Aggregation equations

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Presentation

$$\begin{cases} \partial_t \rho = \operatorname{div}\left(\left(\nabla_x W * \rho\right)\rho\right), & t > 0, x \in \mathbb{R}^d \\ \rho(0,.) = \rho^{\mathrm{ini}} \end{cases}$$

- W is an interaction potential. For example, one can think of $W = \|x\|$.
- ρ^{ini} is a probability measure.
- Possible Dirac masses creation.
- The velocity is not Lipschitz continuous, the characteristic method cannot be used.

Fillipov's Theory

Definition

Let $a = a(t,x) \in \mathbb{R}^d$ be a vector field defined in $[0, T] \times \mathbb{R}^d$, $T > 0$. A Filippov's characteristic $X(t; x, s)$ stemmed from $x \in \mathbb{R}^d$ at time s is a continuous function $X(.; s, x) \in C([0, T], \mathbb{R}^d)$ such that $\partial_t X(t; s, x)$ exists a.e. $t \in [0, T]$ and satisfies :

$$\left| \begin{array}{l} \partial_t X(t; s, x) \in \{\mathrm{Convess}(a)(t, .)\}(X(t; s, x)), \quad a.e.\ t \in [0, T]; \\ \quad X(s; x, s) = x \end{array} \right.$$

Remarks :

- $\{\mathrm{Convess}(a)(t, .)\}(x) = \cap_{r>0} \cap_{N, \mu(N)=0} \mathrm{Conv}\left[a(t, B(x, r) \backslash N)\right]$
- We write $X(t, x) = X(t; 0, x)$

Existence and uniqueness of a Filippov's characteristic

Theorem

Let $T > 0$ and assume that the vector field $a \in L^1_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^d))$ satisfies a one-sided Lipschitz continuity (OSL) estimate, i.e. for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$,

$$(a(t,x) - a(t,y)) \cdot (x - y) \leq \alpha(t)\|x - y\|^2, \quad \alpha \in L^1_{loc}([0, T]).$$

Then, there exists a unique Filippov characteristic X associated with that vector field a.

Application to the linear transport equation

Notations:

- $\mathcal{M}_b(\mathbb{R}^d)$ is the set of finite signed measure on $\mathbb{R}^d$.
- $\mathcal{C}_0(\mathbb{R}^d)$ is the set of continuous functions on $\mathbb{R}^d$ that tend to 0 at ∞.

Theorem (Poupaud & Rascle)

Let $T > 0$ and $a \in L^1_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^d))$ be a vector field satisfying an OSL estimate. Then, for all $u_0 \in \mathcal{M}_b(\mathbb{R}^d)$, there exists a unique measure $u \in C([0,T], \mathcal{M}_b(\mathbb{R}^d))$ solution to the conservative transport equation :

$$\left| \begin{aligned} \partial_t u + div(au) &= 0, \\ u(t=0,.) &= u_0, \end{aligned} \right.$$

such that $u(t) = X(t)_\# u_0$, where X is the unique Filippov's characteristic, i.e. for all ϕ in $\mathcal{C}_0(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} \phi(x) u(t,dx) = \int_{\mathbb{R}^d} \phi(X(t,x)) u_0(dx), \quad t \in [0,T].$$

Back to the aggregation equation

$$\begin{cases} \partial_t \rho = \mathrm{div}\left(\left(\nabla_x W * \rho\right)\rho\right), & t > 0, x \in \mathbb{R}^d \\ \rho(0,.) = \rho^{\mathrm{ini}} \end{cases}$$

- $\rho^{\mathrm{ini}} \in \mathcal{P}_2(\mathbb{R}^d)$, where

$$\mathcal{P}_2(\mathbb{R}^d) = \left\{ \mu \text{ positive measure}, \mu(\mathbb{R}^d) = 1, \int |x|^2 \mu(dx) < \infty \right\},$$

  endowed with the Wasserstein distance,

$$d_W(\mu,\nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |y-x|^2 \gamma(dx,dy) \right)^{1/2}$$

- W satisfies the following properties:

(A0) $W(x) = W(-x)$, and $W(0) = 0$;
(A1) W is λ-convex, $\lambda \in \mathbb{R}$, i.e. $W(x) - \frac{\lambda}{2}|x|^2$ is convex;
(A2) $W \in C^1(\mathbb{R}^d \setminus \{0\})$;
(A3) W is Lipschitz-continuous.

