the impossible
- Entropy aware schemes have increased robustness

Divergence-free condition

- Known numerical issue: It is possible for the approximate flow to not be divergence-free even if it is initially
- Generalized Lagrange multiplier (GLM) terms advect divergence errors away from where they are generated
- Possible to damp errors as well with an additional source term

$$\mathbf{r} = (0\,,\,\vec{0}\,,\,0\,,\,\vec{0}\,,-\alpha\psi)^{T}, \quad \alpha > 0$$

- Entropy conservation and the divergence-free condition are linked
- Several non-conservative terms have been proposed to alter the equations for entropy purposes
- Powell term needed for symmetrization of the advective parts of the PDEs

Resistive GLM-MHD equations

$$\boxed{\mathbf{u}_t + \vec{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{f}}^a(\mathbf{u}) - \vec{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{f}}^v(\mathbf{u},\vec{\nabla}\mathbf{u}) + \boldsymbol{\Upsilon} = \mathbf{0}}$$

- Conservative variables, advective and resistive fluxes

$$\mathbf{u} = (\varrho\,,\,\varrho\vec{v}\,,\,E\,,\,\vec{B}\,,\,\psi)^T$$

$$\overset{\leftrightarrow}{\mathbf{f}}^a(\mathbf{u}) = \overset{\leftrightarrow}{\mathbf{f}}^{a,\mathrm{Euler}}(\mathbf{u}) + \overset{\leftrightarrow}{\mathbf{f}}^{a,\mathrm{MHD}}(\mathbf{u}) + \overset{\leftrightarrow}{\mathbf{f}}^{a,\mathrm{GLM}}(\mathbf{u})$$

$$\overset{\leftrightarrow}{\mathbf{f}}^v(\mathbf{u},\vec{\nabla}\mathbf{u}) = \left(\vec{0}\,,\,\underline{\tau}\,,\,\underline{\tau}\vec{v} - \vec{\nabla}T - \eta\left((\vec{\nabla}\times\vec{B})\times\vec{B}\right)\,,\,\eta\left((\vec{\nabla}\vec{B})^T - \vec{\nabla}\vec{B}\right)\,,\,\vec{0}\right)^T$$

- Non-conservative terms

$$\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}^{\mathrm{MHD}} + \boldsymbol{\Upsilon}^{\mathrm{GLM}}$$

$$\boldsymbol{\Upsilon}^{\mathrm{MHD}} = (\vec{\nabla}\cdot\vec{B})\boldsymbol{\phi}^{\mathrm{MHD}} = \left(\vec{\nabla}\cdot\vec{B}\right)\left(0\,,\,B_1\,,\,B_2\,,\,B_3\,,\,\vec{v}\cdot\vec{B}\,,\,v_1\,,\,v_2\,,\,v_3\,,\,0\right)^T$$

$$\boldsymbol{\Upsilon}^{\mathrm{GLM}} = \overset{\leftrightarrow}{\boldsymbol{\phi}}^{\mathrm{GLM}}\cdot\vec{\nabla}\psi \quad = \boldsymbol{\phi}_1^{\mathrm{GLM}}\frac{\partial\psi}{\partial x} + \boldsymbol{\phi}_2^{\mathrm{GLM}}\frac{\partial\psi}{\partial y} + \boldsymbol{\phi}_3^{\mathrm{GLM}}\frac{\partial\psi}{\partial z}$$

with

$$\boldsymbol{\phi}_\ell^{\mathrm{GLM}} = (0\,,\,0\,,\,0\,,\,0\,,\,v_\ell\psi\,,\,0\,,\,0\,,\,0\,,\,v_\ell)^T, \quad \ell = 1,2,3$$

Entropy definitions

- First examine the ideal parts of the resistive GLM-MHD equations
- Introduce the mathematical entropy function

$$S(\mathbf{u}) = -\frac{\varrho s}{\gamma - 1}, \quad s = \ln(p\varrho^{-\gamma}) = -(\gamma - 1)\ln(\varrho) - \ln(\beta) - \ln(2)$$

where the pressure is

$$p = (\gamma - 1)\left(E - \frac{\varrho}{2}\|\vec{v}\|^2 - \frac{1}{2}\|\vec{B}\|^2 - \frac{1}{2}\psi^2\right)$$

and β is proportional to the inverse temperature

$$\beta = \frac{\varrho}{2p}$$

- Note there is a sign convention difference between mathematics and physics

Entropy definitions I

- Entropy for smooth solutions is conserved

$$S_t + \vec{\nabla} \cdot \vec{f}^{\,S} = 0, \quad \vec{f}^{\,S} = \vec{v}S$$

whereas for discontinuous solutions the entropy decays

$$S_t + \vec{\nabla} \cdot \vec{f}^{\,S} \leq 0$$

- Can move into entropy space with the new set of variables

$$\mathbf{w} = \frac{\partial S}{\partial \mathbf{u}} = \left[\frac{\gamma - s}{\gamma - 1} - \beta\|\vec{v}\|^2\,,\, 2\beta\vec{v}\,,\, -2\beta\,,\, 2\beta\vec{B}\,,\, 2\beta\psi\right]^T$$

