A new model of shoaling and breaking waves

1D solitary wave on a mild beach

Maria Kazakova, G. L. Richard, A. Duran

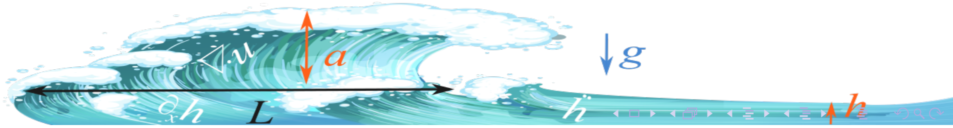
Modelling of surf zone propagation

<u>Main issues</u>

- Dispersive effects (dependence of wave phase velocity on its frequency)
- Energy dissipation
- Turbulent structures generation

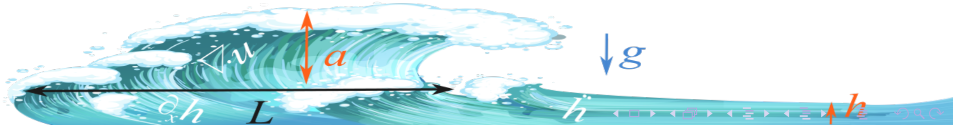
<u>Computational resources</u>

- Navier-Stokes + BC : too expensive
  - Asymptotic models are used
- Depth-averaged equation



Overview

- Past & recent advances on wave breaking modelling :
  $$NSWE \longrightarrow Boussinesq\,type \longrightarrow Shear\,flows$$
- Model derivation key points
- Numerical realization
- Convergence of algorithm
- Experimental data comparison
  - Set up
  - Trial tests
  - Empirical laws
- Conclusions & Perspectives



State of the art

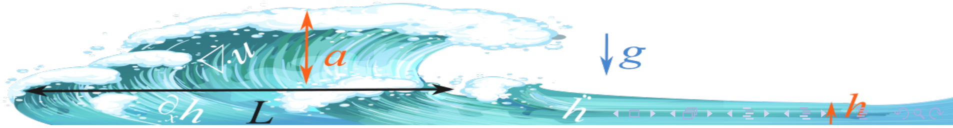
Advances on wave breaking modelling

Depth-average **Euler equation** $\longrightarrow \mu = \frac{h}{L}$ (SW), $\varepsilon = \frac{a}{h}$ (NL)

	$NSWE$	BT	GN
SW	✓	✓	✓
Pressure	Hydrostatic	Not hydrostatic	Not hydrostatic
NL	no assump.	small	no assump.
Shocks	✓	×	×

Dissipation(?)

- ∗ Artificial terms [Zelt, Kennedy, Cienfuegos,...]
- ∗ Switching between model [Bonneton, Marche, Kazolea...]



State of the art

Recent advances, goals for the research

G. Nwogu, **1996**, Y. Zhang *et al*, **2014**

$*$ Empirically add TKE equation.

R. Briganti *et al.*, **2004**, S. L. Gavrilyuk *et al.*, **2016**

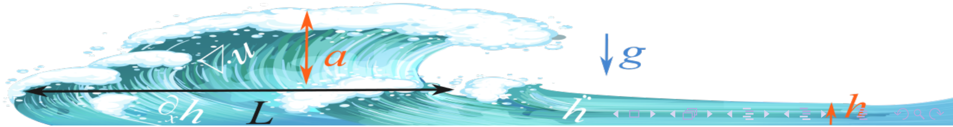
$*$ Use effects of vertical variations of the flow.

Research aims:

$*$ To derive unified model capable to describe wave breaking

$**$ Take into account turbulence structure explicitly

$**$ Use a shear flow hypothesis



Model derivation

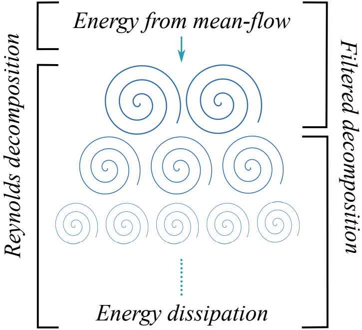
Turbulence : Filtered decomposition

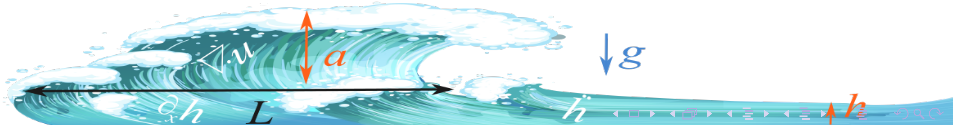
Filtering velocity field decomposition:

$$\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}^r$$

$\overline{\text{(Navier-Stokes)}}$ gives

* Filtered continuity equation
* Filtered momentum equation
* Decomposition of stress tensor
  $\rightarrow$ isotropic and anisotropic part