Definition of the velocity field

For $\rho \in C([0, T], \mathcal{P}_2(\mathbb{R}^d))$, we define the velocity field $\hat{a}_\rho$ with:

$$\hat{a}_\rho(t,x) = -\int_{y \neq x} \nabla W(x-y)\rho(t,dy).$$

We extend the kernel:

$$\widehat{\nabla W}(x) = \begin{cases} \nabla W(x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

so that:

$$\hat{a}_\rho(t,x) = -\int_{\mathbb{R}^d} \widehat{\nabla W}(x-y)\rho(t,dy).$$

Existence and uniqueness of a solution

Theorem (Carillo, James, Lagoutière, Vauchelet)

Let W be a potential satisfying the conditions **(A0)** - **(A3)** *and ρ^{ini} a measure in $\mathcal{P}_2(\mathbb{R}^d)$. Then,*

1. *there exists a unique solution $\rho \in C([0,T], \mathcal{P}_2(\mathbb{R}^d))$, satisfying, in the sense of distribution, the aggregation equation:*

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\hat{a}_\rho \rho) = 0, & t > 0, x \in \mathbb{R}^d \\ \rho(0,.) = \rho^{\mathrm{ini}}. \end{cases}$$

2. *This solution ρ may be represented as the family of pushforward measures $\left(\rho(t) = Z_\rho(t,.)_\# \rho^{ini}\right)_{t\geq 0}$, where $\left(Z_\rho(t,.)\right)_{t\geq 0}$ is the unique Filippov characteristic flow associated to the velocity field $\hat{a}_\rho$.*

3. *If ρ_1 et ρ_2 are two solutions with respective initial conditions ρ_1^{ini} and ρ_2^{ini} in $\mathcal{P}_2(\mathbb{R}^d)$, then:*

$$d_W(\rho_1(t), \rho_2(t)) \leq e^{|\lambda| t} d_W(\rho_1^{ini}, \rho_2^{ini}), \quad t \geq 0.$$

Existence: sketch of the proof

- The velocity $\hat{a}_\rho$ satisfies an OSL estimate
  - W is λ-convex (condition **A1**):
$$(\nabla W(x) - \nabla W(y)) \cdot (x-y) \geq \lambda \|x-y\|^2, \quad x, y \in \mathbb{R}^d \setminus \{0\}$$
  - ∇W is odd (condition **A0**). Taking $y = -x$, the previous inequality holds for $\widehat{\nabla W}$:
$$\left(\widehat{\nabla W}(x) - \widehat{\nabla W}(y)\right) \cdot (x-y) \geq \lambda \|x-y\|^2, \quad x, y \in \mathbb{R}^d$$
  - By definition of the velocity, we have:
$$\hat{a}_\rho(x) - \hat{a}_\rho(y) = -\int_{\mathbb{R}^d} \left(\widehat{\nabla W}(x-z) - \widehat{\nabla W}(y-z)\right) \rho(dz)$$
  - Therefore ($\rho(\mathbb{R}^d) = 1$):
$$(\hat{a}_\rho(x) - \hat{a}_\rho(y)) \cdot (x-y) \leq -\lambda \|x-y\|^2$$

Existence: sketch of the proof

- The velocity $\hat{a}_\rho$ satisfies an OSL estimate
- We consider the case of a finite sum of Dirac masses:
  - Let $\rho^{\mathrm{ini},N} = \sum_{i=1}^N m_i \delta_{x_i}, \quad \sum_{i=1}^N m_i = 1$
  - We look for a solution given by $\rho^N(t,x) = \sum_{i=1}^N m_i \delta_{x_i(t)}$
  - We compute the associated velocity $\hat{a}_{\rho^N}$. It satisfies an OSL estimate so that the associated Filippov's characteristic $\widehat{X}^N$ exist and are unique. The Dirac masses follow these characteristics.
  - Next, we define $\tilde{\rho}^N = \widehat{X}^N{}_{\#}\rho^{\mathrm{ini},N}$. By construction, this measure satisfies the equation:

$$\partial_t \tilde{\rho}^N + \mathrm{div}\left(\hat{a}_{\rho^N} \tilde{\rho}^N\right) = 0.$$

