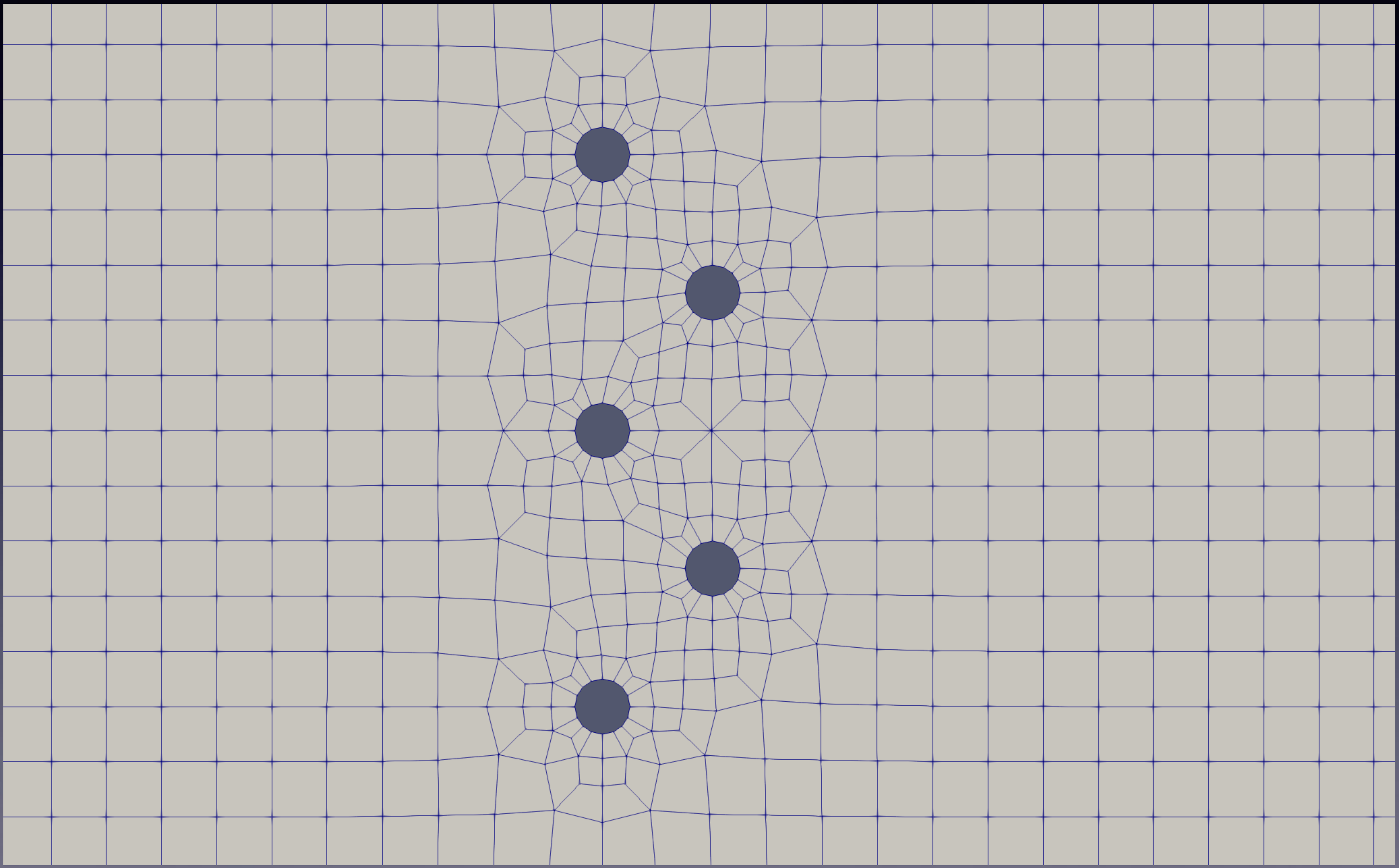
STABLE IMPLEMENTATION OF BOUNDARY CONDITIONS FOR DGSEM APPROXIMATION OF THE COMPRESSIBLE EULER AND NAVIER-STOKES EQUATIONS

David A. Kopriva
Florida State University and San Diego State University

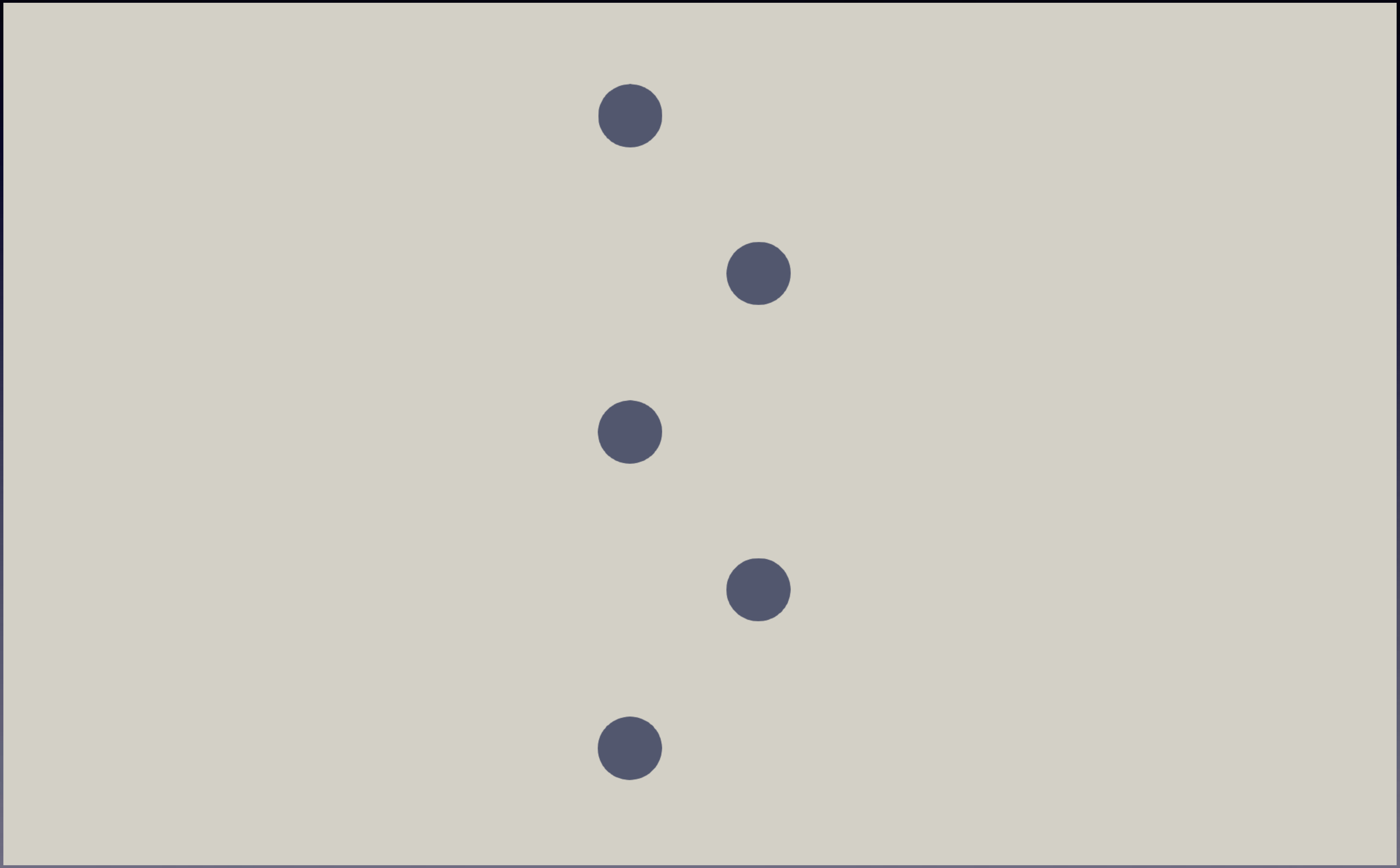
Gregor J. Gassner, U. Cologne and Andrew R. Winters
University of Cologne

Florian Hindenlang
Max Planck Institute

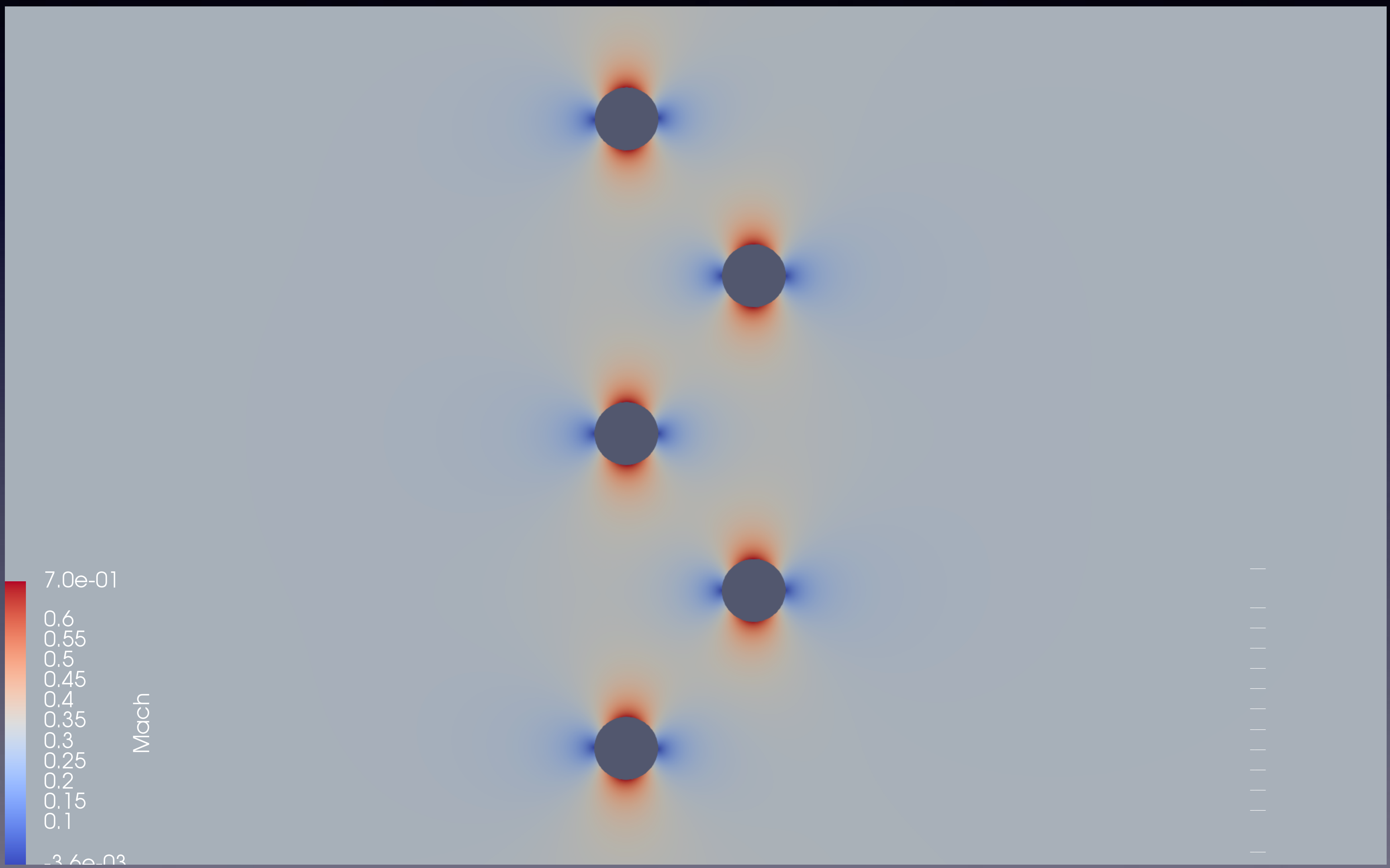
A Flow Geometry



A Compressible Flow



Another Compressible Flow



Boundary Conditions Determine the Flow!