- Contract the PDE system on the left with **w** to obtain conservation law

Entropy behavior for Ideal GLM-MHD

- Contract from the left with $\mathbf{w}$ to determine

$$\mathbf{w}^T\left(\mathbf{u}_t + \vec{\nabla}\cdot\left(\overleftrightarrow{\mathbf{f}}^{a,\mathrm{Euler}}(\mathbf{u}) + \overleftrightarrow{\mathbf{f}}^{a,\mathrm{MHD}}(\mathbf{u}) + \overleftrightarrow{\mathbf{f}}^{a,\mathrm{GLM}}(\mathbf{u})\right) + \boldsymbol{\Upsilon}\right) = 0$$

  where we split the advective flux into three parts for convenience

- From the definition of the entropy variables and many manipulations:

$$\begin{aligned}
\mathbf{w}^T\mathbf{u}_t &= S_t\\
\mathbf{w}^T(\vec{\nabla}\cdot\overleftrightarrow{\mathbf{f}}^{a,\mathrm{Euler}}) &= \vec{\nabla}\cdot\vec{f}^{S}\\
\mathbf{w}^T\left(\vec{\nabla}\cdot\overleftrightarrow{\mathbf{f}}^{a,\mathrm{MHD}} + \boldsymbol{\Upsilon}^{\mathrm{MHD}}\right) &= 0\\
\mathbf{w}^T\left(\vec{\nabla}\cdot\overleftrightarrow{\mathbf{f}}^{a,\mathrm{GLM}} + \boldsymbol{\Upsilon}^{\mathrm{GLM}}\right) &= 0
\end{aligned}$$

- Entropy conservation not possible when $\vec{\nabla}\cdot\vec{B}\neq 0$ unless a non-conservative term is included

Entropy behavior for resistive GLM-MHD

- Examine how viscous and resistive effects change entropy
- Know for smooth solutions that

$$\mathbf{w}^T\left(\mathbf{u}_t + \vec{\nabla}\cdot\overleftrightarrow{\mathbf{f}}^a - \vec{\nabla}\cdot\overleftrightarrow{\mathbf{f}}^v + \boldsymbol{\Upsilon}\right) = 0$$

$$\Updownarrow$$

$$S_t + \vec{\nabla}\cdot\vec{f}^{\,S} - \mathbf{w}^T\vec{\nabla}\cdot\overleftrightarrow{\mathbf{f}}^v = 0$$

- Integrate over the domain to obtain a variational form of the entropy evolution for the DG scheme to mimic

$$\int\limits_{\Omega} S_t + \vec{\nabla}\cdot\vec{f}^{\,S} - \mathbf{w}^T\vec{\nabla}\cdot\overleftrightarrow{\mathbf{f}}^v\,\mathrm{dV} = 0$$

Entropy behavior for resistive GLM-MHD I

- Apply Divergence Theorem (advective) and integration-by-parts (viscous)

$$\int_{\Omega} S_t \,\mathrm{dV} + \int_{\partial\Omega} (\vec{f}^{\,S} \cdot \vec{n}) - \mathbf{w}^T (\overleftrightarrow{\mathbf{f}}^v \cdot \vec{n}) \,\mathrm{dS} = - \int_{\Omega} (\vec{\nabla} \mathbf{w})^T \overleftrightarrow{\mathbf{f}}^v \,\mathrm{dV}$$

- Possible to re-formulate viscous fluxes as

$$\overleftrightarrow{\mathbf{f}}^v(\mathbf{u}, \vec{\nabla}\mathbf{u}) = \underline{\mathsf{K}} \vec{\nabla} \mathbf{w}$$

with a symmetric, positive semi-definite block matrix $\underline{\mathsf{K}} \in \mathbb{R}^{27\times 27}$

- Term on the right hand side can be bounded!

$$- \int_{\Omega} (\vec{\nabla} \mathbf{w})^T \overleftrightarrow{\mathbf{f}}^v \,\mathrm{dV} = - \int_{\Omega} (\vec{\nabla} \mathbf{w})^T \underline{\mathsf{K}} \, \vec{\nabla} \mathbf{w} \,\mathrm{dV} \le 0.$$

Entropy behavior for resistive GLM-MHD II

- Satisfy the entropy inequality up to the prescription of proper boundary conditions

$$\int_{\Omega} S_t \,\mathrm{dV} + \int_{\partial\Omega} (\vec{f}^{\,S} \cdot \vec{n}) - \mathbf{w}^T (\overleftrightarrow{\mathbf{f}}^v \cdot \vec{n}) \,\mathrm{dS} \leq 0$$

- For periodic boundaries (closed systems) we see that the entropy decays in time

$$\int_{\Omega} S_t \,\mathrm{dV} \leq 0$$

DGSEM: Mapping the equations

- Subdivide domain Ω into N_{el} non-overlapping, conforming, curved hexahedral elements $E_\nu, \nu = 1, 2, \ldots, N_{\mathrm{el}}$
- Transform into computational coordinates $\vec{\xi} = (\xi, \eta, \zeta)^T$ in the reference element $E = [-1, 1]^3$ by mapping $\vec{x} = \vec{X}(\vec{\xi})$
- Element mapping defines Jacobian, J, covariant and contravariant basis vectors $\vec{a}_i, \vec{a}^i, i = 1, 2, 3$
- Basis vectors vary on curved elements
- Important that the contravariant vectors satisfy the metric identities

$$\sum_{i=1}^{3} \frac{\partial \left(J a_n^i\right)}{\partial \xi^i} = 0\,, \quad n = 1, 2, 3$$

DGSEM: Mapping the equations I

- Transform divergence of block vectors, divergence of space vectors, and the gradient of state vectors or a scalar into reference space

$$\vec{\nabla}_x \cdot \overleftrightarrow{\mathbf{g}} = \frac{1}{J}\vec{\nabla}_\xi \cdot \left(\underline{\mathsf{M}}^T \overleftrightarrow{\mathbf{g}}\right), \quad \vec{\nabla}_x \cdot \vec{h} = \frac{1}{J}\vec{\nabla}_\xi \cdot \left(\underline{M}^T \vec{h}\right)$$

$$\vec{\nabla}_x \mathbf{u} = \frac{1}{J}\underline{\mathsf{M}}\vec{\nabla}_\xi \mathbf{u}, \quad \vec{\nabla}_x h = \frac{1}{J}\underline{M}\vec{\nabla}_\xi h$$