Model derivation

Turbulence : Filtered decomposition

Continuity equation: $\operatorname{div} \overline{\boldsymbol{v}} = 0$,

Momentum equation: $\dfrac{\partial \overline{\boldsymbol{v}}}{\partial t} + \mathbf{div}\left(\overline{\boldsymbol{v} \otimes \boldsymbol{v}}\right) = \boldsymbol{g} - \dfrac{1}{\rho}\,\mathbf{grad}\,\overline{p} + \nu \Delta \overline{\boldsymbol{v}}$

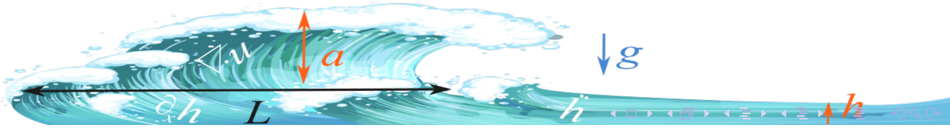
$\overline{\boldsymbol{v} \otimes \boldsymbol{v}}$ should be defined!

$$\boldsymbol{\sigma}^r = -\rho\left(\overline{\boldsymbol{v} \otimes \boldsymbol{v}} - \overline{\boldsymbol{v}} \otimes \overline{\boldsymbol{v}}\right) = -\frac{2}{3}\rho k^r \mathbf{I} + \mathbf{A}^r$$

<u>Turbulent viscosity hypothesis</u>

$\mathbf{A}^r = 2\rho\nu_T \overline{D} = \rho\nu_T \left[\mathbf{grad}\,\overline{\boldsymbol{v}} + \left(\mathbf{grad}\,\overline{\boldsymbol{v}}\right)^T\right]$

$+$ *Equation for kinetic energy*



Model derivation

2D case

The filtered mass conservation

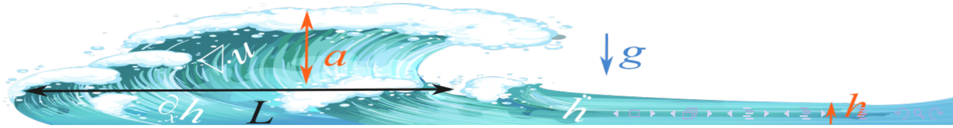
$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{w}}{\partial z} = 0.$$

Momentum Ox balance equation :

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \overline{u}^2}{\partial x} + \frac{\partial \overline{uw}}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left(\frac{\partial A^r_{xx}}{\partial x} + \frac{\partial A^r_{xz}}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Oz :

$$\frac{\partial \overline{w}}{\partial t} + \frac{\partial \overline{uw}}{\partial x} + \frac{\partial \overline{w}^2}{\partial z} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\left(\frac{\partial A^r_{xz}}{\partial x} + \frac{\partial A^r_{zz}}{\partial z}\right) + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right).$$



Model derivation

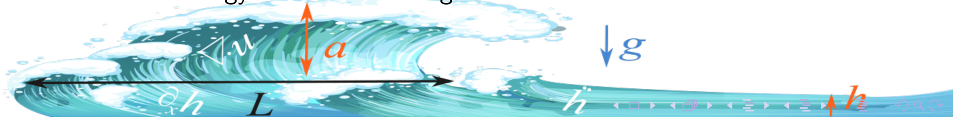
2D case

Equation for the kinetic energy :

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\overline{u}^2}{2}+\frac{\overline{w}^2}{2}\right)\\
&+\frac{\partial}{\partial x}\left[\overline{u}\left(\frac{\overline{u}^2}{2}+\frac{\overline{w}^2}{2}+gz\right)+\frac{pu}{\rho}-\frac{A^r_{xx}u}{\rho}-\frac{A^r_{xz}w}{\rho}-\frac{\tau_{xx}u}{\rho}-\frac{\tau_{xz}w}{\rho}\right]\\
&+\frac{\partial}{\partial z}\left[\overline{w}\left(\frac{\overline{u}^2}{2}+\frac{\overline{w}^2}{2}+gz\right)+\frac{pw}{\rho}-\frac{A^r_{xz}u}{\rho}-\frac{A^r_{zz}w}{\rho}-\frac{\tau_{xz}u}{\rho}-\frac{\tau_{zz}w}{\rho}\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=-\varepsilon_f-P^r.
\end{aligned}
$$