  - We proove that $\hat{a}_{\tilde{\rho}^N} = \hat{a}_{\rho^N}$

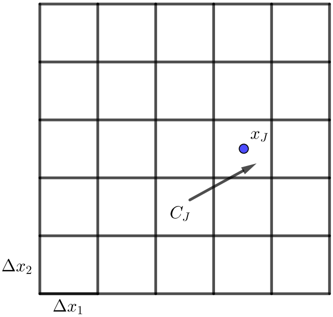
Existence: sketch of the proof

- The velocity $\hat{a}_\rho$ satisfies an OSL estimate
- We consider the case of a finite sum of Dirac masses:
- The result is extended to the general case by passing to the limit. We consider an initial condition ρ^{ini} that we approximate by a finite sum of Dirac masses $\rho^{\mathrm{ini},N}$ such that $d_W(\rho^{\mathrm{ini},N}, \rho^{\mathrm{ini}}) \to 0$.

Discretization with cartesian meshes

- We denote by Δt the time step and by Δx_i the space step along the i-th direction, $i = 1, \ldots, d$.
- For $J \in \mathbb{Z}^d$, we denote by x_J the center of the cell J.
- For $\rho^{\mathrm{ini}} \in \mathcal{P}_2(\mathbb{R}^d)$, we define, for $J \in \mathbb{Z}^d$, the initial condition in the following way:

$$\rho_J^0 = \int_{C_J} \rho^{\mathrm{ini}}(dx) \geq 0.$$



An upwind scheme

We consider the following upwind scheme:

$$\rho_J^{n+1} = \rho_J^n - \sum_{i=1}^{d} \frac{\Delta t}{\Delta x_i} \left((a_{iJ}^n)^+ \rho_J^n + (a_{iJ}^n)^- \rho_J^n - (a_{iJ+e_i}^n)^- \rho_{J+e_i}^n - (a_{iJ-ei}^n)^+ \rho_{J-e_i}^n \right)$$

where,

$$(a)^+ = \max\{0, a\}, \qquad (a)^- = \max\{0, -a\}$$

The discrete velocity is defined by:

$$a_{iJ}^n = -\sum_{K\in\mathbb{Z}^d} \rho_K^n \widehat{\partial_{x_i} W}(x_J - x_K),$$

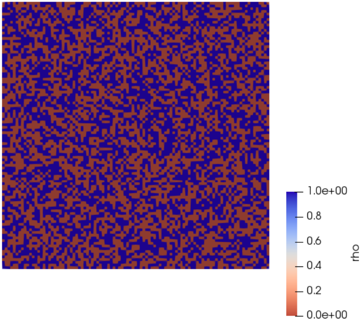
Properties of the scheme

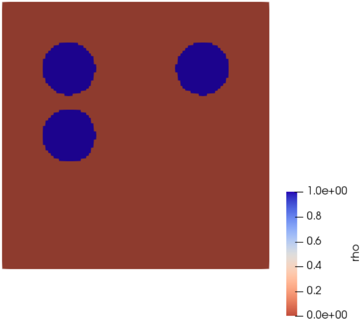
- Mass conservation.
- Bounded velocity: $|a_{iJ}^n| \leq w_\infty$, where $|\widehat{\nabla W}| \leq w_\infty$.
- Positivity: by induction, assuming the CFL $w_\infty \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \leq 1$ and writing the scheme as:

$$\rho_J^{n+1} = \rho_J^n \left[1 - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} |a_{iJ}^n|\right] + \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \rho_{J+e_i}^n (a_{iJ+e_i}^n)^- + \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \rho_{J-e_i}^n (a_{iJ-ei}^n)^+$$

- Conservation of the center of mass.