Yet:

- Hardly discussed except in passing
- Often dealt with in ad-hoc manner
- Published proposals not stable

We study…

- Conditions under which DGSEMs are stable
- Examples of stable BC implementations
- General Analysis
  - 3D
  - Curved Elements
  - Linear and nonlinear equations

Compressible Flow Model

Navier-Stokes Equations: Conservative form

$$\mathbf{u}_t + \sum_{i=1}^{3} \frac{\partial \mathbf{f}_i}{\partial x_i} = \frac{1}{\mathrm{Re}} \sum_{i=1}^{3} \frac{\partial \mathbf{f}_{v,i}\left(\mathbf{u}, \nabla_x \mathbf{u}\right)}{\partial x_i}$$

Conservative Variables

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho \vec{v} \\ \rho E \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}$$

Compact Version

$$\mathbf{u}_t + \vec{\nabla}_x \cdot \overleftrightarrow{\mathbf{f}} = \frac{1}{\mathrm{Re}} \vec{\nabla}_x \cdot \overleftrightarrow{\mathbf{f}}_v \left(\mathbf{u}, \vec{\nabla}_x \mathbf{u} \right)$$

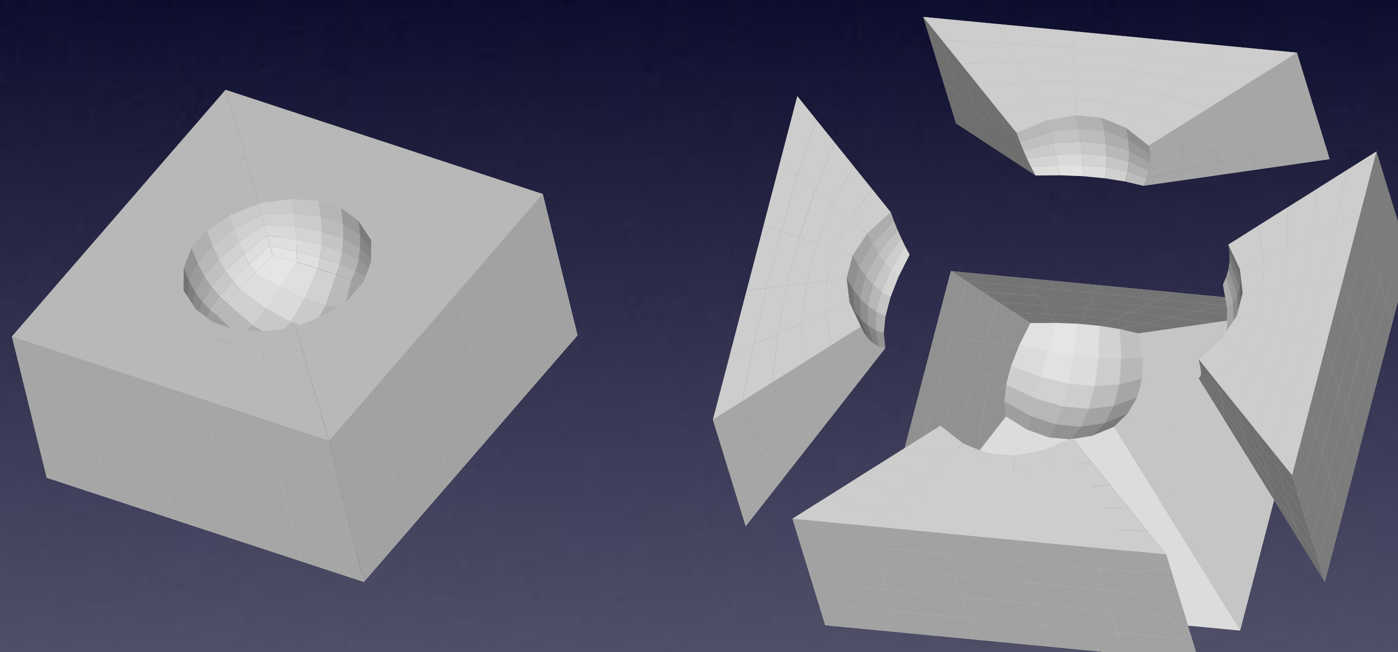
Vector of Vectors: $\overleftrightarrow{\mathbf{f}} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{bmatrix}$