Dissipation :

* ε_f – viscous dissipation due to filtered velocity field
* P^r – energy transfer from large scale to small scale



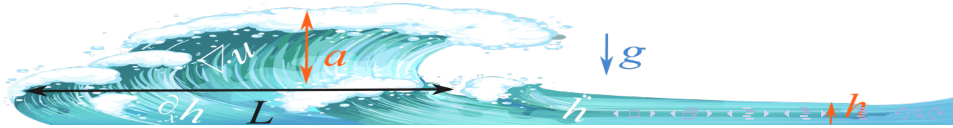
Numerical realization

Virtual enstrophy : breaking criteria

$$\overline{u}(t,x,z) = U(t,x) + u'(t,x,z),\ \varphi = \frac{<< u'^2 >>}{h^2}$$

Teshukov, **2007**, Richard, Gavrilyuk, **2012**,**2013**,**2015**
Integration over the depth and hypothesis of mild topography give:

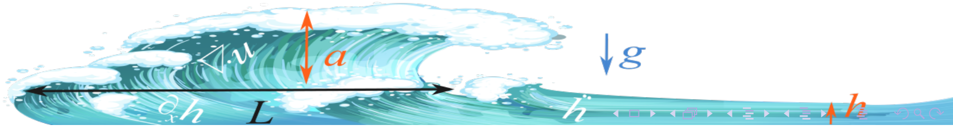
$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial hU}{\partial x} = 0, \\[2ex] \dfrac{\partial hU}{\partial t} + \dfrac{\partial}{\partial x}\left(hU^2 + \dfrac{gh^2}{2} + h^3\varphi + \dfrac{h^2\ddot{h}}{3} \right) = \dfrac{\partial}{\partial x}\left(\dfrac{4}{R} h^3 \sqrt{\varphi} \dfrac{\partial U}{\partial x} \right) - ghb', \\[2ex] \dfrac{\partial h\varphi}{\partial t} + \dfrac{\partial hU\varphi}{\partial x} = \dfrac{8h\sqrt{\varphi}}{R}\left(\dfrac{\partial U}{\partial x} \right)^2 - C_r h^3 \varphi^{3/2} \end{cases}$$



Numerical realization

Hyperbolic & Elliptic stage separation

$$
\begin{cases}
\dfrac{\partial h}{\partial t} + \dfrac{\partial (hU)}{\partial x} = 0, \\[2ex]
\dfrac{\partial (hU)}{\partial t} + \dfrac{\partial}{\partial x}\left(hU^2 + \dfrac{gh^2}{2} + \boxed{\dfrac{h^2 \ddot{h}}{3}} + h^3 \varphi \right) = \dfrac{\partial}{\partial x}\left(\dfrac{4h^3\sqrt{\varphi}}{R}\dfrac{\partial U}{\partial x} \right) - b' h g, \\[2ex]
\dfrac{\partial h\varphi}{\partial t} + \dfrac{\partial (h\varphi U)}{\partial x} = \dfrac{8h\sqrt{\varphi}}{R}\left(\dfrac{\partial U}{\partial x}\right)^2 - C_r h^3 \varphi^{3/2},
\end{cases}
$$



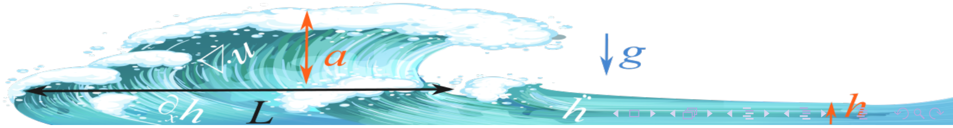
Numerical realization

Hyperbolic & Elliptic stage separation

$$
\begin{cases}
\dfrac{\partial h}{\partial t} + \dfrac{\partial (hU)}{\partial x} = 0, \\[2ex]
\dfrac{\partial (hK)}{\partial t} + \dfrac{\partial}{\partial x}\left(\boxed{hKU} + \dfrac{gh^2}{2} + \quad \boxed{\alpha} + h^3\varphi \right) = \dfrac{\partial}{\partial x}\left(\dfrac{4h^3\sqrt{\varphi}}{R}\dfrac{\partial U}{\partial x} \right) - b' h g, \\[2ex]
\dfrac{\partial h\varphi}{\partial t} + \dfrac{\partial (h\varphi U)}{\partial x} = \dfrac{8h\sqrt{\varphi}}{R}\left(\dfrac{\partial U}{\partial x} \right)^2 - C_r h^3 \varphi^{3/2},
\end{cases}
$$