Some examples





Convergence order of the scheme

Theorem (Delarue, Lagoutière, Vauchelet)

For $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$, we denote by $\rho = (\rho(t))_{t\geq 0}$ the unique measure solution to the aggregation equation.
Assume that W satisfies **(A0)**-**(A3)** *and that the CFL condition, $w_\infty \sum_{i=1}^{d} \frac{\Delta t}{\Delta x_i} \leq 1$, holds. Defining $((\rho_J^n)_{J\in\mathbb{Z}^d})_{n\in\mathbb{N}}$ with the upwind scheme and $\rho_{\Delta x}^n$ the associated measure,*

$$\rho_{\Delta x}^n = \sum_{J\in\mathbb{Z}^d} \rho_J^n \delta_{x_J},$$

there exists a constant $C > 0$, depending only on λ, w_∞ and d, such that, for all $n \in \mathbb{N}$,

$$d_W(\rho(t^n), \rho_{\Delta x}^n) \leq C e^{|\lambda|(1+\Delta t)t^n} \left(\sqrt{t^n \Delta x} + \Delta x\right).$$

Toy problem

We consider the following initial condition ρ^{ini}:

$$\rho^{\mathrm{ini}} = \frac{1}{2}\delta_{x_-(0)} + \frac{1}{2}\delta_{x_+(0)},$$

where $x_-(0) = -1/2$ et $x_+(0) = 1/2$. The exact solution is of the form:

$$\rho(t,x) = \frac{1}{2}\delta_{x_-(t)} + \frac{1}{2}\delta_{x_+(t)},$$

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for the pointy potential $W = \|x\|$:

$$\begin{cases} x_-(t) = x_-(0) + \dfrac{1}{2}t, \\ x_+(t) = x_+(0) - \dfrac{1}{2}t, \end{cases}$$

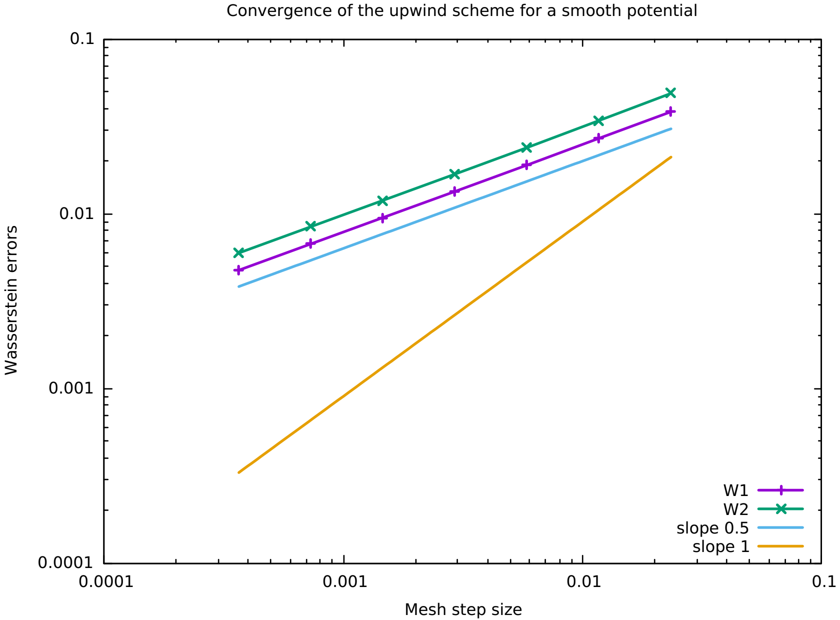
for $t < 1$ and x_- and x_+ are glued together at 0 for $t \geq 1$.

for the smooth potential $W = \frac{1}{2}\|x\|^2$:

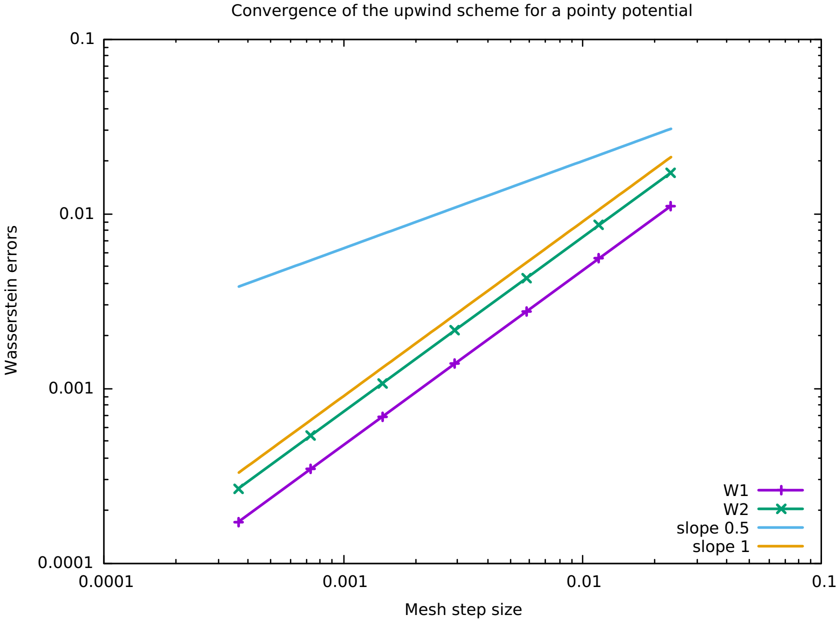
$$\begin{cases} x_-(t) = x_-(0)e^{-t}, \\ x_+(t) = x_+(0)e^{-t}, \end{cases}$$

for all $t > 0$.

Convergence results - smooth potential



Convergence results - pointy potential



Schemes for non-cartesian meshes

We consider two different schemes that can both be written as:

$$\rho_K^{n+1} = \rho_K^n - \frac{\Delta t}{|K|} \sum_{L \in \mathcal{V}(K)} |L \cap K| g(\rho_K^n, \rho_L^n, \nu_{KL}).$$

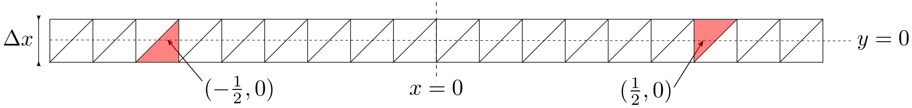
- *Lax-Friedrichs*:

$$g(\rho_K^n, \rho_L^n, \nu_{KL}) = \frac{1}{2}\left(\rho_K^n a_K^n \cdot \nu_{KL} + \rho_L^n a_L^n \cdot \nu_{KL} + a_\infty(\rho_L^n - \rho_K^n)\right).$$

- *Upwind*:

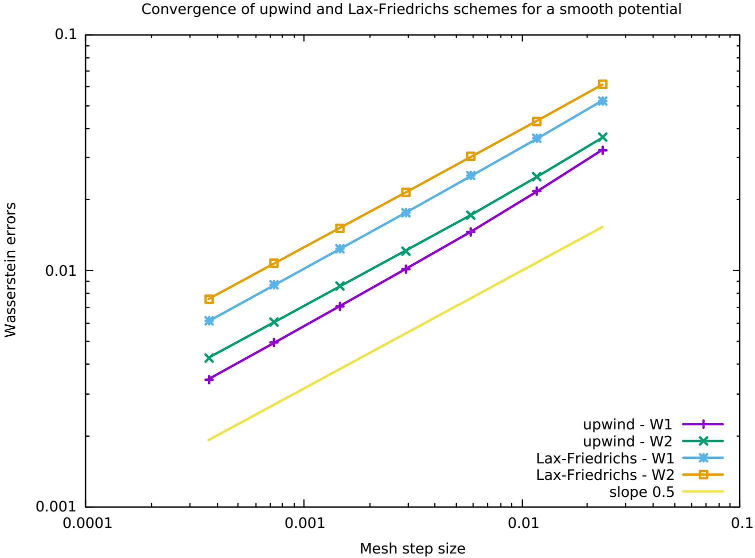
$$g(\rho_K^n, \rho_L^n, \nu_{KL}) = \rho_K^n (a_K^n \cdot \nu_{KL})^+ - \rho_L^n (a_L^n \cdot \nu_{KL})^-.$$