Write as 1st order system with

$$\overleftrightarrow{\mathbf{q}} = \vec{\nabla}_x \mathbf{u}$$

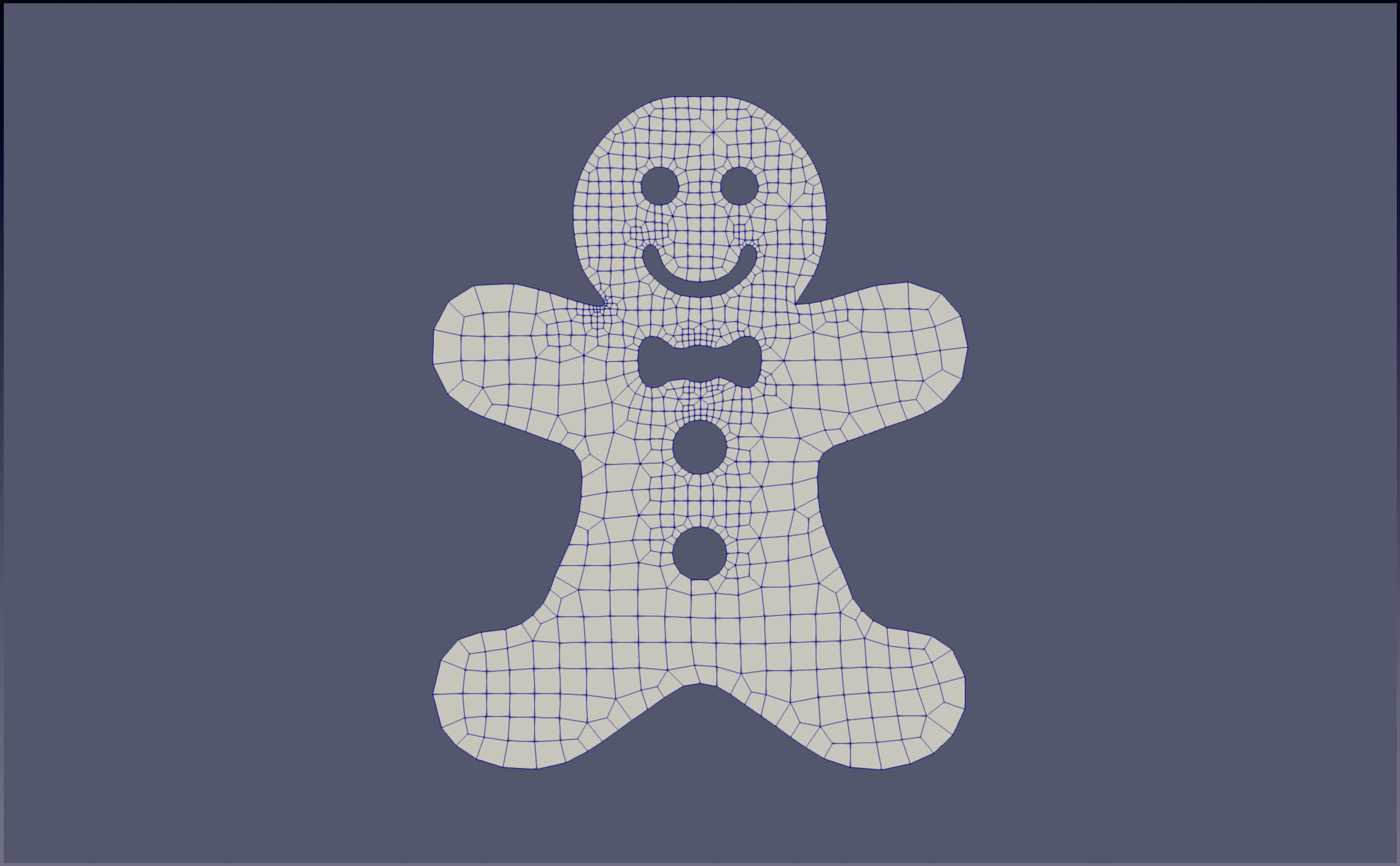
DGSEM Approximation

Subdivide domain into elements

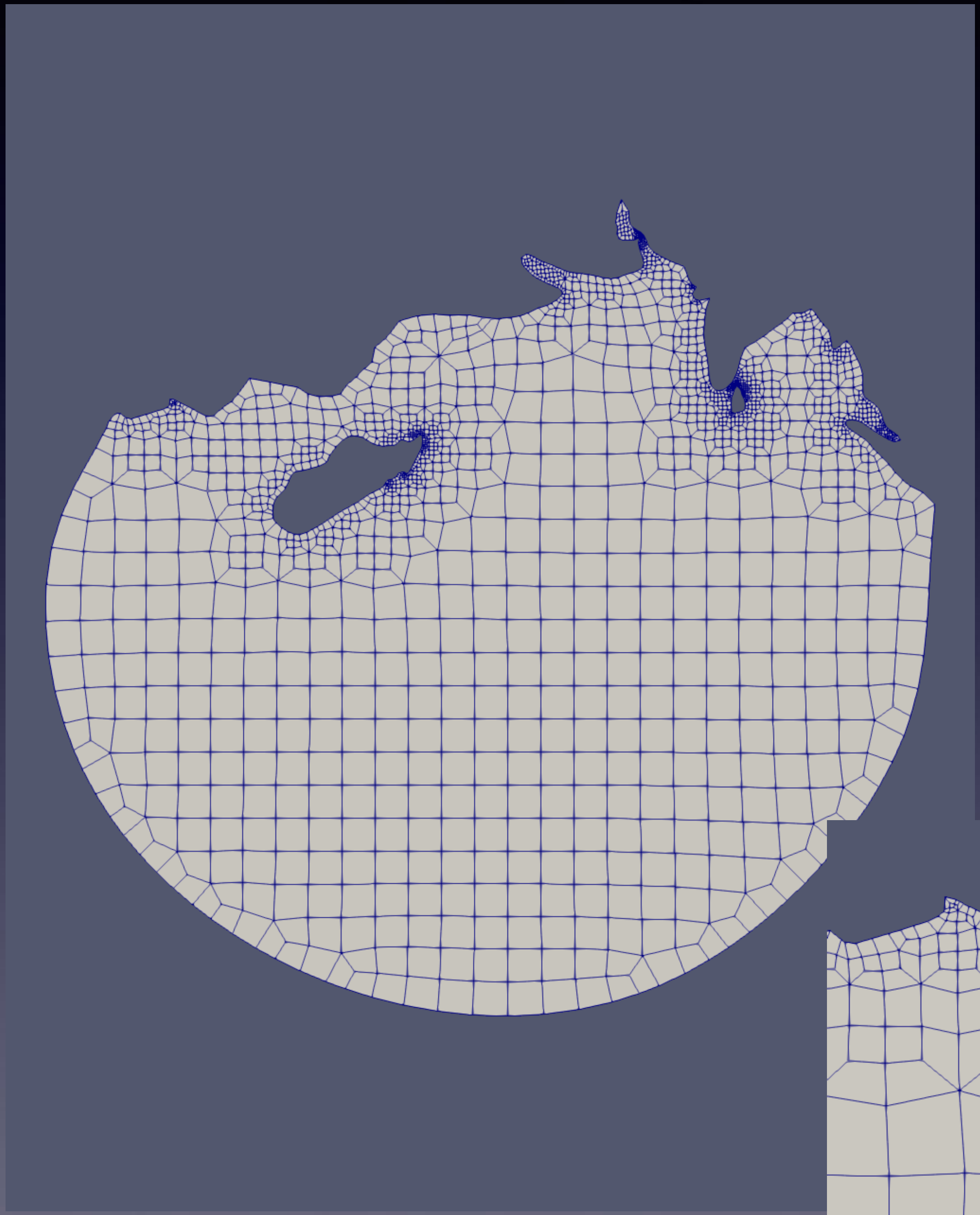


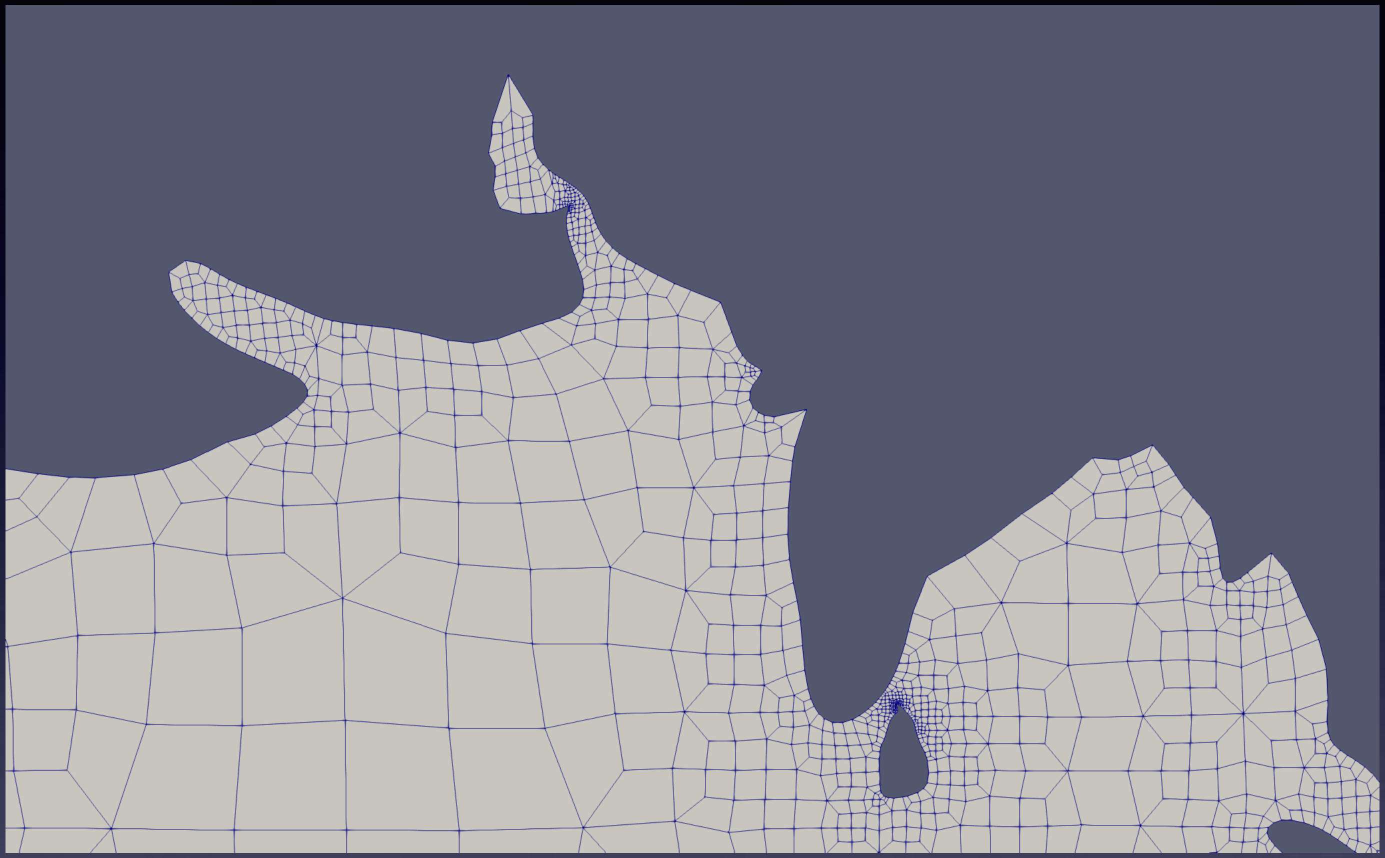
Curved Elements OK!

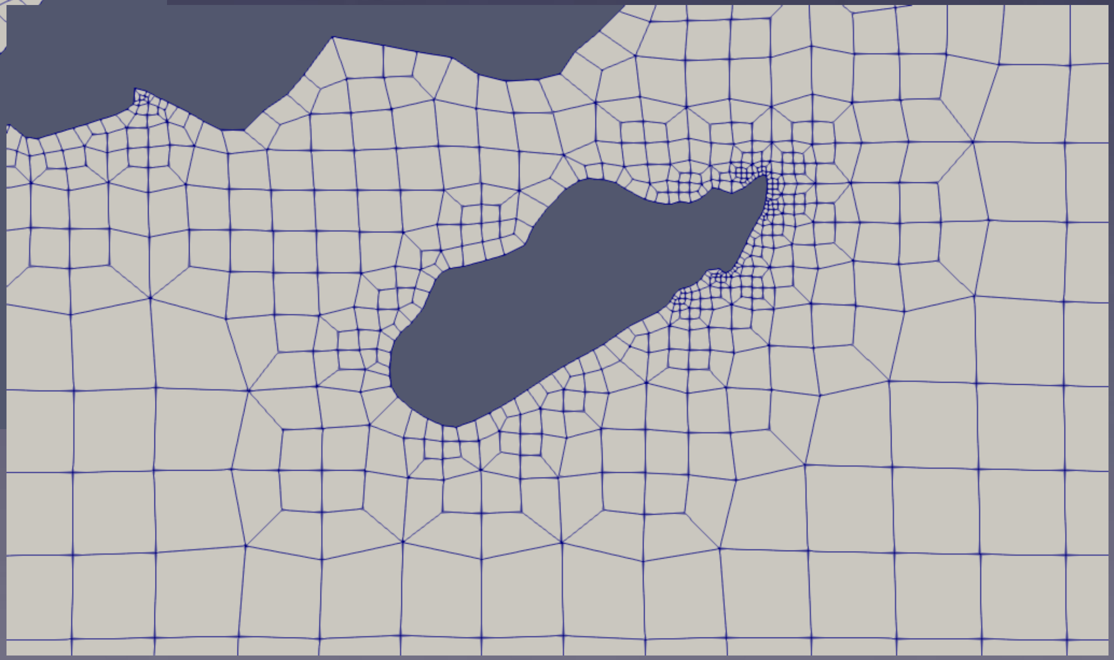
Unstructured Grids OK



Unstructured Grids OK

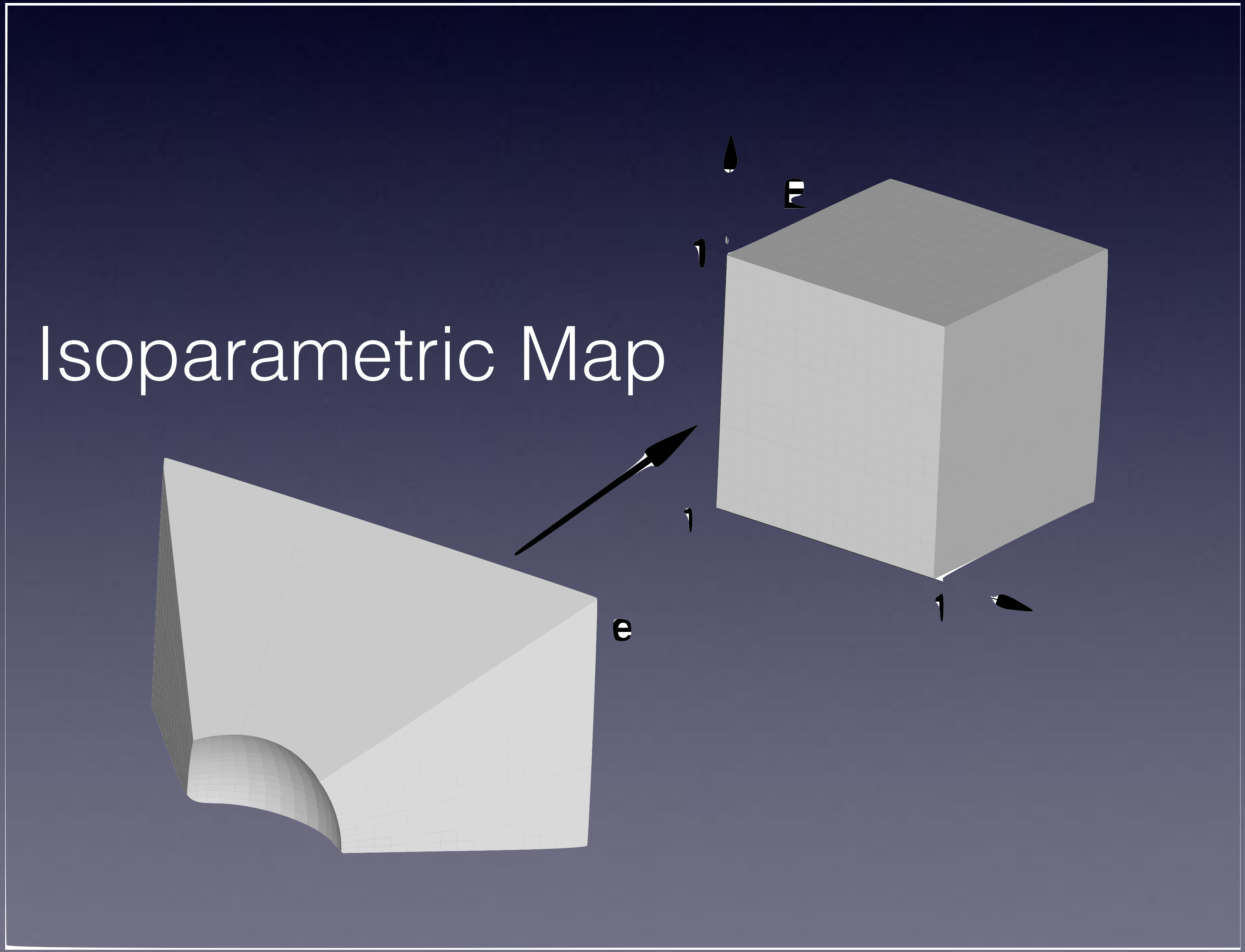






DGSEM Approximation

Map Element to Reference Element



Transformation of Operators

$$\vec{a}_i = \frac{\partial \vec{X}}{\partial \xi^i} \quad i = 1,2,3$$

$$\mathcal{J}\vec{a}^i = \vec{a}_j \times \vec{a}_j, \quad (i,j,k) \text{ cyclic}$$