- Compact notation due to two matrices dependent on the metric terms

$$\underline{\mathsf{M}} = \begin{bmatrix} Ja_1^1\,\mathsf{I}_9 & Ja_1^2\,\mathsf{I}_9 & Ja_1^3\,\mathsf{I}_9 \\ Ja_2^1\,\mathsf{I}_9 & Ja_2^2\,\mathsf{I}_9 & Ja_2^3\,\mathsf{I}_9 \\ Ja_3^1\,\mathsf{I}_9 & Ja_3^2\,\mathsf{I}_9 & Ja_3^3\,\mathsf{I}_9 \end{bmatrix}, \qquad \underline{M} = \begin{bmatrix} Ja_1^1 & Ja_1^2 & Ja_1^3 \\ Ja_2^1 & Ja_2^2 & Ja_2^3 \\ Ja_3^1 & Ja_3^2 & Ja_3^3 \end{bmatrix}$$

DGSEM: Mapping the equations II

- Define compact tilde notation for contravariant block and spatial vectors

$$\overset{\leftrightarrow}{\tilde{\mathbf{g}}} = \underline{\mathsf{M}}^T \overset{\leftrightarrow}{\mathbf{g}} = \overset{\leftrightarrow}{\mathbf{g}}\,\underline{M}\,, \quad \vec{\tilde{h}} = \underline{M}^T \vec{h}$$

- Obtain the transformed resistive GLM-MHD equations

$$J\mathbf{u}_t + \vec{\nabla}_\xi \cdot \overset{\leftrightarrow}{\tilde{\mathbf{f}}}^a + \left(\vec{\nabla}_\xi \cdot \vec{\tilde{B}}\right)\boldsymbol{\phi}^{\mathrm{MHD}} + \overset{\leftrightarrow}{\tilde{\boldsymbol{\phi}}}^{\mathrm{GLM}} \cdot \vec{\nabla}_\xi \psi = \vec{\nabla}_\xi \cdot \overset{\leftrightarrow}{\tilde{\mathbf{f}}}^v\left(\mathbf{u}, \overset{\leftrightarrow}{\mathbf{q}}\right)$$

$$J\overset{\leftrightarrow}{\mathbf{q}} = \underline{\mathsf{M}}\vec{\nabla}_\xi \mathbf{w}$$

- Introduce auxiliary variable, $\overset{\leftrightarrow}{\mathbf{q}}$, that is the gradient of the entropy variables

DGSEM: Variational formulation

- Multiply by test functions $\boldsymbol{\varphi}$ and $\overleftrightarrow{\boldsymbol{\vartheta}}$ and integrate over the reference element

$$\left\langle J\mathbf{u}_t + \vec{\nabla}_\xi \cdot \overleftrightarrow{\tilde{\mathbf{f}}}^a + \left(\vec{\nabla}_\xi \cdot \vec{\tilde{B}}\right)\boldsymbol{\phi}^{\mathrm{MHD}} + \overleftrightarrow{\tilde{\boldsymbol{\phi}}}^{\mathrm{GLM}} \cdot \vec{\nabla}_\xi \psi, \boldsymbol{\varphi}\right\rangle = \left\langle \vec{\nabla}_\xi \cdot \overleftrightarrow{\tilde{\mathbf{f}}}^v\left(\mathbf{u}, \overleftrightarrow{\mathbf{q}}\right), \boldsymbol{\varphi}\right\rangle$$

$$\left\langle J\overleftrightarrow{\mathbf{q}}, \overleftrightarrow{\boldsymbol{\vartheta}}\right\rangle = \left\langle \underline{\mathsf{M}}\vec{\nabla}_\xi \mathbf{w}, \overleftrightarrow{\boldsymbol{\vartheta}}\right\rangle$$

- Introduce inner product notation on the reference element for state and block vectors

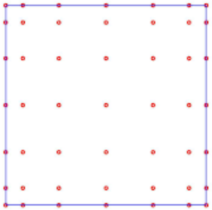
$$\langle \mathbf{u}, \mathbf{v}\rangle = \int_E \mathbf{u}^T \mathbf{v}\, \mathrm{d}\vec{\xi} \quad \text{and} \quad \left\langle \overleftrightarrow{\mathbf{f}}, \overleftrightarrow{\mathbf{g}}\right\rangle = \int_E \sum_{i=1}^{3} \mathbf{f}_i^T \mathbf{g}_i\, \mathrm{d}\vec{\xi}$$

DGSEM: Nodal DG, LGL, and collocation

- Lagrange basis with degree N

$$u_1(x,y,z,t)\big|_e \approx U_1(\xi,\eta,\zeta,t) := \sum_{i,j,k=0}^{N} U_{1,ijk}(t)\,\ell_i(\xi)\,\ell_j(\eta)\ell_k(\zeta)$$

- Legendre-Gauss-Lobatto nodes (because they include the boundary)



- Collocation of flux and solution, e.g., velocity

$$V_{2,ijk}(t) := \frac{U_{3,ijk}(t)}{U_{1,ijk}(t)}$$

- Collocation of interpolation and integration: $(N+1)^3$ LGL nodes/weights

$$\langle \mathbf{f}, \mathbf{1} \rangle = \int_E \mathbf{f}(\xi,\eta,\zeta)\,\mathrm{d}\vec{\xi} \approx \sum_{i,j,k=0}^{N} \mathbf{F}(\xi_i,\eta_j,\zeta_k)\,\omega_i\,\omega_j\,\omega_k = \langle \mathbf{F}, \mathbf{1} \rangle_N$$

DGSEM: Property of the derivative matrix

- On the continuous level have integration-by-parts

$$\int_{-1}^{1} uv' \,\mathrm{d}x = uv\Big|_{-1}^{1} - \int_{-1}^{1} u'v \,\mathrm{d}x$$