The toy problem in two dimensions

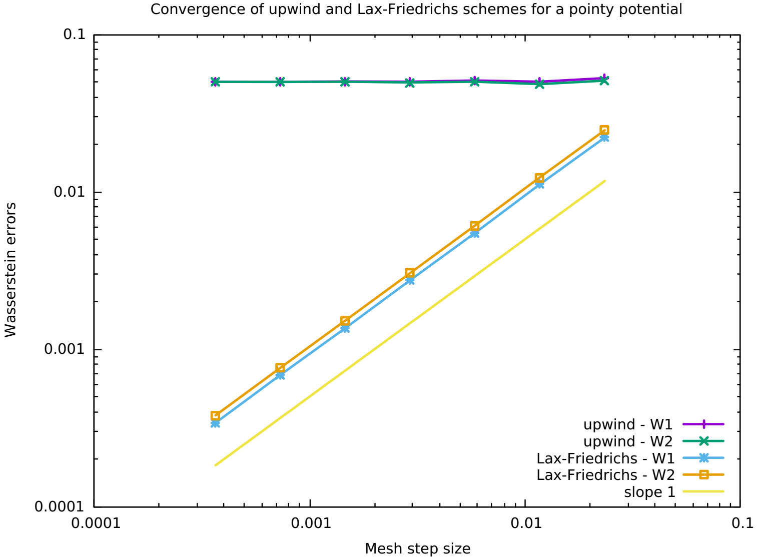


- Computation of the Wasserstein distances for both schemes (Lax-Friedrichs and upwind).
- Convergence test for the smooth and the pointy potential.
- Height of the domain is Δx, where Δx is the mesh step size.
- The exact solutions are the same as in the one dimensional setting.

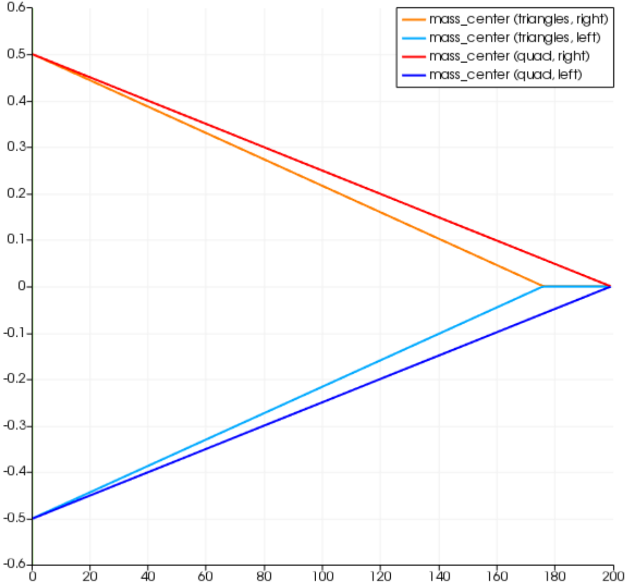
Convergence results - non-cartesian meshes, smooth potential



Convergence results - non-cartesian meshes, pointy potential

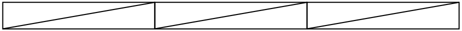


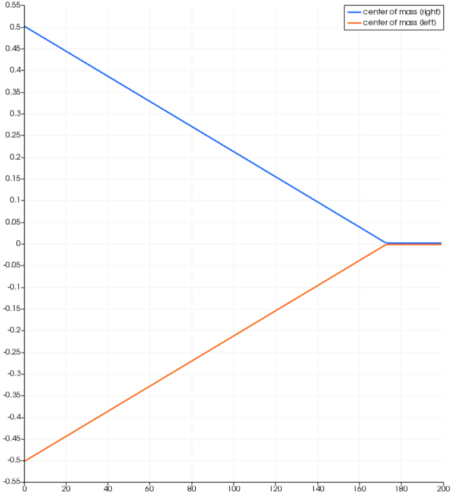
Position of the left and right center of mass in time



Toy problem with a horizontaly stretched mesh

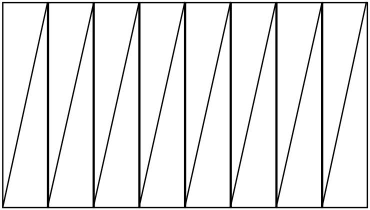
- We use the potential $W(x) = \|x\|$.
- The evolution in time of the position of the left and right center of mass are represented
- The mesh is streched horizontaly.