Change of variables $K = u + \dfrac{1}{3h}\nabla\left(h^2 \dot{h}\right)$

O. Le Métayer, S. Gavrilyuk, S. Hank, (2010)
A numerical scheme for the Green–Naghdi model,JCP



Numerical realization

Hyperbolic Stage h, hK, $h\varphi$

Hyperbolic stage $\rightarrow$ Godunov's Type Scheme of 2^{nd} order

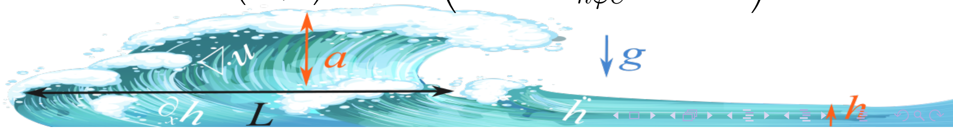
$*$ 2^{nd} order : MUSCL Reconstruction

$*$ Riemann Solver : HLL

$*$ Time discretization : RK-2 (Heun's method)

$$\bar{\mathbf{U}}_i = \mathbf{U}_i^n + \frac{dt}{dx}\left(\mathbf{F}_{i-1/2}^n - \mathbf{F}_{i+1/2}^n\right)$$

$$\mathbf{U} = \begin{pmatrix} h \\ hK \\ h\varphi \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} hU \\ hKU + \alpha + \dfrac{gh^2}{2} + h^3\varphi \\ h\varphi U \end{pmatrix}$$



Numerical realization

Elliptic operator inversion

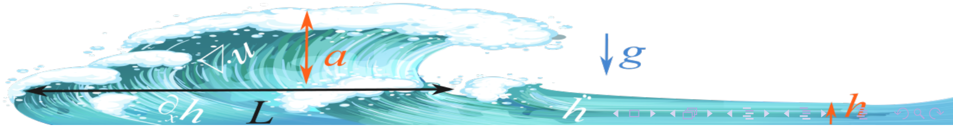
Elliptic stage $\rightarrow$ System of linear equations

Elliptic equation for velocity

$$hK = hU - \frac{1}{3}\nabla\left(h^2\frac{\partial hU}{\partial}\right) + \frac{1}{6}\nabla\left(hU\nabla(h^2)\right)$$

In discrete form with three-diagonal matrix A

$$\begin{pmatrix} \vdots \\ hU \\ \vdots \end{pmatrix}_i^{n+1} = \left[\quad A\left(h, h^2, \frac{\partial h^2}{\partial x}, \frac{\partial^2 h}{\partial x^2}\right)_i^{n+1} \quad\right]\begin{pmatrix} \vdots \\ hK \\ \vdots \end{pmatrix}_i^{n+1}$$



Well-balanced algorithm

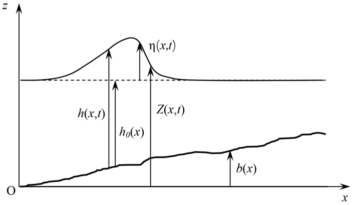
$$\frac{\partial(hK)}{\partial t} + \frac{\partial}{\partial x}\left(hKU + \frac{gh^2}{2} + \alpha + h^3\varphi\right) = \boxed{-b'hg}$$

$$b_c = max(b_l, b_r)$$

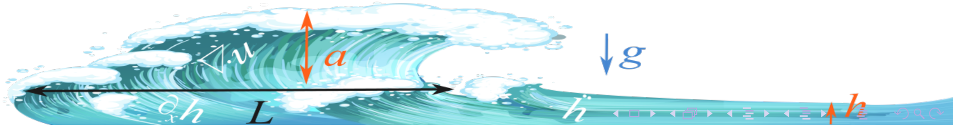
$$h_l = h_l^{MUSCL} + (b_l - b_c)$$

$$h_r = h_r^{MUSCL} + (b_r - b_c)$$

$$h_l, h_r \rightarrow Fluxes$$



E. Audusse et *al*, (2004)
A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows, J Sci Comput, 25(6), 2050-2065.

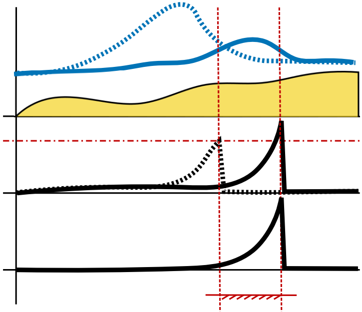


Numerical realization

Virtual enstrophy : breaking criteria

$\forall \varepsilon < 0.05$, or at the initial wave propagation ($t < t^*$)