Gradient

$$\vec{\nabla}_x \mathbf{u} = \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \\ \mathbf{u}_z \end{bmatrix} = \frac{1}{\mathcal{J}} \begin{bmatrix} \mathcal{J}a_1^1 \underline{\mathrm{I}}_5 & \mathcal{J}a_1^2 \underline{\mathrm{I}}_5 & \mathcal{J}a_1^3 \underline{\mathrm{I}}_5 \\ \mathcal{J}a_2^1 \underline{\mathrm{I}}_5 & \mathcal{J}a_2^2 \underline{\mathrm{I}}_5 & \mathcal{J}a_2^3 \underline{\mathrm{I}}_5 \\ \mathcal{J}a_3^1 \underline{\mathrm{I}}_5 & \mathcal{J}a_3^2 \underline{\mathrm{I}}_5 & \mathcal{J}a_3^3 \underline{\mathrm{I}}_5 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\xi \\ \mathbf{u}_\eta \\ \mathbf{u}_\zeta \end{bmatrix} = \frac{1}{\mathcal{J}} \mathcal{M} \vec{\nabla}_\xi \mathbf{u}$$

Divergence

$$\vec{\nabla}_x \cdot \overleftrightarrow{\mathbf{g}} = \frac{1}{\mathcal{J}} \vec{\nabla}_\xi \cdot (\mathcal{M}^T \overleftrightarrow{\mathbf{g}})$$

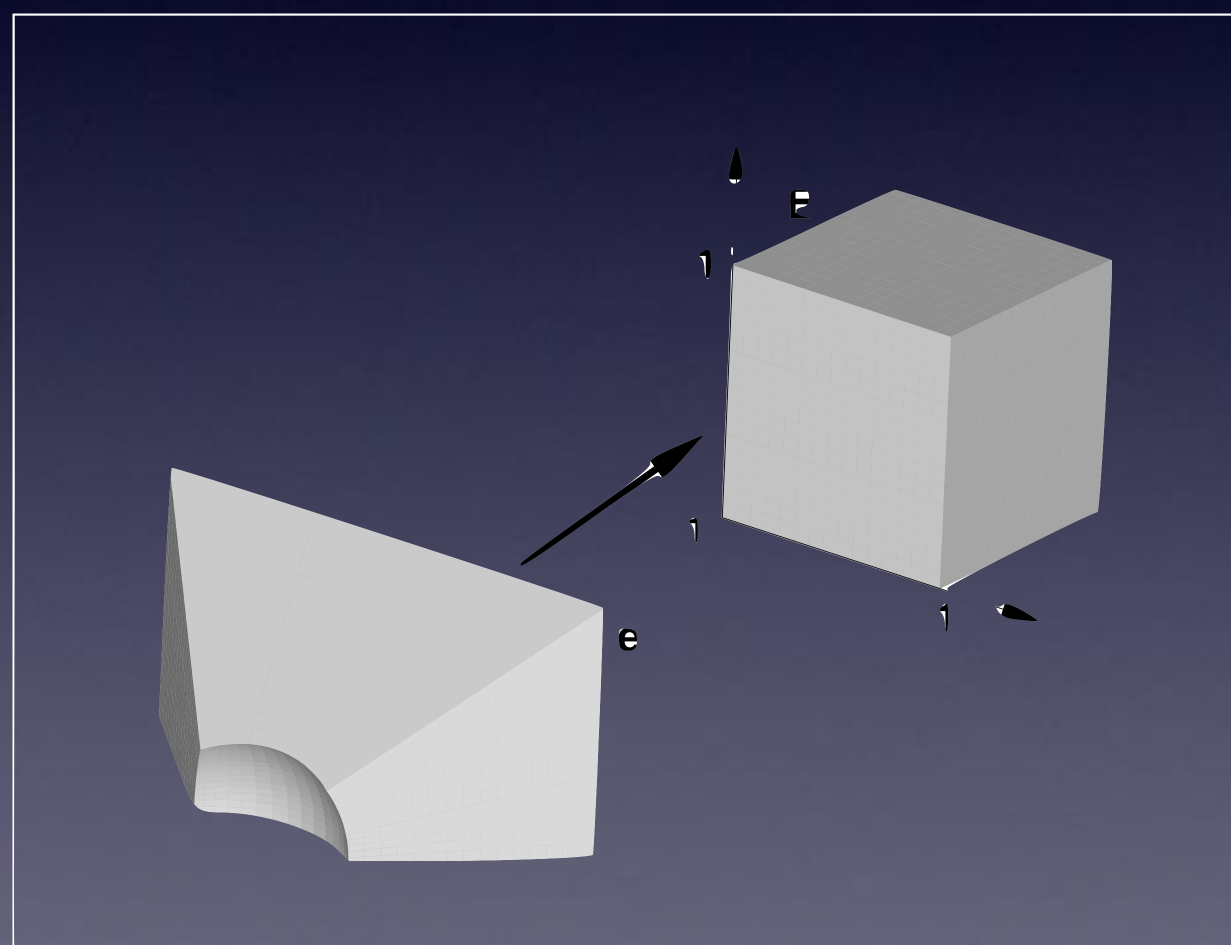
Define Contravariant fluxes:

$$\overleftrightarrow{\tilde{\mathbf{f}}} = \mathcal{M}^T \overleftrightarrow{\mathbf{f}}$$

NS in Reference Coordinates

$$\mathcal{J}\mathbf{u}_t + \vec{\nabla}_\xi \cdot \tilde{\overleftrightarrow{\mathbf{f}}} = \frac{1}{\mathrm{Re}}\vec{\nabla}_\xi \cdot \tilde{\overleftrightarrow{\mathbf{f}}}_v\left(\mathbf{u}, \overleftrightarrow{\mathbf{q}}\right)$$

$$\mathcal{J}\overleftrightarrow{\mathbf{q}} = \mathcal{M}\vec{\nabla}_\xi \mathbf{u}$$



$$\mathbf{u}_t + \vec{\nabla}_x \cdot \overleftrightarrow{\mathbf{f}} = \frac{1}{\mathrm{Re}}\vec{\nabla}_x \cdot \overleftrightarrow{\mathbf{f}}_v\left(\mathbf{u}, \vec{\nabla}_x \mathbf{u}\right)$$

$$\overleftrightarrow{\mathbf{q}} = \vec{\nabla}_x \mathbf{u}$$

Weak Form Construction

$$\mathcal{J}\mathbf{u}_t + \vec{\nabla}_\xi \cdot \overset{\leftrightarrow}{\tilde{\mathbf{f}}} = \frac{1}{\mathrm{Re}} \vec{\nabla}_\xi \cdot \overset{\leftrightarrow}{\tilde{\mathbf{f}}}_v \left(\mathbf{u}, \overset{\leftrightarrow}{\mathbf{q}}\right)$$

$$\mathcal{J}\overset{\leftrightarrow}{\mathbf{q}} = \mathcal{M}\vec{\nabla}_\xi \mathbf{u}$$

(1) Take inner product of equations with test functions

$$\langle u, v \rangle = \int_E uv dE$$

(2) Apply Gauss Law

$$\langle \mathcal{J}\mathbf{u}, \boldsymbol{\phi} \rangle + \int_{\partial E} \boldsymbol{\phi}^T \left\{ \overset{\leftrightarrow}{\tilde{\mathbf{f}}} - \frac{1}{\mathrm{Re}} \overset{\leftrightarrow}{\tilde{\mathbf{f}}}_v \right\} \cdot \hat{n} \; \mathrm{dS} - \left\langle \overset{\leftrightarrow}{\tilde{\mathbf{f}}}, \vec{\nabla}_\xi \boldsymbol{\phi} \right\rangle = -\frac{1}{\mathrm{Re}} \left\langle \overset{\leftrightarrow}{\tilde{\mathbf{f}}}_v, \vec{\nabla} \boldsymbol{\phi} \right\rangle$$

$$\left\langle \mathcal{J}\overset{\leftrightarrow}{\mathbf{q}}, \overset{\leftrightarrow}{\boldsymbol{\psi}} \right\rangle = \int_{\partial E} \mathbf{u}^T \left\{ \mathcal{M}^T \overset{\leftrightarrow}{\boldsymbol{\psi}} \right\} \cdot \hat{n} \; \mathrm{dS} - \left\langle \mathbf{u}, \vec{\nabla} \cdot \left(\mathcal{M}^T \overset{\leftrightarrow}{\boldsymbol{\psi}} \right) \right\rangle$$

Approximate

Functions with polynomials

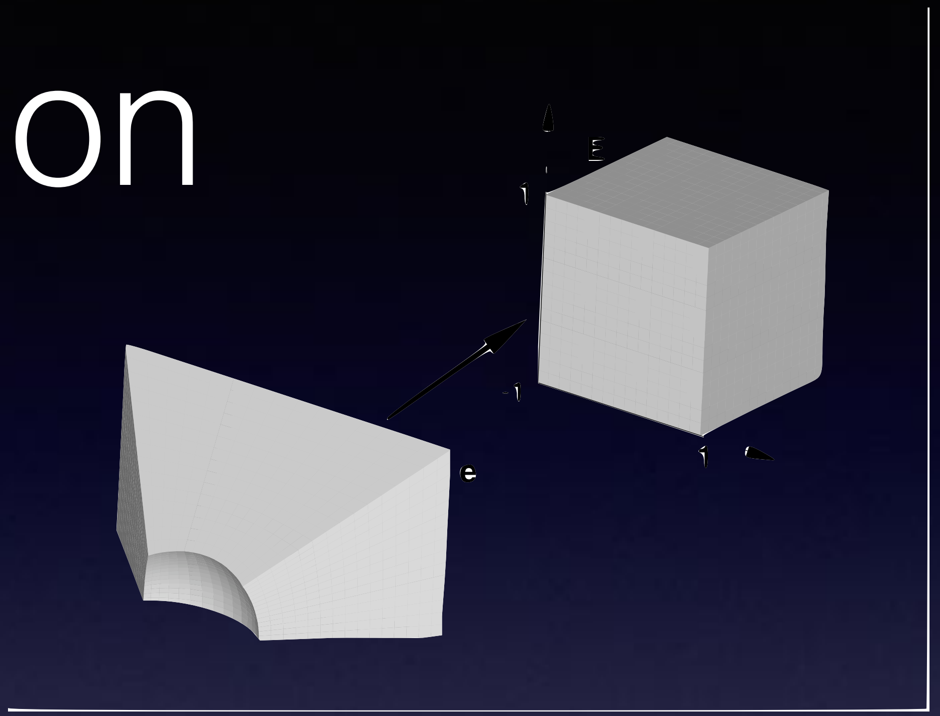
Boundary quantities
with numerical ones

$$\langle \mathcal{J}\mathbf{u}, \boldsymbol{\phi} \rangle + \int_{\partial E} \boldsymbol{\phi}^T \left\{ \tilde{\overleftrightarrow{\mathbf{f}}} - \frac{1}{\mathrm{Re}} \tilde{\overleftrightarrow{\mathbf{f}}}_v \right\} \cdot \hat{n}\ \mathrm{dS} - \left\langle \tilde{\overleftrightarrow{\mathbf{f}}}, \vec{\nabla}_\xi \boldsymbol{\phi} \right\rangle = -\frac{1}{\mathrm{Re}} \left\langle \tilde{\overleftrightarrow{\mathbf{f}}}_v, \vec{\nabla} \boldsymbol{\phi} \right\rangle$$

$$\left\langle \mathcal{J}\overleftrightarrow{\mathbf{q}}, \overleftrightarrow{\boldsymbol{\psi}} \right\rangle = \int_{\partial E} \mathbf{u}^T \left\{ \mathcal{M}^T \overleftrightarrow{\boldsymbol{\psi}} \right\} \cdot \hat{n}\ \ \mathrm{dS} - \left\langle \mathbf{u}, \vec{\nabla} \cdot \left(\mathcal{M}^T \overleftrightarrow{\boldsymbol{\psi}} \right) \right\rangle$$