- Define the differentiation and mass matrices

$$\mathcal{D}_{ij} := \frac{\partial \ell_j}{\partial \xi}\bigg|_{\xi=\xi_i} \quad (i,j = 0,\dots,N), \qquad \mathcal{M} = \mathrm{diag}(\omega_0,\dots,\omega_N)$$

- For LGL nodes the DG derivative matrix satisfies summation-by-parts (SBP) property

$$(\mathcal{M}\mathcal{D}) + (\mathcal{M}\mathcal{D})^T = \mathcal{B} := \mathrm{diag}(-1,0,\dots,0,1)$$

- Used to discretely mimic integration-by-parts

$$\begin{aligned} u^T\mathcal{M}\mathcal{D}v &= u^T(\mathcal{B} - \mathcal{D}^T\mathcal{M})v \\ &= u^T\mathcal{B}v - u^T\mathcal{D}^T\mathcal{M}v \\ &= u^T\mathcal{B}v - (\mathcal{D}u)^T\mathcal{M}v \end{aligned}$$

DGSEM: Discrete metric identities

- Compute the metric terms as a curl using the DG derivative matrix

$$Ja_n^i = -\hat{x}_i \cdot \nabla_\xi \times \left(\mathbb{I}^N(X_l \nabla_\xi X_m)\right), \; i = 1,2,3, \; n = 1,2,3, \; (n,m,l) \text{ cyclic}$$

- Ensures that the discrete metric identities (DMI) hold

$$\sum_{i=1}^{3} \frac{\partial \mathbb{I}^N\left(Ja_n^i\right)}{\partial \xi^i} = 0, \quad n = 1,2,3.$$

- Discrete metric identities are crucial for entropy conservation/stability

DGSEM: General strong form

- Apply the SBP property once on all spatial derivative terms to generate boundary terms
- Resolve the discontinuity at the interface with numerical surface fluxes for the advective and viscous components (denoted with *)
- Non-conservative terms also discontinuous and contribute at the boundary (denoted with $^\Diamond$)
- Apply the SBP property again to arrive at the strong form DG method

DGSEM: General strong form I

$$
\begin{aligned}
&\langle J\mathbf{U}_t, \boldsymbol{\varphi}\rangle_N + \left\langle \vec{\nabla}_\xi \cdot \mathbb{I}^N\left(\tilde{\overleftrightarrow{\mathbf{F}}}^a\right), \boldsymbol{\varphi}\right\rangle_N + \int\limits_{\partial E,N} \boldsymbol{\varphi}^T \left\{\left(\mathbf{F}_n^{a,*} - \mathbf{F}_n^a\right)\right\} \hat{s}\, \mathrm{d}S \\
&\quad + \left\langle \boldsymbol{\Phi}^{\mathrm{MHD}} \vec{\nabla}_\xi \cdot \mathbb{I}^N\left(\tilde{\vec{B}}\right), \boldsymbol{\varphi}\right\rangle_N + \int\limits_{\partial E,N} \boldsymbol{\varphi}^T \left\{\left(\boldsymbol{\Phi}^{\mathrm{MHD}} B_n\right)^\diamond - \boldsymbol{\Phi}^{\mathrm{MHD}} B_n\right\} \hat{s}\, \mathrm{d}S \\
&\quad + \left\langle \tilde{\overleftrightarrow{\boldsymbol{\Phi}}}^{\mathrm{GLM}} \cdot \vec{\nabla}_\xi \mathbb{I}^N(\psi), \boldsymbol{\varphi}\right\rangle_N + \int\limits_{\partial E,N} \boldsymbol{\varphi}^T \left\{\left(\boldsymbol{\Phi}_n^{\mathrm{GLM}} \psi\right)^\diamond - \boldsymbol{\Phi}_n^{\mathrm{GLM}} \psi\right\} \hat{s}\, \mathrm{d}S \\
&\quad = \left\langle \vec{\nabla}_\xi \cdot \mathbb{I}^N\left(\tilde{\overleftrightarrow{\mathbf{F}}}^v\right), \boldsymbol{\varphi}\right\rangle_N + \int\limits_{\partial E,N} \boldsymbol{\varphi}^T \left\{\mathbf{F}_n^{v,*} - \mathbf{F}_n^v\right\} \hat{s}\, \mathrm{d}S
\end{aligned}
$$

$$
\left\langle J\overleftrightarrow{\mathbf{Q}}, \overleftrightarrow{\boldsymbol{\vartheta}}\right\rangle_N = \int\limits_{\partial E,N} \mathbf{W}^{*,T}\left(\overleftrightarrow{\boldsymbol{\vartheta}} \cdot \vec{n}\right) \hat{s}\, \mathrm{d}S - \left\langle \mathbf{W}, \vec{\nabla}_\xi \cdot \mathbb{I}^N\left(\underline{M}^T \overleftrightarrow{\boldsymbol{\vartheta}}\right)\right\rangle_N
$$

- Know how to handle the conservative terms (top and bottom)
- What about non-conservative terms (middle two)?

DGSEM: Conservative terms (viscous)

- Viscous volume contributions use the standard LGL-DGSEM, e.g., in the ξ direction at each LGL node

$$\left(\tilde{\mathbf{F}}^{v}_{1,ijk}(\mathbf{U})\right)_{\xi} = \frac{1}{\mathcal{M}_{ii}}\left(\delta_{iN}\left[\tilde{\mathbf{F}}^{v,*}_{1} - \tilde{\mathbf{F}}^{v}_{1}\right]_{Njk} - \delta_{i0}\left[\tilde{\mathbf{F}}^{v,*}_{1} - \tilde{\mathbf{F}}^{v}_{1}\right]_{0jk}\right) + \sum_{m=0}^{N}\mathcal{D}_{im}\tilde{\mathbf{F}}^{v}_{1,mjk}$$

- Use the Bassi-Rebay (BR1) viscous interface coupling in terms of the discrete entropy variables and gradients

$$\mathbf{F}^{v,*}_{n} = \left\{\!\!\left\{\overleftrightarrow{\mathbf{F}}^{v}\right\}\!\!\right\}\cdot\vec{n} \qquad \mathbf{W}^{*} = \{\!\{\mathbf{W}\}\!\}$$

Conservative terms (advective): Why care about split forms??