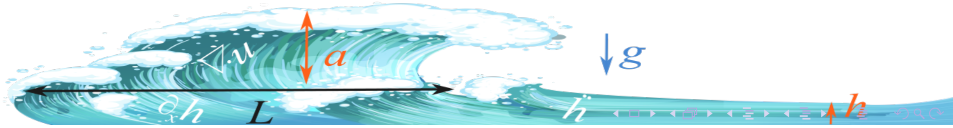
$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial hU}{\partial x} = 0, \\ \dfrac{\partial hK}{\partial t} + \dfrac{\partial}{\partial x}\left(hUK + \dfrac{gh^2}{2} + \alpha \right) = -ghb', \\ \dfrac{\partial h\psi}{\partial t} + \dfrac{\partial (hU\psi)}{\partial x} = \dfrac{8h\sqrt{\psi}}{R}\left(\dfrac{\partial U}{\partial x}\right)^2 - \mathcal{D}, \end{cases} \qquad \begin{array}{c} \forall t < t^*, \forall x \in [0, L] : \\ \max\limits_x \psi(t,x) < \psi_0, \end{array}$$



Numerical realization

Virtual enstrophy : breaking criteria

$$
\begin{cases}
\begin{array}{ll}
 & \dfrac{\partial h}{\partial t} + \dfrac{\partial hU}{\partial x} = 0, \\
\forall t > t_*, & \dfrac{\partial hK}{\partial t} + \dfrac{\partial}{\partial x}\left(hUK + \dfrac{gh^2}{2} + h^3\varphi + \alpha \right) = \dfrac{\partial}{\partial x}\left(\dfrac{4}{R} h^3 \sqrt{\varphi} \dfrac{\partial U}{\partial x} \right) \\
\forall x \in [0, L] & \dfrac{\partial h\psi}{\partial t} + \dfrac{\partial (hU\psi)}{\partial x} = \dfrac{8h\sqrt{\psi}}{R}\left(\dfrac{\partial U}{\partial x}\right)^2 - \mathcal{D}, \\
 & \\
\forall t > t_*, & \\
\forall x \in \bigcup\limits_{\ell = t^*}^{t} \{x : \max\limits_x \psi(\ell, x) > \psi_0\}, & \dfrac{\partial h\varphi}{\partial t} + \dfrac{\partial (hU\varphi)}{\partial x} = \dfrac{8h\sqrt{\varphi}}{R}\left(\dfrac{\partial U}{\partial x}\right)^2 - \mathcal{D},
\end{array}
\end{cases}
$$



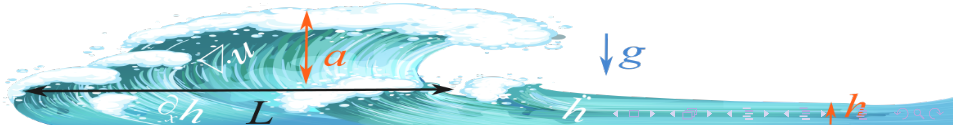
Numerical Simulations

Soliton test & Convergence

$$h(x,t) = h_0 + \xi(x,t), \quad u(x,t) = c_0 \left(1 - h_0/h(x,t)\right)$$

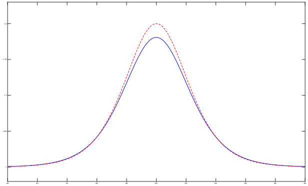
$$\xi(x,t) = \frac{2a\left(Fr^2 - 1 - 3\tilde{\varphi}\right)}{Fr^2 - 1 - (3+a^2)\tilde{\varphi} + \left(Fr^2 - 1 - (3-a^2)\tilde{\varphi}\right)\cosh(\kappa(x - c_0 t - x_0))};$$

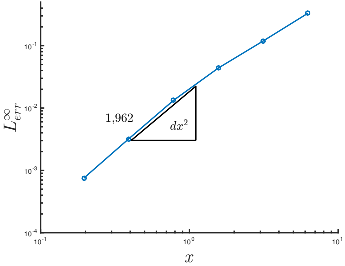
$$\kappa = \sqrt{\frac{3(Fr^2 - 1 - 3\tilde{\varphi})}{Fr^2}}, c_0 = \sqrt{g\left(h_0 + a + \tilde{\varphi}(3h_0 + a)\right)}$$