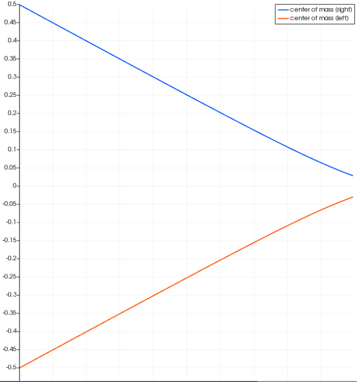


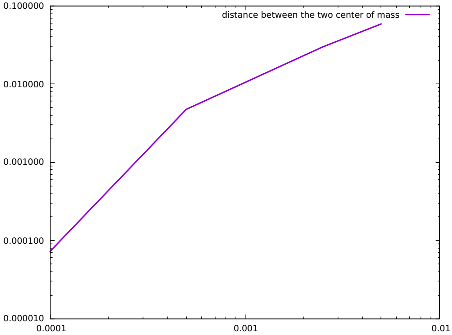


Toy problem with a verticaly stretched mesh

- Again, we use the potential $W(x) = \|x\|$.
- The mesh is streched verticaly.



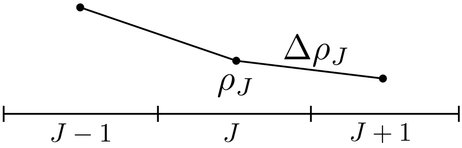




An order 2 Lax-Friedrichs type scheme

We reconstruct $(a\rho)_J$ and ρ_J in an affine way, on each cell, using a minmod.

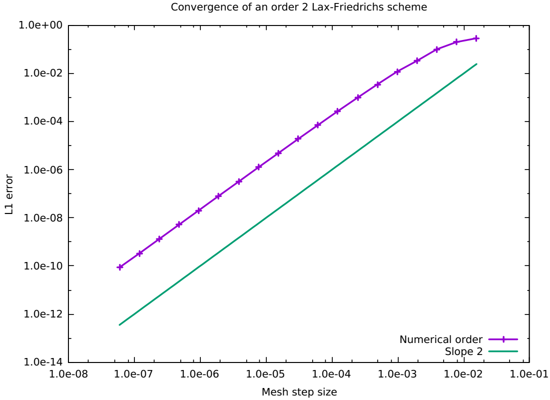
$$\begin{aligned}\rho_J^{n+1} = \rho_J^n - \frac{\Delta t}{2\Delta x}\Bigg[&\left(a\rho_J^n + \frac{\Delta a\rho_J^n}{2} + a\rho_{J+1}^n - \frac{\Delta a\rho_{J+1}^n}{2}\right)\\ &- \left(a\rho_{J-1}^n + \frac{\Delta a\rho_{J-1}^n}{2} + a\rho_J^n - \frac{\Delta a\rho_J^n}{2}\right)\Bigg]\\ +\frac{c\Delta t}{2\Delta x}\Bigg[&\left(\rho_{J+1}^n - \frac{\Delta\rho_{J+1}^n}{2} - \left(\rho_J^n + \frac{\Delta\rho_J^n}{2}\right)\right)\\ &- \left(\rho_J^n - \frac{\Delta\rho_J^n}{2} - \left(\rho_{J-1}^n + \frac{\Delta\rho_{J-1}^n}{2}\right)\right)\Bigg]\end{aligned}$$



Order with smooth data

The domain is the interval $[0, 1]$ and we choose:

- $\rho^{\mathrm{ini}}(x) = e^{-40(x-0.25)^2} + e^{-40(x-0.75)^2}$
- $W(x) = \|x\|^2$
- Mesh step sizes: $\Delta x = 2^{-k}, k \in \{6, \dots, 24\}$
- L^1 error.



Order for the one dimensional toy problem

Mesh step sizes: $\Delta x = 2^{-k}, k \in \{6, \ldots, 19\}$