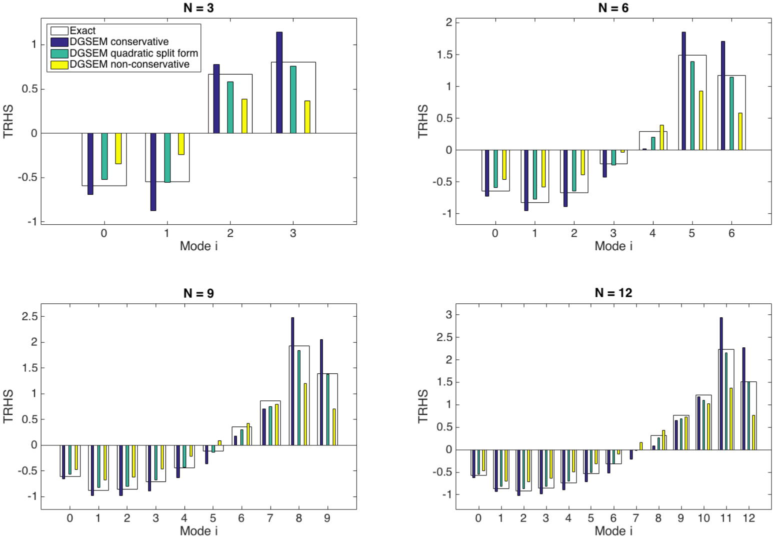
- One interpretation of split forms is the average of conservative and advective forms, e.g.

$$(ab)_x = \frac{1}{2}\left((ab)_x + a_x b + a b_x\right)$$

- Split forms have known beneficial dealiasing properties!
- Could address geometric dealiasing by splitting apart mapping terms from physical fluxes
- Can further add physical dealiasing depending on how one interprets the non-linearities in the PDE

Conservative terms (advective): Quadratic flux example

- Consider a simple on dimensional quadratic flux $f = \frac{1}{2}u^2$
- Analyze the modal energy at different orders to heuristically explain split form dealiasing



DGSEM: Split form advective terms

- Advective volume contributions use the split form LGL-DGSEM, e.g., in the ξ direction at each LGL node

$$\left(\tilde{\mathbf{F}}^a_{1,ijk}(\mathbf{U})\right)_\xi = \frac{1}{\mathcal{M}_{ii}}\left(\delta_{iN}\left[\tilde{\mathbf{F}}^{a,*}_1 - \tilde{\mathbf{F}}^a_1\right]_{Njk} - \delta_{i0}\left[\tilde{\mathbf{F}}^{a,*}_1 - \tilde{\mathbf{F}}^a_1\right]_{0jk}\right) + 2\sum_{m=0}^{N}\mathcal{D}_{im}\ \{\{J\vec{a}^1\}\}_{(i,m)jk}\cdot\mathbf{F}^\#_1(\mathbf{U}_{ijk},\mathbf{U}_{mjk})$$

- Introduces a two-point numerical volume flux denoted by a $\#$ symbol
- Only conditions on the volume flux are consistency and symmetry

$$\mathbf{F}^\#_1(\mathbf{U},\mathbf{U}) = \mathbf{F}_1 \quad \text{and} \quad \mathbf{F}^\#_1(\mathbf{U}_{ijk},\mathbf{U}_{mjk}) = \mathbf{F}^\#_1(\mathbf{U}_{mjk},\mathbf{U}_{ijk})$$

DGSEM: Sharp fluxes

- Great deal of freedom selecting the form of the sharp fluxes
- Sharp fluxes can be designed to build other physical properties into the discretization
- Numerical volume flux can recover split formulations of the PDEs
- Entropy conservative formulations generate a specific split form
- Don't need to know this form explicitly in the DG framework

DGSEM: Entropy conservative sharp flux

- Design an entropy conservative with Tadmor's finite volume condition
- Discrete entropy conservation condition (DECC)

$$[\![\mathbf{W}]\!]^T\,\mathbf{F}_\ell^{\#,\mathrm{EC}}(\mathbf{U}_L,\mathbf{U}_R) = [\![\Psi_\ell]\!] - \{\!\{B_\ell\}\!\}\,[\![\theta]\!]\,,\quad \ell = 1,2,3$$

  with the entropy flux potential

$$\vec{\Psi} = \mathbf{W}^T\overleftrightarrow{\mathbf{F}}^a - \vec{F}^S + \theta\vec{B}$$

  and $\theta = \mathbf{W}^T\phi^{\mathrm{MHD}} = 2\beta(\vec{V}\cdot\vec{B})$
- Low-order flux extends to high-order in the split form DG framework