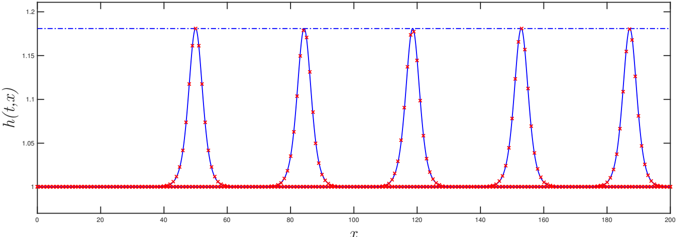


Numerical Simulations

Soliton test & Convergence



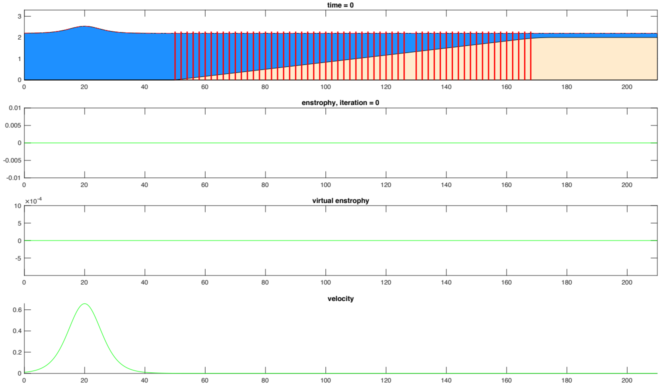




Numerical Simulations

Experimental Data Comparison

S. C. Hsiao *et al.*, **2008**, $tg\beta = 0.017$



Numerical Simulations

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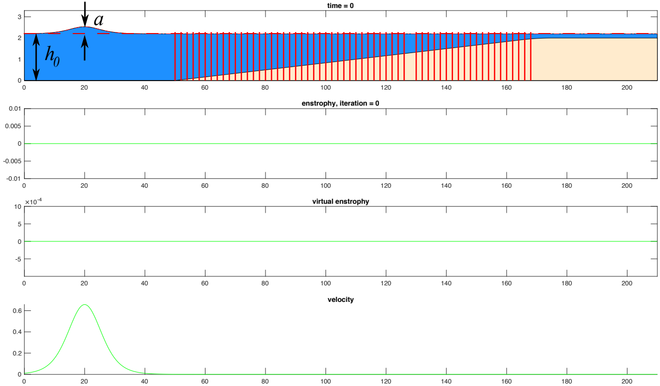

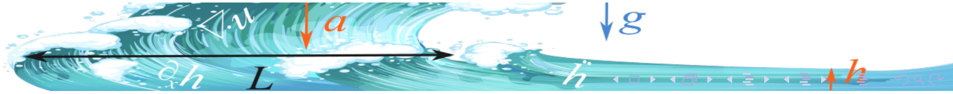

Numerical Simulations

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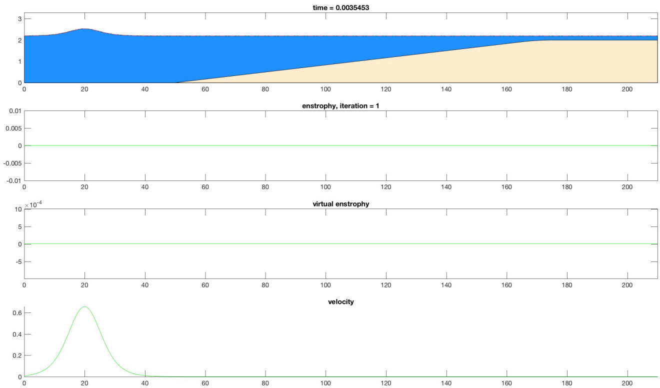
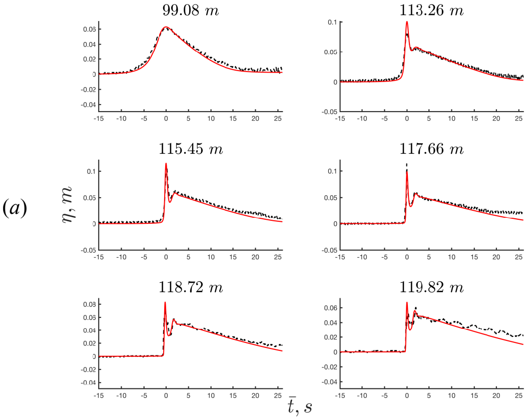
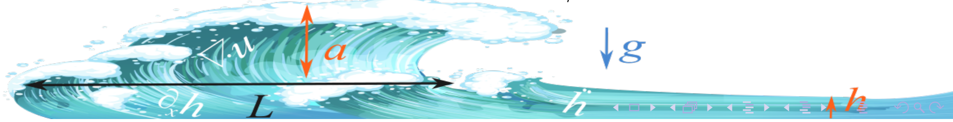