Integrals
with quadrature

Continuous Function Approximation



Approximation by Polynomial Interpolant

$$U\left(\vec{\xi}\right) = \mathbb{I}^N(u) = \sum_{i,j,k=0}^{N} u\left(\xi_i, \eta_j, \zeta_k\right) \ell_i\left(\xi\right) \ell_j\left(\eta\right) \ell_k\left(\zeta\right)$$

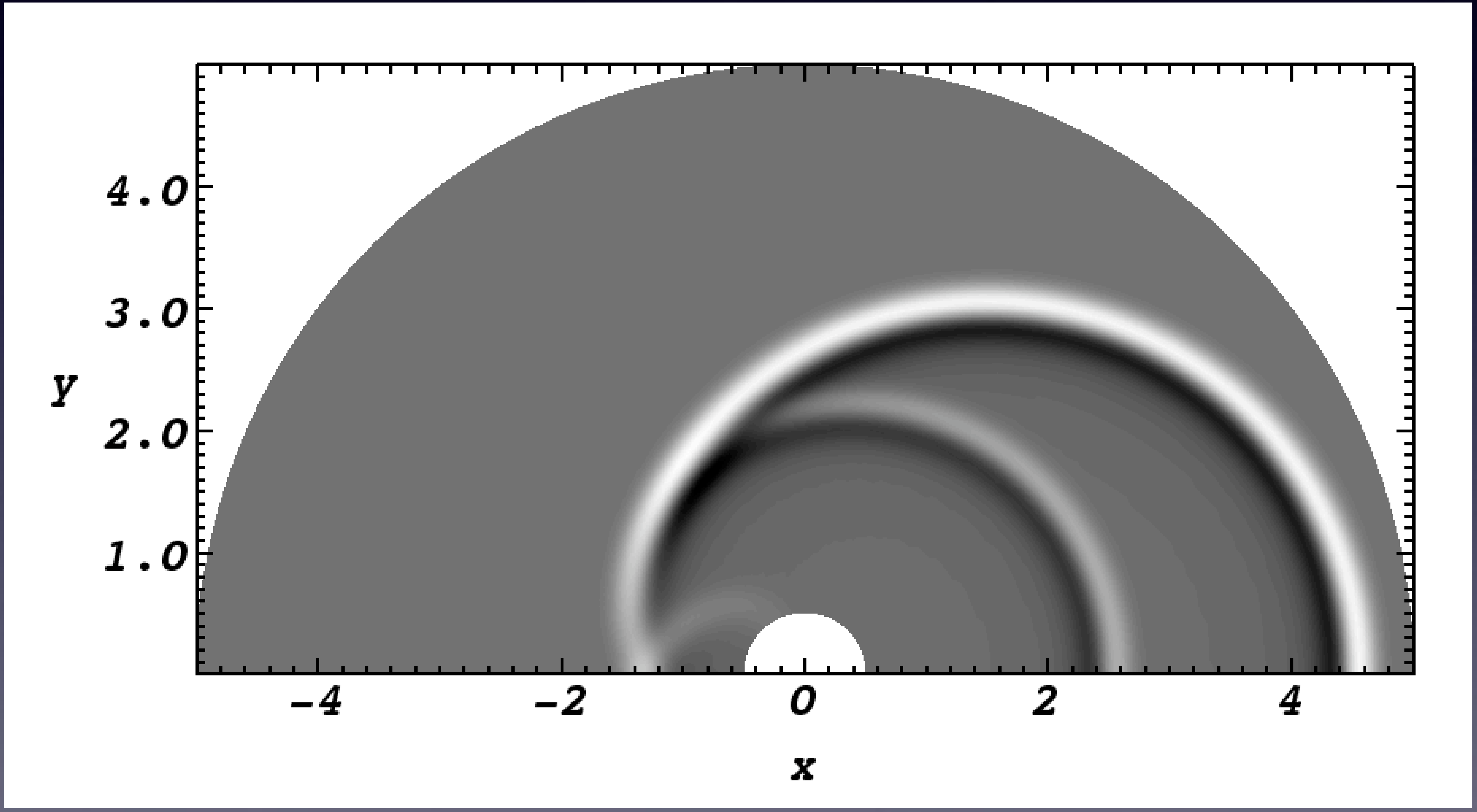
$$\ell_j(x) = \prod_{i=0; i \neq j}^{N} \frac{x - x_i}{x - x_j} = \text{Lagrange Interpolating Polynomial of degree } N$$

$$x_j = \text{Gauss Lobatto points}$$

$$\ell_j\left(x_i\right) = \delta_{ij}$$

Arbitrary High Order

Acoustic Scattering from a Cylinder



74th Order

Differentiation

Differentiate interpolant, evaluate at quadrature points

$$\left.\frac{\partial U}{\partial \xi}\right|_{nml} = \sum_{i,j,k=0}^{N} u_{ijk}\ell'_i\left(\xi_n\right)\ell_j\left(\eta_m\right)\ell_k\left(\zeta_l\right)$$

$$= \sum_{i=0}^{N} u_{ijk}\ell'_i\left(\xi_n\right) = \sum_{i=0}^{N} u_{ijk}\mathcal{D}_{ni}$$

Differentiation

Gradient

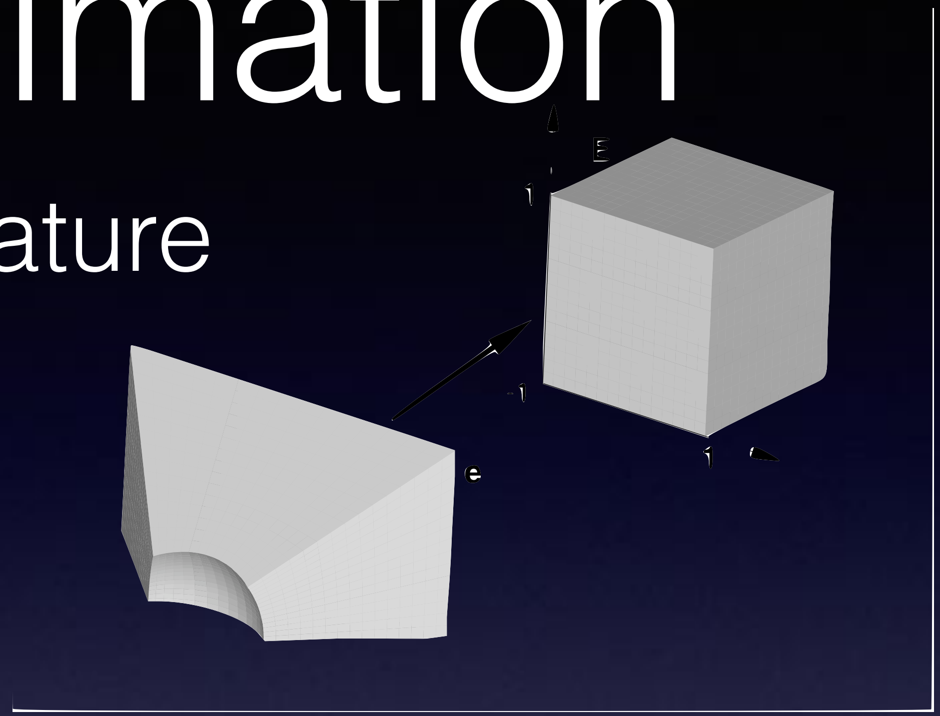
$$\nabla U_{ijk} = \sum_{n=0}^{N} U_{njk}\mathcal{D}_{in}\hat{\xi} + \sum_{n=0}^{N} U_{ink}\mathcal{D}_{jn}\hat{\eta} + \sum_{n=0}^{N} U_{ijn}\mathcal{D}_{kn}\hat{\zeta}$$

Divergence

$$\nabla \cdot \vec{F}_{ijk} = \sum_{n=0}^{N} F^{(\xi)}_{njk}\mathcal{D}_{in} + \sum_{n=0}^{N} F^{(\eta)}_{ink}\mathcal{D}_{jn} + \sum_{n=0}^{N} F^{(\zeta)}_{ijn}\mathcal{D}_{kn}$$

Integral Approximation

Gauss-Lobatto Quadrature



Integration over Volume

$$\int_{E,N} g\,d\xi d\eta d\zeta \equiv \sum_{i,j,k=0}^{N} g_{ijk} w_{ijk}, \qquad (w_{ijk} = w_i w_j w_k)$$

Defines discrete inner product/Norm

$$\langle U, V \rangle_N \equiv \int_{E,N} UV\,d\xi d\eta d\zeta = \sum_{i,j,k=0}^{N} U_{ijk} V_{ijk} w_{ijk}$$

Summation-By-Parts

Exactness of Gauss Quadrature implies

Integration By Parts

$$(u, v_\xi) = \int_{\partial E} UV\big|_{\xi=-1}^{1} \, d\eta d\zeta - (u_\xi, v)$$

$$\Downarrow$$

Summation by Parts

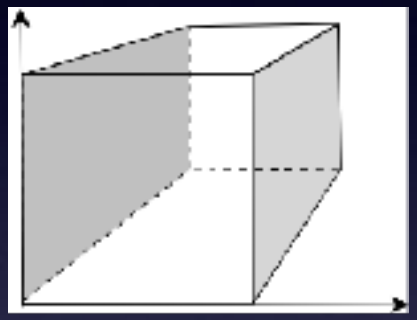
$$(U, V_\xi)_N = \int_{\partial E, N} UV\big|_{\xi=-1}^{1} \, d\eta d\zeta - (U_\xi, V)_N$$

Summation by Parts

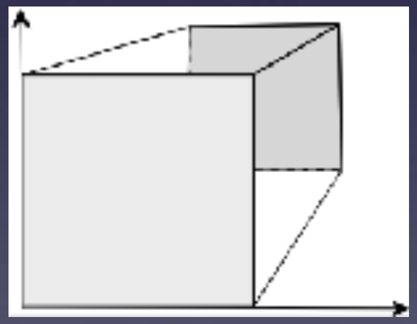
works in each direction

F_1

$$\langle U_\xi, V\rangle_N = \int_N UV\Big|_{\xi=-1}^{1} d\eta d\zeta - \langle U, V_\xi\rangle_N$$