DGSEM: Entropy conservative sharp flux I

- Entropy conservative flux in first spatial direction

$$\mathbf{F}_1^{\#,\mathrm{EC}}(\mathbf{U}_L, \mathbf{U}_R) = \begin{pmatrix} \varrho^{\ln} \{\{v_1\}\} \\ \varrho^{\ln} \{\{v_1\}\}^2 - \{\{B_1\}\}^2 + \overline{p} + \frac{1}{2}\Big(\{\{B_1^2\}\} + \{\{B_2^2\}\} + \{\{B_3^2\}\} \Big) \\ \varrho^{\ln} \{\{v_1\}\} \{\{v_2\}\} - \{\{B_1\}\} \{\{B_2\}\} \\ \varrho^{\ln} \{\{v_1\}\} \{\{v_3\}\} - \{\{B_1\}\} \{\{B_3\}\} \\ f_{1,5}^{\mathrm{EC}} \\ c_h \{\{\psi\}\} \\ \{\{v_1\}\} \{\{B_2\}\} - \{\{v_2\}\} \{\{B_1\}\} \\ \{\{v_1\}\} \{\{B_3\}\} - \{\{v_3\}\} \{\{B_1\}\} \\ c_h \{\{B_1\}\} \end{pmatrix}$$

with

$$\overline{p} = \frac{\{\{\varrho\}\}}{2\,\{\{\beta\}\}}$$

DGSEM: Advective numerical surface flux

- $\tilde{\mathbf{F}}^{a,*}$ is the numerical surface flux
- We link the choice of the numerical volume flux and the numerical surface flux, e.g.,

$$\tilde{\mathbf{F}}^{a,*}(\mathbf{U}_L, \mathbf{U}_R) = \tilde{\mathbf{F}}^{\#,\mathrm{EC}}(\mathbf{U}_L, \mathbf{U}_R)$$

$$\tilde{\mathbf{F}}^{a,*}(\mathbf{U}_L, \mathbf{U}_R) = \tilde{\mathbf{F}}^{\#,\mathrm{EC}}(\mathbf{U}_L, \mathbf{U}_R) - \frac{\lambda_{max}}{2}\left[\mathbf{U}_R - \mathbf{U}_L\right]$$

- First choice leads to an entropy conservative (EC) method
- Second choice yields an entropy stable (ES) scheme

DGSEM: Non-conservative MHD terms

- Compute non-conservative MHD volume contributions as a partial split form

$$
\begin{aligned}
\boldsymbol{\Phi}^{\mathrm{MHD}}\vec{\nabla}_{\xi}\cdot\mathbb{I}^{N}\left(\vec{\tilde{B}}\right) \approx \boldsymbol{\Phi}^{\mathrm{MHD}}\vec{\mathbb{D}}^{\mathrm{NC}}_{\mathrm{div}}\cdot\vec{\tilde{B}} = \quad & \sum_{m=0}^{N}\mathcal{D}_{im}\left(\boldsymbol{\Phi}^{\mathrm{MHD}}_{ijk}\left(\vec{B}_{mjk}\cdot\left\{\!\!\left\{J\vec{a}^{\,1}\right\}\!\!\right\}_{(i,m)jk}\right)\right) \\
+ & \sum_{m=0}^{N}\mathcal{D}_{jm}\left(\boldsymbol{\Phi}^{\mathrm{MHD}}_{ijk}\left(\vec{B}_{imk}\cdot\left\{\!\!\left\{J\vec{a}^{\,2}\right\}\!\!\right\}_{i(j,m)k}\right)\right) \\
+ & \sum_{m=0}^{N}\mathcal{D}_{km}\left(\boldsymbol{\Phi}^{\mathrm{MHD}}_{ijk}\left(\vec{B}_{ijm}\cdot\left\{\!\!\left\{J\vec{a}^{\,3}\right\}\!\!\right\}_{ij(k,m)}\right)\right)
\end{aligned}
$$

- Define non-conservative MHD surface coupling with

$$
\left(\boldsymbol{\Phi}^{\mathrm{MHD}}B_n\right)^{\Diamond} = \left(\boldsymbol{\Phi}^{\mathrm{MHD}}\right)^{-}\left(\left\{\!\!\left\{\vec{B}\right\}\!\!\right\}\cdot\vec{n}\right)
$$

where $(\cdot)^{-}$ denotes the interior value of the considered element

DGSEM: Non-conservative GLM terms

- Compute non-conservative GLM volume contributions with a standard gradient form

$$\overset{\leftrightarrow}{\tilde{\boldsymbol{\Phi}}}^{\mathrm{GLM}} \cdot \vec{\nabla}_\xi \mathbb{I}^N(\psi) \approx \overset{\leftrightarrow}{\tilde{\boldsymbol{\Phi}}}^{\mathrm{GLM}} \cdot \vec{\mathbb{D}}^{\mathrm{NC}}_{\mathrm{grad}} \psi = \sum_{m=0}^{N} \mathcal{D}_{im} \left(\left(J\vec{a}^{1}_{ijk} \cdot \overset{\leftrightarrow}{\boldsymbol{\Phi}}^{\mathrm{GLM}}_{ijk} \right) \psi_{mjk} \right) + \sum_{m=0}^{N} \mathcal{D}_{jm} \left(\left(J\vec{a}^{2}_{ijk} \cdot \overset{\leftrightarrow}{\boldsymbol{\Phi}}^{\mathrm{GLM}}_{ijk} \right) \psi_{imk} \right) + \sum_{m=0}^{N} \mathcal{D}_{km} \left(\left(J\vec{a}^{3}_{ijk} \cdot \overset{\leftrightarrow}{\boldsymbol{\Phi}}^{\mathrm{GLM}}_{ijk} \right) \psi_{ijm} \right)$$

- Define non-conservative GLM surface coupling with

$$\left(\boldsymbol{\Phi}^{\mathrm{GLM}}_n \psi \right)^{\Diamond} = \left(\left(\overset{\leftrightarrow}{\boldsymbol{\Phi}}^{\mathrm{GLM}} \right)^{-} \cdot \vec{n} \right) \{\!\{\psi\}\!\}$$

where $(\cdot)^{-}$ denotes the interior value of the considered element

DGSEM: Entropy conservative steps

- **We use these DG discretization principles for the volume and surface contributions to demonstrate entropy stability for the resistive GLM-MHD equations**
- Due to the construction of the DGSEM we discretely mimic the continuous analysis:

  1. Contract the strong DGSEM formulation into entropy space taking $\varphi = \mathbf{W}$, $\overleftrightarrow{\boldsymbol{\vartheta}} = \overleftrightarrow{\tilde{\mathbf{F}}}^{v}$
  2. Advective and non-conservative volume contributions generate the entropy flux at the interfaces (SBP, DECC, DMI)
  3. Sum over all elements cancels extraneous MHD surface terms in entropy space (DECC, definition of $\left(\boldsymbol{\Phi}^{\mathrm{MHD}} B_n\right)^{\Diamond}$ and $\left(\boldsymbol{\Phi}_n^{\mathrm{GLM}} \psi\right)^{\Diamond}$)
  4. Include the viscous BR1 boundary coupling and rewrite the viscous flux volume contributions
  $$\overleftrightarrow{\mathbf{F}}^{v}(\mathbf{U}, \vec{\nabla}\mathbf{U}) = \underline{\mathsf{K}}\vec{\nabla}\mathbf{W}$$
  5. Assume periodic boundary conditions

- Determine that the total discrete entropy decays

$$\frac{d\overline{S}}{dt} \equiv \sum_{\nu=1}^{N_{\mathrm{el}}} \langle J^{\nu} S_t^{\nu}, 1 \rangle_N \leq 0$$

Numerical results: Convergence

- Use the method of manufactured solutions to test convergence
- Consider a solution of the form

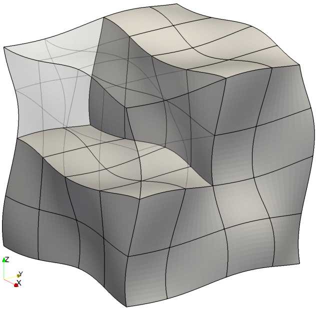
$$\mathbf{u} = \left[h, h, h, 0, 2h^2 + h, h, -h, 0, 0\right]^T$$

  where

$$h = h(x, y, z, t) = 0.5 \sin(2\pi(x + y + z - t)) + 2$$

- Introduces an additional source term into the approximation

Numerical results: Convergence I



N_{el}	$L^2(\varrho)$	$L^2(v_1)$	$L^2(p)$	$L^2(B_1)$
4^3	1.62E-01	1.74E-01	3.42E-01	1.19E-01
8^3	6.11E-03	8.38E-03	1.59E-02	3.51E-03
16^3	2.40E-04	5.02E-04	1.18E-03	1.39E-04
32^3	1.93E-05	2.51E-05	7.42E-05	7.56E-06
avg EOC	4.34	4.25	4.06	4.65

Table : L^2-errors and EOC of manufactured solution test for $N=3$

Numerical results: Entropy conservation

- Entropy is conserved for well-resolved simulations
- Purposely choose a challenging spherical blast wave test case with discontinuities to demonstrate entropy conservation
- Inner and outer states given by

	ϱ	v_1	v_2	v_3	p	B_1	B_2	B_3	ψ
inner	1.2	0.1	0.0	0.1	0.9	1.0	1.0	1.0	0.0
outer	1.0	0.2	-0.4	0.2	0.3	1.0	1.0	1.0	0.0

Table : Inner and outer primitive states for the entropy conservation test.

which are blended over δ_0 with the function

$$\mathbf{u} = \frac{\mathbf{u}_{\text{inner}} + \lambda \mathbf{u}_{\text{outer}}}{1+\lambda}\,, \quad \lambda = \exp\left[\frac{5}{\delta_0}(r - r_0)\right]\,, \quad r = \|\vec{x} - \vec{x}_c\|$$

Numerical results: Entropy conservation I

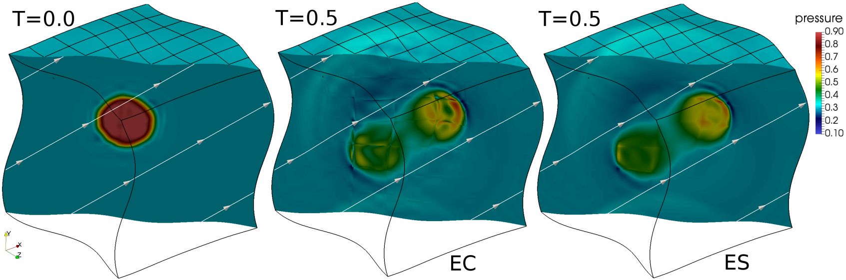


Figure : Evolution of 3D blast wave for entropy conservative and entropy stable approximations

Numerical results: Entropy conservation II

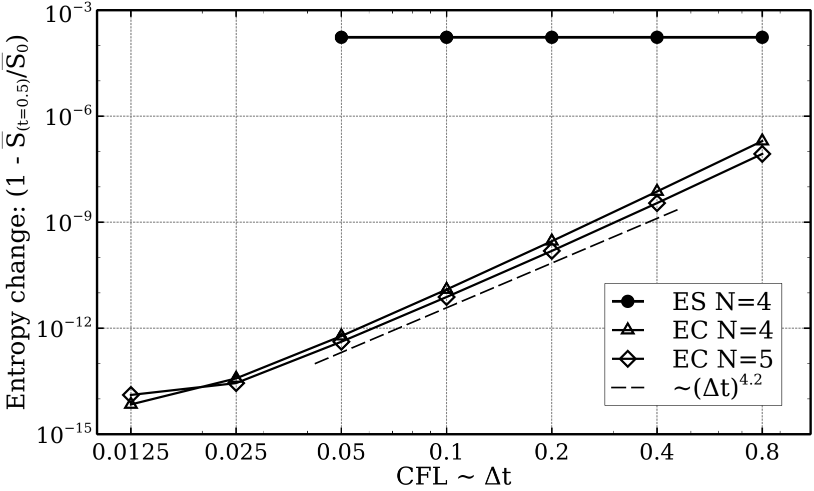


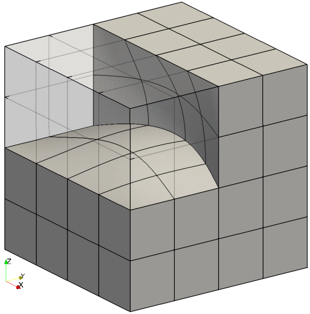
Figure : Log-log plot of entropy change from the initial entropy $\overline{S}_0$ to $\overline{S}(t = 0.5)$ over the timestep for a 3D spherical blast wave