Numerical Simulations

Experimental Data Comparison

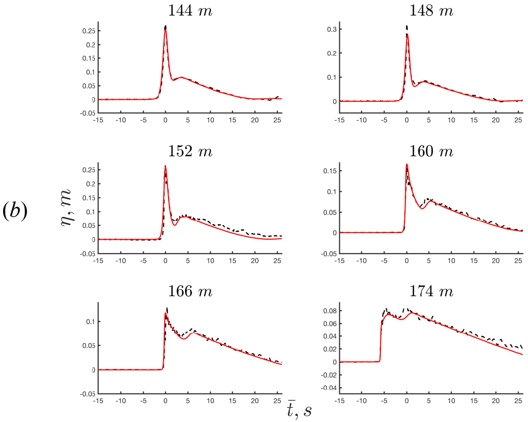


$$N_{trial} = 3,\ h = 1.2m,\ \varepsilon = a_0/h_0 = 0.048$$



Numerical Simulations

Experimental Data Comparison

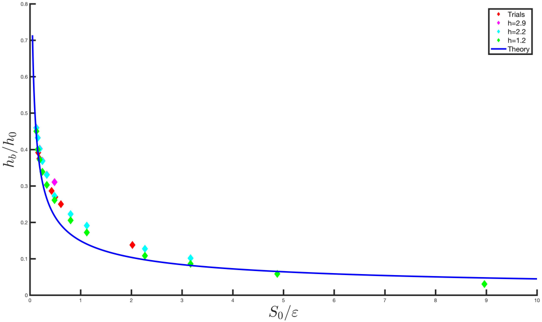


$$N_{trial} = 41,\ h = 2.2m,\ \varepsilon = a_0/h_0 = 0.137$$

Numerical Simulations

Empirical Laws for breaking depth

$$S_0 = 1.521 \frac{tan\beta}{\sqrt{\varepsilon}}, \qquad \frac{h_b}{h_0} = \frac{0.149}{(S_0/\varepsilon)^{0.523}}$$



S.T. Grilli,et al. *Breaking criteria and characteristics for solitary waves on slopes,* J. Waterw. Port Coast. Ocean Eng. 123 (3), 102–112, (1994)

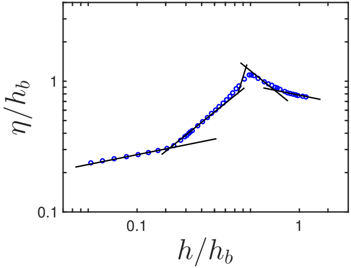
Numerical Simulations

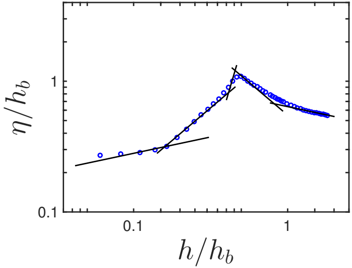
Empirical Laws for breaking depth ()

$$\frac{\eta_{max}}{h_b} \sim \left(\frac{h_{loc}}{h_b}\right)^n$$

$n = -1/4$ – gradual shoaling, $n = -1$ – rapid shoaling

$n = 4$ – rapid decay, $n = 1$ – zone of gradual decay





C.E. Synolakis, J. Skjelbreia, *Evolution of maximum amplitude of solitary waves on plane beaches,* J. Waterw. Port Coast. Ocean Eng. 119 (3), 323–342, (1993).