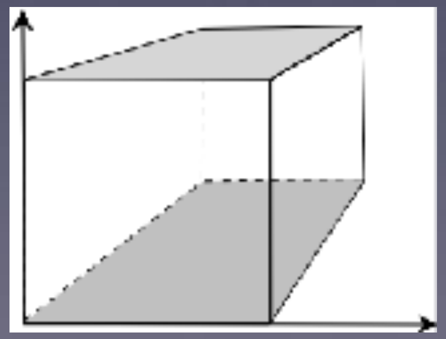


$$\langle U_\eta, V\rangle_N = \int_N UV\Big|_{\eta=-1}^{1} d\xi d\zeta - \langle U, V_\eta\rangle_N$$



F_2

$$\langle U_\zeta, V\rangle_N = \int_N UV\Big|_{\zeta=-1}^{1} d\xi d\zeta - \langle U, V_\zeta\rangle_N$$



F_3

Discrete Gauss Law

$$\left(\nabla \cdot \vec{F}, \phi\right)_N = \int_{\partial E, N} \vec{F} \cdot \hat{n} dS - \left(\vec{F}, \nabla \phi\right)_N$$

Discrete Integral Calculus

$$\left(\nabla\cdot\vec{F},\phi\right)_N=\int_{\partial E,N}\vec{F}\cdot\hat{n}dS-\left(\vec{F},\nabla\phi\right)_N$$

$$\int_{E,N}\nabla\cdot\vec{F}d\vec{\xi}=\int_{\partial E,N}\vec{F}\cdot\hat{n}dS$$

$$\left(\nabla^2\Phi,V\right)_N+\left(\nabla\Phi,\nabla V\right)_N=\int_{\partial E,N}\nabla\Phi\cdot\hat{n}V\,dS$$

$$\left(\nabla^2\Phi,V\right)_N-\left(\nabla^2V,\Phi\right)_N=\int_{\partial E,N}\left(\nabla\Phi\cdot\hat{n}V-\nabla V\cdot\hat{n}\Phi\right)dS$$

From Exactness

$$\int_{E,N}\nabla V\,d\xi d\eta d\zeta=\int_{\partial E,N}V\hat{n}dS$$

$$\int_{E,N}\nabla\times\vec{F}d\xi d\eta d\zeta=\int_{\partial E,N}\hat{n}\times\vec{F}dS$$

Coupling-Advective

Riemann Solver

$$\mathbf{F}^*\left(\mathbf{U}^L, \mathbf{U}^R\right) = \frac{\overrightarrow{\mathbf{f}}\left(\mathbf{U}^L\right)\cdot\hat{n} + \overrightarrow{\mathbf{f}}\left(\mathbf{U}^L\right)\cdot\hat{n}}{2} + \frac{\sigma}{2}\left|\bar{A}\right|\left(\mathbf{U}^L - \mathbf{U}^R\right)$$

$$= \left\{\left\{\overrightarrow{\mathbf{f}}\cdot\hat{n}\right\}\right\} - \frac{\sigma}{2}\left|\bar{A}\right|[[\mathbf{U}]]$$

Roe
Lax-Friedrichs
van Leer
...

Coupling - Diffusive

Bassi-Rebay-1:

$$\mathbf{U}^{*}\left(\mathbf{U}^{L},\mathbf{U}^{R}\right)=\frac{\mathbf{U}^{L}+\mathbf{U}^{R}}{2}=\{\!\{\mathbf{U}\}\!\}$$

$$\mathbf{F}_v^{*}\left(\overleftrightarrow{\tilde{\mathbf{F}}}_v^{L}\cdot\hat{n},\overleftrightarrow{\tilde{\mathbf{F}}}_v^{R}\cdot\hat{n}\right)=\{\!\{\overleftrightarrow{\tilde{\mathbf{F}}}_v\cdot\hat{n}\}\!\}$$

Others…
Bassi-Rebay-2
Interior Penalty
…

Weak Form Construction…

Apply Gauss Law again…

$\approx \mathbf{U} \in \mathbb{P}^N$ $\quad \in \mathbb{P}^N$ $\quad \approx \langle \cdot, \cdot \rangle_N$

$$\langle \mathcal{J}\mathbf{u}_t, \boldsymbol{\phi} \rangle + \int_{\partial E} \boldsymbol{\phi}^T \left\{ \overset{\leftrightarrow}{\tilde{\mathbf{F}}}{}^{*} - \overset{\leftrightarrow}{\tilde{\mathbf{f}}} \cdot \hat{n} - \frac{1}{\mathrm{Re}} \overset{\leftrightarrow}{\tilde{\mathbf{F}}}{}^{*}_{v} \right\} \mathrm{dS} - \left\langle \vec{\nabla}_\xi \cdot \overset{\leftrightarrow}{\tilde{\mathbf{f}}}, \boldsymbol{\phi} \right\rangle = -\frac{1}{\mathrm{Re}} \left\langle \overset{\leftrightarrow}{\tilde{\mathbf{f}}}_v, \vec{\nabla} \boldsymbol{\phi} \right\rangle$$

$\in \mathbb{P}^N$ $\quad \approx \int_{\partial E, N}$ $\quad \approx \mathbb{D}\left(\mathbf{F}^{\#}\right)$???

$$\left\langle \mathcal{J}\overset{\leftrightarrow}{\mathbf{q}}, \overset{\leftrightarrow}{\boldsymbol{\psi}} \right\rangle = \int_{\partial E} \{\mathbf{U}^{*}{}^{,T} - \mathbf{u}\} \left\{ \mathcal{M}^T \overset{\leftrightarrow}{\boldsymbol{\psi}} \right\} \cdot \hat{n} \; \mathrm{dS} - \left\langle \mathbf{u}, \vec{\nabla} \cdot \left(\mathcal{M}^T \overset{\leftrightarrow}{\boldsymbol{\psi}} \right) \right\rangle$$

$\approx Q \in \mathbb{P}^N$

Split Form/Two Point Flux

$$\vec{f} = \vec{A}(x)\mathbf{u}$$

Split Form: $\nabla \cdot \vec{f} = \frac{1}{2}\left\{\nabla \cdot \vec{f} + \vec{A} \cdot \nabla u + \nabla \cdot \vec{A} u\right\}$

$$\frac{1}{2}\left(\nabla \cdot \overrightarrow{\mathbf{F}}\left(\mathbf{U}\right), \phi\right)_N + \frac{1}{2}\left(\mathbb{I}^N\left(\vec{A}\right) \cdot \nabla\mathbf{U}, \phi\right)_N + \frac{1}{2}\left(\nabla \cdot \mathbb{I}^N\left(\vec{A}\right)\mathbf{U}, \phi\right)_N$$

$$\phi = \ell_i \ell_j \ell_k$$

Volume Terms

$$\left(\nabla\cdot\overleftrightarrow{\mathbf{F}}\left(\mathbf{U}\right),\phi\right)_N \longrightarrow w_{ijk}\left\{\boxed{\sum_{n=0}^{N} F_{njk}^{(\xi)}\mathcal{D}_{in}} + \sum_{n=0}^{N} F_{ink}^{(\eta)}\mathcal{D}_{jn} + \sum_{n=0}^{N} F_{ijn}^{(\zeta)}\mathcal{D}_{kn}\right\}$$

$$\left(\mathbb{I}^N\left(\vec{\mathcal{A}}\right)\cdot\nabla\mathbf{U},\phi\right)_N \longrightarrow$$

$$w_{ijk}\left\{\boxed{\mathcal{A}_{ijk}^{(\xi)}\sum_{n=0}^{N}\mathbf{U}_{njk}\mathcal{D}_{in}} + \mathcal{A}_{ijk}^{(\eta)}\sum_{n=0}^{N}\mathbf{U}_{ink}\mathcal{D}_{jn} + \mathcal{A}_{ijk}^{(\zeta)}\sum_{n=0}^{N}\mathbf{U}_{ijn}\mathcal{D}_{kn}\right\}$$

$$\left(\nabla\cdot\mathbb{I}^N\left(\vec{\mathcal{A}}\right)\mathbf{U},\phi\right)_N \longrightarrow$$

$$w_{ijk}\left\{\boxed{\sum_{n=0}^{N}\mathcal{A}_{njk}^{(\xi)}\mathcal{D}_{in}} + \sum_{n=0}^{N}\mathcal{A}_{ink}^{(\eta)}\mathcal{D}_{jn} + \mathcal{A}_{ijk}^{(\zeta)}\sum_{n=0}^{N}\mathcal{A}_{ijn}^{(\zeta)}\mathcal{D}_{kn}\right\}\mathbf{U}_{ijk}$$

Volume Terms

$$\frac{1}{2}\left(\nabla\cdot\overrightarrow{\mathbf{F}}\left(\mathbf{U}\right),\phi\right)_N+\frac{1}{2}\left(\mathbb{I}^N\left(\vec{\mathcal{A}}\right)\cdot\nabla\mathbf{U},\phi\right)_N+\frac{1}{2}\left(\nabla\cdot\mathbb{I}^N\left(\vec{\mathcal{A}}\right)\mathbf{U},\phi\right)_N$$

$$\frac{1}{2}\sum_{n=0}^{N}\left\{\mathbf{F}^{(\xi)}_{njk}+\mathcal{A}^{(\xi)}_{ijk}\mathbf{U}_{njk}+\mathcal{A}^{(\xi)}_{njk}\mathbf{U}_{ijk}\right\}\mathcal{D}_{in}\,w_{ijk}$$

$$+\frac{1}{2}\sum_{n=0}^{N}\left\{\mathbf{F}^{(\eta)}_{ink}+\mathcal{A}^{(\eta)}_{ijk}\mathbf{U}_{ink}+\mathcal{A}^{(\eta)}_{ink}\mathbf{U}_{ijk}\right\}\mathcal{D}_{jn}\;w_{ijk}$$

$$+\frac{1}{2}\sum_{n=0}^{N}\left\{\mathbf{F}^{(\zeta)}_{njk}+\mathcal{A}^{(\zeta)}_{ijk}\mathbf{U}_{njk}+\mathcal{A}^{(\zeta)}_{njk}\mathbf{U}_{ijk}\right\}\mathcal{D}_{kn}\,w_{ijk}$$

$$=\sum_{n=0}^{N}\bar{\mathbf{F}}^{(\xi)}_{(njk,i)}\mathcal{D}_{in}w_{ijk}+\sum_{n=0}^{N}\bar{\mathbf{F}}^{(\eta)}_{(ink,j)}\mathcal{D}_{jn}w_{ijk}+\sum_{n=0}^{N}\bar{\mathbf{F}}^{(\zeta)}_{(ijn,k)}\mathcal{D}_{kn}w_{ijk}$$

Special Averages

$$\frac{1}{2}\sum_{n=0}^{N}\left\{\mathbf{F}^{(\xi)}_{njk}+\mathcal{A}^{(\xi)}_{ijk}\mathbf{U}_{njk}+\mathcal{A}^{(\xi)}_{njk}\mathbf{U}_{ijk}\right\}\mathcal{D}_{in}$$

$$+$$

$$\frac{1}{2}\qquad\mathcal{A}^{(\xi)}_{ijk}\mathbf{U}_{ijk}\boxed{\sum_{n=0}^{N}\mathcal{D}_{in}}=\sum_{n=0}^{N}\mathcal{A}^{(\xi)}_{ijk}\mathbf{U}_{ijk}\mathcal{D}_{in}=0$$

$$=0$$

$$=\frac{1}{2}\sum_{n=0}^{N}\left\{\mathcal{A}^{(\xi)}_{njk}\mathbf{U}_{njk}+\mathcal{A}^{(\xi)}_{ijk}\mathbf{U}_{njk}+\mathcal{A}^{(\xi)}_{njk}\mathbf{U}_{ijk}+\mathcal{A}^{(\xi)}_{ijk}\mathbf{U}_{ijk}\right\}\mathcal{D}_{in}$$