Numerical results: Divergence cleaning test

- Demonstrate effect of GLM divergence cleaning with malicious initial condition
- Explicitly defined to not be divergence-free
- For periodic boundaries explore the use of damping in the GLM anstaz
- Define initial conditions

$$\varrho(x,y,0) = 1, \quad E(x,y,0) = 6, \quad B_1(x,y,0) = \exp\left(-\frac{1}{8}\,\frac{(x-0.5)^2+(y-0.5)^2+(z-0.5)^2}{0.0275^2}\right)$$

on a curved domain $\Omega = [0,1]^3$ given by



Numerical results: Divergence cleaning test I

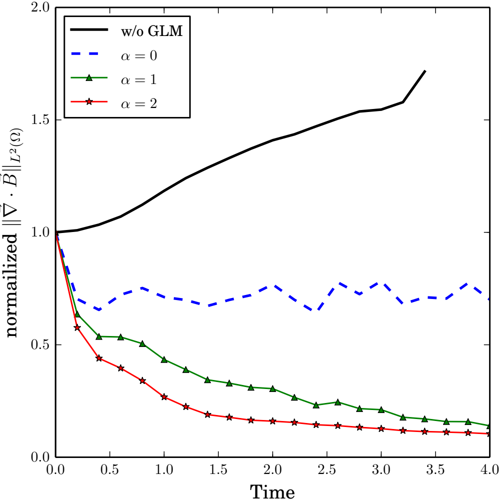


Figure : Evolution of divergence error for maliciously chosen test case with periodic boundary conditions. Without divergence cleaning crashes while GLM divergence with/without divergence cleaning controls the errors

Numerical results: Robustness

- Demonstrate the increased robustness of entropy aware approximations
- Use a generalization of the well-known Orszag-Tang vortex to 3D flows with initial conditions

$$\begin{pmatrix} \varrho \\ v_1 \\ v_2 \\ v_3 \\ p \\ B_1 \\ B_2 \\ B_3 \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{25}{36\pi} \\ -\sin(2\pi z) \\ \sin(2\pi x) \\ \sin(2\pi y) \\ \frac{5}{12\pi} \\ -\frac{1}{4\pi}\sin(2\pi z) \\ \frac{1}{4\pi}\sin(4\pi x) \\ \frac{1}{4\pi}\sin(4\pi y) \\ 0 \end{pmatrix}$$

- Choose viscosity parameters such that $\mathrm{Re}_\mathrm{k} \approx 1000$, $\mathrm{Re}_\mathrm{m} \approx 1667$
- Find that standard DGSEM crashes while the entropy stable DGSEM runs!

Numerical results: Robustness I

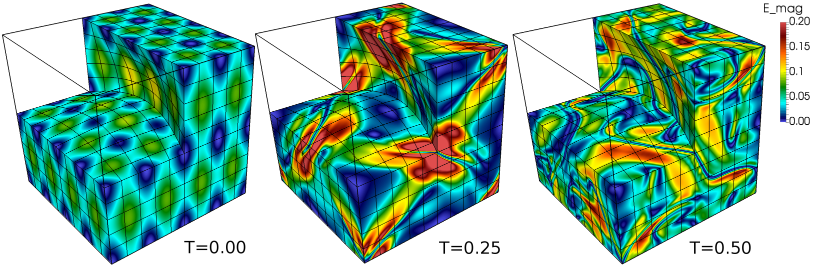


Figure : Visualization of the time evolution of the magnetic energy for a 3D version of the viscous Orszag-Tang vortex with $N = 7$ on a 10^3 internally curved hexahedral mesh

Numerical results: Robustness II

- Demonstrate the increased robustness of entropy aware approximations
- Use an insulating version of the inviscid Taylor-Green vortex
- Domain $\Omega = [0, 2\pi]^3$ with primitive variable initial conditions

$$
\begin{aligned}
\varrho &= 1 \\
\vec{v} &= (\sin(x)\cos(y)\cos(z),\ -\cos(x)\sin(y)\cos(z),\ 0)^T \\
p &= \frac{100}{\gamma} + \frac{1}{16}\left(\cos(2x) + \cos(2y)\right)\left(2 + \cos(2z)\right) \\
&\quad + \frac{1}{16}\left(\cos(4x) + \cos(4y)\right)\left(2 - \cos(4z)\right) \\
\vec{B} &= (\cos(2x)\sin(2y)\sin(2z),\ -\sin(2x)\cos(2y)\sin(2z),\ 0)^T
\end{aligned}
$$

- Even more strenuous test because there is no viscosity!
- Use 64^3 degrees of freedom with polynomial orders $N = 3, 7, 15$
- Standard DGSEM crashes while all configurations of entropy stable DGSEM run!

Conclusions

- Showed that re-writing the viscous fluxes in terms of the gradient of the entropy variables was important for entropy stability
- Building an entropy stable DGSEM involved several important components:
  1. Derivative matrix needed the SBP property
  2. Design of a two-point entropy conserving finite volume flux
  3. Discrete metric identities must be satisfied
  4. Discretization of two non-conservative terms one for PDE symmetrization and another for Galilean invariance
- Entropy stable DG method remains high-order and has demonstrably improved robustness
- Further investigations: shock capturing (artificial viscosity), efficient implementation to mitigate increased computational effort