Numerical Simulations

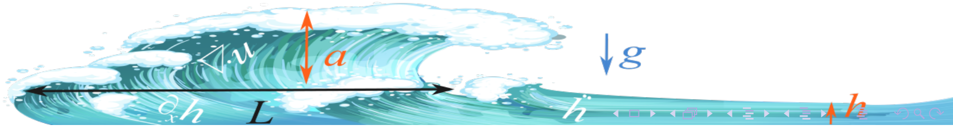
Empirical Laws for breaking depth ()

$$\frac{\eta_{max}}{h_b} \sim \left(\frac{h_{loc}}{h_b}\right)^n$$

$n = -1/4$ – gradual shoaling, $n = -1$ – rapid shoaling

$n = 4$ – rapid decay, $n = 1$ – zone of gradual decay

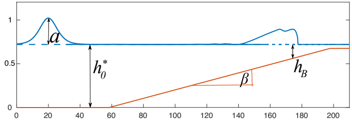
$$\psi_0 = \frac{g}{h_0^*}\widetilde{\psi}_0, \widetilde{\psi}_0 = \begin{cases} \left(0.1 + \frac{0.031}{\varepsilon}\right), & \varepsilon > 0.05, \\ 0, & \varepsilon < 0.05, \end{cases} \quad R = \begin{cases} 1.7, & \varepsilon > 0.05, \\ 6, & \varepsilon < 0.05, \end{cases}$$



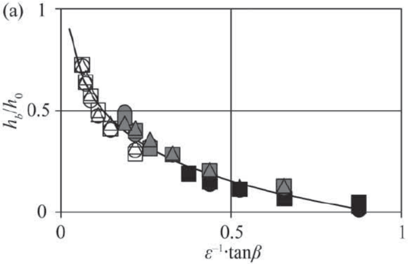
Wave transformation over different slopes

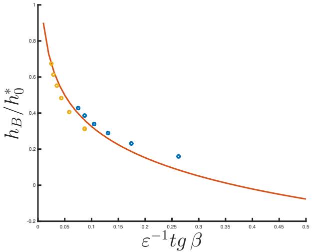
Strategies : 1. $R = R(\beta)$, $\psi_0 = \psi_0(\beta)$; 2. $R = R(\beta)$ 3. $\psi_0 = \psi_0(\beta)$

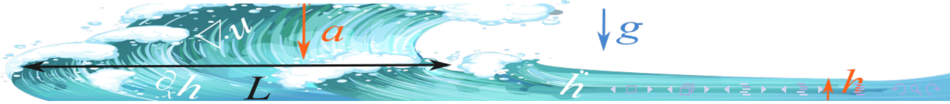
H. J. Hafsteinsson *et al.*, **2017**



$$R = \begin{cases} 1.75, & \beta = 1^\circ, \\ 4, & \beta = 3^\circ, \\ 8, & \beta = 6^\circ. \end{cases}$$







Wave transformation over different slopes

Strategies : 1. $R = R(\beta)$, $\psi_0 = \psi_0(\beta)$; 2. $R = R(\beta)$ 3. $\psi_0 = \psi_0(\beta)$

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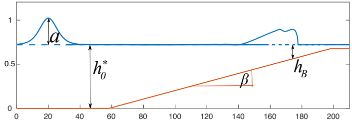
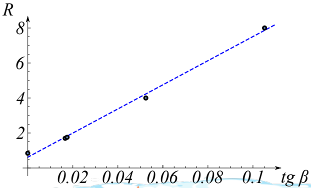


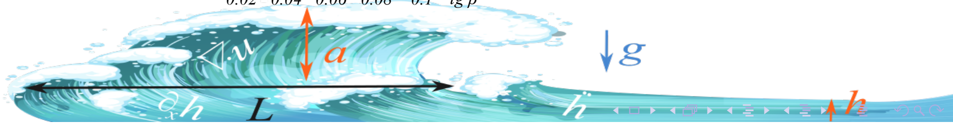
$$R = \begin{cases} 1.75, & \beta = 1^\circ, \\ 4, & \beta = 3^\circ, \\ 8, & \beta = 6^\circ. \end{cases}$$

Dependence of Reynolds number R on topography $tg\,\beta$

$*\ tg\,\beta = 0$ from stability of solitary wave $\varepsilon = 0.78$.



$$R(\beta) = 0.85 + 60\,tg\,\beta$$



Conclusions & Perspectives

- Modelling
  - Dispersive model for shoaling and breaking waves is derived
  - Natural procedure for the breaking wave description is proposed
- Numerics
  - The well-balanced $1D$ code is developed
  - Comparison with exact solution, experimental data is performed
  - Numerical simulations are confirmed with empirical laws for breaking depth

Purpose for the further investigation:

1D Non-uniform topography with dry zones (in prep. Richard, Duran)
Cnoidal waves propagation
2D simulations (in prep. Richard, Duran, Fabrèges)