Special Averages

$$= \frac{1}{2}\sum_{n=0}^{N}\left\{\mathcal{A}^{(\xi)}_{njk}\mathbf{U}_{njk} + \mathcal{A}^{(\xi)}_{ijk}\mathbf{U}_{njk} + \mathcal{A}^{(\xi)}_{njk}\mathbf{U}_{ijk} + \mathcal{A}^{(\xi)}_{ijk}\mathbf{U}_{ijk}\right\}\mathcal{D}_{in}$$

$$= 2\sum_{n=0}^{N}\left\{\left(\frac{\mathcal{A}^{(\xi)}_{njk} + \mathcal{A}^{(\xi)}_{ijk}}{2}\right)\left(\frac{\mathbf{U}_{njk} + \mathbf{U}_{ijk}}{2}\right)\right\}\mathcal{D}_{in}$$

$$= 2\sum_{n=0}^{N}\left\{\left\{\mathcal{A}^{(\xi)}\right\}\right\}_{(n,i)jk}\{\{\mathbf{U}\}\}_{(n,i)jk}\mathcal{D}_{in}$$

$$= 2\sum_{n=0}^{N}\mathbf{F}^{\#}_{(n,i)jk}D_{in}$$

Key Ingredient

Summation-By-Parts and form of $\mathbf{F}^{\#}$ implies

$$\left\langle \mathbb{D}\mathbf{F}^{\#},\mathbf{U}\right\rangle = \frac{1}{2}\int_{\partial E,N} \mathbf{U}^{T}\overleftrightarrow{\mathbf{F}}\cdot\hat{n}\,dS$$

Volume term replaced by surface quadrature

Control with numerical flux

DGSEM

$$\left\langle \mathcal{J}\mathbf{U}_t, \boldsymbol{\phi} \right\rangle_N + \int_{\partial E,N} \boldsymbol{\phi}^T \left\{ \overset{\leftrightarrow}{\tilde{\mathbf{F}}}{}^{*} - \overset{\leftrightarrow}{\tilde{\mathbf{f}}} \cdot \hat{n} - \frac{1}{\mathrm{Re}} \overset{\leftrightarrow}{\tilde{\mathbf{F}}}{}_v^{*} \right\} \mathrm{dS} - \left\langle \mathbb{D}\left(\overset{\leftrightarrow}{\mathbf{F}}{}^{\#} \right), \boldsymbol{\phi} \right\rangle_N = -\frac{1}{\mathrm{Re}} \left\langle \overset{\leftrightarrow}{\tilde{\mathbf{F}}}{}_v, \vec{\nabla}\boldsymbol{\phi} \right\rangle_N$$

$$\left\langle \mathcal{J}\overset{\leftrightarrow}{\mathbf{Q}}, \overset{\leftrightarrow}{\boldsymbol{\psi}} \right\rangle_N = \int_{\partial E,N} \left\{ \mathbf{U}^{**,T} - \mathbf{U} \right\} \left\{ \mathcal{M}^T \overset{\leftrightarrow}{\boldsymbol{\psi}} \right\} \cdot \hat{n} \ \mathrm{dS} - \left\langle \mathbf{U}, \vec{\nabla} \cdot \left(\mathcal{M}^T \overset{\leftrightarrow}{\boldsymbol{\psi}} \right) \right\rangle_N$$

Stability Analysis

$$\langle \mathcal{J}\mathbf{U}_t, \boldsymbol{\phi} \rangle_N + \int_{\partial E,N} \boldsymbol{\phi}^T \left\{ \tilde{\overleftrightarrow{\mathbf{F}}}^* - \tilde{\overleftrightarrow{\mathbf{f}}} \cdot \hat{n} - \frac{1}{\mathrm{Re}} \tilde{\overleftrightarrow{\mathbf{F}}}_v^* \right\} \mathrm{dS} - \left\langle \mathbb{D}\left(\overleftrightarrow{\mathbf{F}}^{\#} \right), \boldsymbol{\phi} \right\rangle_N = -\frac{1}{\mathrm{Re}} \left\langle \tilde{\overleftrightarrow{\mathbf{F}}}_v, \vec{\nabla}\boldsymbol{\phi} \right\rangle_N$$

$$\left\langle \mathcal{J}\overleftrightarrow{\mathbf{Q}}, \overleftrightarrow{\psi} \right\rangle_N = \int_{\partial E,N} \left\{ \mathbf{U}^{**,T} - \mathbf{U} \right\} \left\{ \mathcal{M}^T \overleftrightarrow{\psi} \right\} \cdot \hat{n} \; \mathrm{dS} - \left\langle \mathbf{U}, \vec{\nabla} \cdot \left(\mathcal{M}^T \overleftrightarrow{\psi} \right) \right\rangle_N$$

Linear

- Linearize equations
- Replace $\phi = \left(S^{-1}\right)^T S^{-1}\mathbf{U}$
  $\psi \leftarrow \mathcal{B}\mathbf{Q}$
- Sum over all elements

Entropy

(entropy variable)

- Replace $\phi \leftarrow \mathbf{W}$
  $\psi \leftarrow \mathcal{B}\nabla\mathbf{W}$
- Sum over all elements

Linear Energy Bound

$$\frac{d}{dt}\sum_{elements} ||\mathbf{U}||^2_{J,N} \le -2 \sum_{\substack{\text{Boundary}\\ \text{faces}}} \int_{\partial E,N} \left\{ \left[\tilde{\mathbf{F}}^* - \frac{1}{2}\left(\overleftrightarrow{\tilde{\mathbf{F}}}\cdot\hat{n}\right)\right]^T \mathbf{U} - \frac{1}{\mathrm{Re}}\left[\tilde{\mathbf{F}}_v^{*,T}\mathbf{U} + \mathbf{U}^{*,T}\left(\overleftrightarrow{\tilde{\mathbf{F}}}_v\cdot\hat{n}\right) - \mathbf{U}^T\left(\overleftrightarrow{\tilde{\mathbf{F}}}_v\cdot\hat{n}\right)\right]\right\} \mathrm{dS}$$

Sufficient Condition for Stability:

$$\boxed{\left[\tilde{\mathbf{F}}^* - \frac{1}{2}\left(\overleftrightarrow{\tilde{\mathbf{F}}}\cdot\hat{n}\right)\right]^T \mathbf{U} \; - \frac{1}{\mathrm{Re}}\left[\tilde{\mathbf{F}}_v^{*,T}\mathbf{U} + \mathbf{U}^{*,T}\left(\overleftrightarrow{\tilde{\mathbf{F}}}_v\cdot\hat{n}\right) - \mathbf{U}^T\left(\overleftrightarrow{\tilde{\mathbf{F}}}_v\cdot\hat{n}\right)\right] \ge 0}$$

- 3D
- Curved Hex Elements
- Any Polynomial Order

BC Implementation

BC Implementations are stable if

$$\text{(SSC)}\quad \left[\boxed{\tilde{\mathbf{F}}^*} - \frac{1}{2}\left(\overleftrightarrow{\tilde{\mathbf{F}}}\cdot\hat{n}\right)\right]^T \mathbf{U} \;-\frac{1}{\mathrm{Re}}\left[\boxed{\tilde{\mathbf{F}}_v^{*,T}}\,\mathbf{U} + \boxed{\mathbf{U}^{*,T}}\left(\overleftrightarrow{\tilde{\mathbf{F}}}_v\cdot\hat{n}\right) - \mathbf{U}^T\left(\overleftrightarrow{\tilde{\mathbf{F}}}_v\cdot\hat{n}\right)\right] \geq 0$$

Dirichlet-Type

Neumann-Type

Robin-Type Conditions

Typical Implementation

(E) $\left[\tilde{\mathbf{F}}^{*}-\frac{1}{2}\left(\overleftrightarrow{\tilde{\mathbf{F}}}\cdot\hat{n}\right)\right]^{T}\mathbf{U}\geq\mathbf{0}$ Euler Part

(D) $\left[\tilde{\mathbf{F}}_{v}^{*,T}\mathbf{U}+\mathbf{U}^{*,T}\left(\overleftrightarrow{\tilde{\mathbf{F}}}_{v}\cdot\hat{n}\right)-\mathbf{U}^{T}\left(\overleftrightarrow{\tilde{\mathbf{F}}}_{v}\cdot\hat{n}\right)\right]\leq\mathbf{0}$ Navier-Stokes Part

Sufficient, but not necessary

Examples

- Euler Inflow/Outlfow
- Euler Free-Slip Wall
- Navier-Stokes Inflow-Outflow
- Navier-Stokes Wall

Linear-Symmetric Equations

$$\mathbb{N} = (U, V, W) \cdot \hat{n}$$

$$\mathbf{U} = \begin{bmatrix} \rho \\ U \\ V \\ W \\ P \end{bmatrix} \qquad \overleftrightarrow{\mathbf{F}} \cdot \hat{n} = \begin{bmatrix} b\mathbb{N} \\ n_x(\rho b + aP) \\ n_y(\rho b + aP) \\ n_z(\rho b + aP) \\ a\mathbb{N} \end{bmatrix}$$

$$a^2 + b^2 = c^2$$

Euler Inflow-Outflow

Specify Free Stream in Upwind Riemann Solver

$$\tilde{\mathbf{F}}^*\left(\mathbf{U}^L,\mathbf{U}^R\right)=\left\{\left\{\tilde{A}\cdot\hat{n}\mathbf{U}\right\}\right\}-\frac{\left|\tilde{A}\cdot\hat{n}\right|}{2}\left[\left[\mathbf{U}\right]\right]=A^+\mathbf{U}^L+A^-\mathbf{U}^R$$

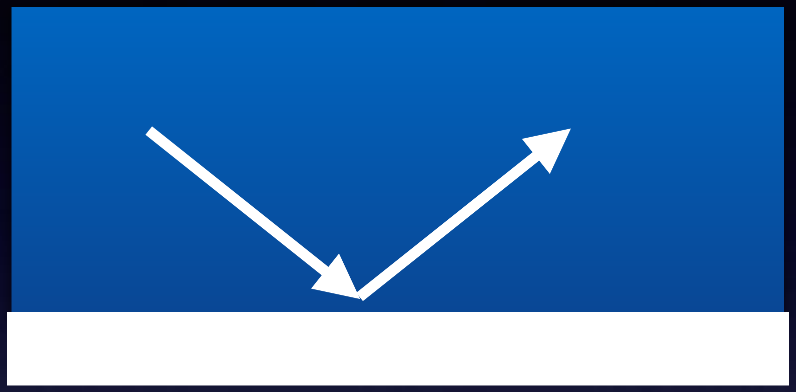
(E) $$\left(\tilde{\mathbf{F}}^*-\frac{1}{2}\overset{\leftrightarrow}{\tilde{\mathbf{F}}}\cdot\hat{n}\right)^T\mathbf{U}\ge\frac{1}{2}\mathbf{U}^T\underline{\mathrm{A}}^+\mathbf{U}\ge 0$$

Euler Free-Slip Wall

Specify No Normal Velocity

$$\mathbb{N} = (U, V, W) \cdot \hat{n} = 0$$

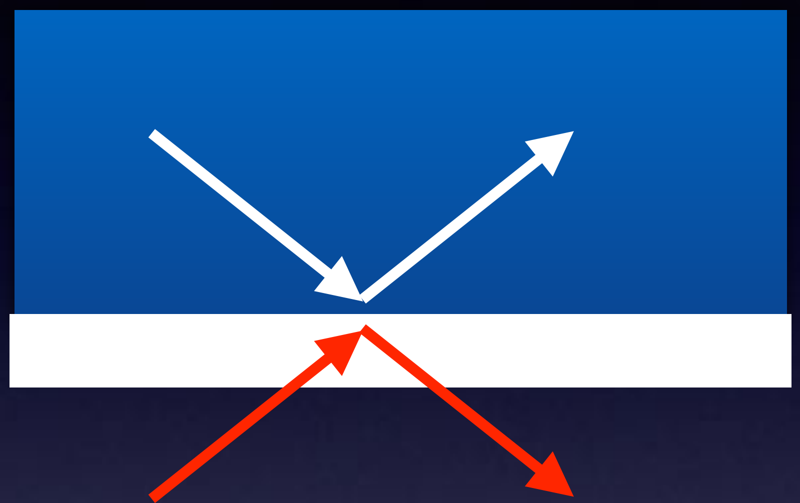
$$\mathbf{F}^* = \begin{bmatrix} 0 \\ n_x\left(b\rho + aP\right) \\ n_y\left(b\rho + aP\right) \\ n_z\left(b\rho + aP\right) \\ 0 \end{bmatrix}$$



$$\text{(E)} \quad \mathbf{U}^T\left(\mathbf{F}^* - \frac{1}{2}\mathbf{F}\cdot\hat{n}\right) = \begin{bmatrix} \rho & U & V & W & P \end{bmatrix} \begin{bmatrix} -b\mathbb{N} \\ n_x\left(b\rho + aP\right) \\ n_y\left(b\rho + aP\right) \\ n_z\left(b\rho + aP\right) \\ -a\mathbb{N} \end{bmatrix} = 0 \quad ✓$$

Euler Free-Slip Wall

Equal & Opposite in Upwind Riemann Solver



$$\left(\mathbf{v}\cdot\hat{n}\right)^{ext} \equiv \mathbb{N}^{ext} = -\left(\mathbf{v}\cdot\hat{n}\right)^{\text{int}} \equiv -\mathbb{N}^{\text{int}}$$

$$\tilde{\mathbf{F}}^{*}\left(\mathbf{U}^{L},\mathbf{U}^{R}\right) = \left\{\left\{\tilde{A}\cdot\hat{n}\mathbf{U}\right\}\right\} - \frac{\left|\tilde{A}\cdot\hat{n}\right|}{2}\left[\left[\mathbf{U}\right]\right] = A^{+}\mathbf{U}^{L} + A^{-}\mathbf{U}^{R}$$

(E) $$\left(\tilde{\mathbf{F}}^{*} - \frac{1}{2}\overset{\leftrightarrow}{\tilde{\mathbf{F}}}\cdot\hat{n}\right)^{T}\mathbf{U} = c\mathbb{N}^{2} \geq 0$$ ✓

Navier-Stokes Inflow

Free
Stream
$\mathbf{U}=0$

Specify Ext State

(E) $$\left(\tilde{\mathbf{F}}^* - \frac{1}{2}\overset{\leftrightarrow}{\tilde{\tilde{\mathbf{F}}}}\cdot\hat{n}\right)^T \mathbf{U} \geq \frac{1}{2}\mathbf{U}^T\underline{\mathrm{A}}^+\mathbf{U} \geq 0$$

(D) $$\left[\tilde{\mathbf{F}}_v^{*,T}\mathbf{U} + \mathbf{U}^{*,T}\left(\overset{\leftrightarrow}{\tilde{\tilde{\mathbf{F}}}}_v\cdot\hat{n}\right) - \mathbf{U}^T\left(\overset{\leftrightarrow}{\tilde{\tilde{\mathbf{F}}}}_v\cdot\hat{n}\right)\right]$$

Navier-Stokes Inflow

(D) $\left[\tilde{\mathbf{F}}_v^{*,T}\mathbf{U} + \mathbf{U}^{*,T}\left(\tilde{\overleftrightarrow{\mathbf{F}}}_v \cdot \hat{n}\right) - \mathbf{U}^T\left(\tilde{\overleftrightarrow{\mathbf{F}}}_v \cdot \hat{n}\right)\right]$ =0 ✓

$\tilde{\mathbf{F}}_v^* = \tilde{\overleftrightarrow{\mathbf{F}}}_v \cdot \hat{n}$

Compute Flux from Interior

$\mathbf{U}^* = 0$

Specify Solution

$$(E) + (D) = \left(\tilde{\mathbf{F}}^* - \frac{1}{2}\tilde{\overleftrightarrow{\mathbf{F}}} \cdot \hat{n}\right)^T \mathbf{U} \geq \frac{1}{2}\mathbf{U}^T \underline{\mathbf{A}}^+ \mathbf{U} \geq 0$$

Navier-Stokes Outflow

(E) $\left(\tilde{\mathbf{F}}^* - \frac{1}{2}\overset{\leftrightarrow}{\tilde{\mathbf{F}}}\cdot\hat{n}\right)^T \mathbf{U} \ge \frac{1}{2}\mathbf{U}^T \underline{A}^+\mathbf{U} \ge 0$

(D) $\left[\tilde{\mathbf{F}}_v^{*,T}\mathbf{U} + \mathbf{U}^{*,T}\left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v\cdot\hat{n}\right) - \mathbf{U}^T\left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v\cdot\hat{n}\right)\right]$ =0

$\tilde{\mathbf{F}}_v^* = 0$

Specify Flux

$\mathbf{U}^* = \mathbf{U}$

Use interior Solution

Navier-Stokes Wall

Satisfy (E) through Riemann Solver

Re-Write (D)

$$\mathbf{U}^T\tilde{\mathbf{F}}_v^* + \mathbf{U}^{*,T}\left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v \cdot \hat{n}\right) - \mathbf{U}^T\left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v \cdot \hat{n}\right) = \boxed{\mathbf{U}^T\left\{\tilde{\mathbf{F}}_v^* - \overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v \cdot \hat{n}\right\} + \mathbf{U}^{*,T}\left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v \cdot \hat{n}\right) \leq 0}$$

Written out…

$$\begin{bmatrix} \rho & U & V & W & P \end{bmatrix} \begin{bmatrix} 0 \\ \tau_1^* - \tau_1 \\ \tau_2^* - \tau_2 \\ \tau_3^* - \tau_3 \\ r\left(\frac{\partial P^*}{\partial n} - \frac{\partial P}{\partial n}\right) \end{bmatrix} + \begin{bmatrix} \rho^* & U^* & V^* & W^* & P^* \end{bmatrix} \begin{bmatrix} 0 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ r\frac{\partial P}{\partial n} \end{bmatrix} \leqslant 0,$$

$$\tau_i = n_x\tau_{i1} + n_y\tau_{i2} + n_z\tau_{i3}$$

Navier-Stokes Wall

$$\begin{bmatrix} \rho & U & V & W & P \end{bmatrix} \begin{bmatrix} 0 \\ \tau_1^* - \tau_1 \\ \tau_2^* - \tau_2 \\ \tau_3^* - \tau_3 \\ r\left(\frac{\partial P^*}{\partial n} - \frac{\partial P}{\partial n}\right) \end{bmatrix} + \begin{bmatrix} \rho^* & U^* & V^* & W^* & P^* \end{bmatrix} \begin{bmatrix} 0 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ r\frac{\partial P}{\partial n} \end{bmatrix} \leqslant 0,$$

Vanishes if

$$\tau_i^* = \tau_i \qquad U^* = V^* = W^* = 0$$

$$+$$

Isothermal

$$P^* = 0$$

$$\frac{\partial P^*}{\partial n} = \frac{\partial P}{\partial n}$$

Adiabatic

$$P^* = P$$

$$\frac{\partial P^*}{\partial n} = 0$$

Not so fast…

A Guide to the Implementation of Boundary Conditions… AIAA 2014

Use interior Solution $\mathbf{U}^* = \mathbf{U}$

Viscous Flux from Interior: $\tilde{\mathbf{F}}_v^* = \overleftrightarrow{\tilde{\mathbf{F}}}_v \cdot \hat{n}$

(SSC) $$\underbrace{\left\{\tilde{\mathbf{F}}^* - \frac{1}{2}\left(\overleftrightarrow{\tilde{\mathbf{F}}} \cdot \hat{n}\right)\right\}^T \mathbf{U}}_{\geq 0} - \frac{1}{\mathrm{Re}}\underbrace{\left\{\mathbf{U}^T \left(\overleftrightarrow{\tilde{\mathbf{F}}}_v \cdot \hat{n}\right)\right\}}_{???} \overset{???}{\geq 0}$$

Nonlinear: Entropy Bound

$$\frac{d}{dt}\overline{S} \leqslant -\sum_{\substack{\text{Boundary}\\\text{faces}}} \int_N \left\{ (\mathbf{W})^T \left(\tilde{\mathbf{F}}^{\mathrm{ec},*} - \left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}} \cdot \hat{n} \right) \right) + \left(\vec{\tilde{F}}^{\,\epsilon} \cdot \hat{n} \right) \right\} \, \mathrm{dS}$$

$$+ \frac{1}{\mathrm{Re}} \sum_{\substack{\text{Boundary}\\\text{faces}}} \int_N \left\{ \mathbf{W}^T \left(\tilde{\mathbf{F}}_v^* - \left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v \cdot \hat{n} \right) \right) + (\mathbf{W}^*)^T \left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v \cdot \hat{n} \right) \right\} \, \mathrm{dS} \,,$$

Sufficient Condition for Stability

$$-\left\{ (\mathbf{W})^T \left(\tilde{\mathbf{F}}^{\mathrm{ec},*} - \left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}} \cdot \hat{n} \right) \right) + \left(\vec{\tilde{F}}^{\,\epsilon} \cdot \hat{n} \right) \right\} + \frac{1}{\mathrm{Re}} \left\{ \mathbf{W}^T \left(\tilde{\mathbf{F}}_v^* - \left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v \cdot \hat{n} \right) \right) + (\mathbf{W}^*)^T \left(\overset{\leftrightarrow}{\tilde{\mathbf{F}}}_v \cdot \hat{n} \right) \right\} \leq 0$$

TO DO

- Robin Conditions
- Entropy BCs

Issues: Linear theory well understood

Linear Stability $\not\Leftrightarrow$ Nonlinear Stability

Entropy function not unique

A stable procedure with one entropy function may not be stable with another.

Summary

Have linear and entropy stability bounds to establish stability of DGSEM Approximations for

- Arbitrary 3D geometries
- Curved elements
- Any polynomial order

Bounds establish stable boundary procedures

Approach extends to, Shallow water, MHD eqns., …

See Andrew’s talk