東京工業大学
Tokyo Institute of Technology

The Silver SHARK workshop

Limiting-Free Discontinuity Capturing Schemes for Compressible Multi-components Flow

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Outlines

Section I. Introduction and Research Purpose

Section II. Formulation of Boundary Variation Diminishing Algorithm

Section III. Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

Section IV. Implementation on Compressible-multiphase flow

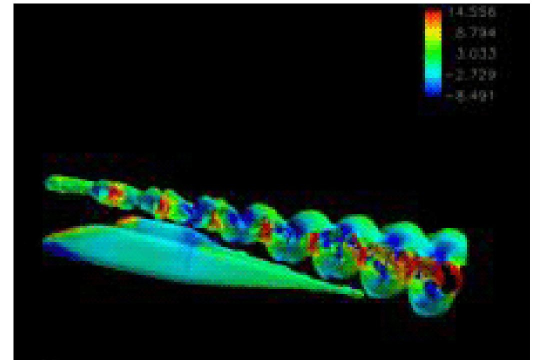
Section V. Works in progress

1.1Rearch Background

Flow Structures in Compressible Multi-components Flows

Smooth Solutions

1. Acoustic Waves
2. Turbulence
3. Vortex Dominated Flow
4. Rarefaction Fan



Vortex-dominated flow near helicopter (Advanced Dynamics Inc.)

F/A18-F in transonic flight (NASA Gallery)

Discontinuous Solutions

1. Shock Waves
2. Contact Discontinuities
3. Material Interface
4. Detonation Front

Numerical schemes should be able to solve both smooth and discontinuous solutions

Difficulties in Designing Numerical Schemes

Smooth Solution

High Order Schemes based on polynomials:
High order FVM, DG

Sufficient

Discontinuous Solution

High Order Schemes equipped Limiting Processes

Lots of efforts have been made so far

years

Finite Volume Method (FVM):

TVD ENO WENO WENOM WENOZ WENOCU TENO

High Local Reconstruction (HLR):

DG-TVB DG-WENO DG-AD DG-MOOD

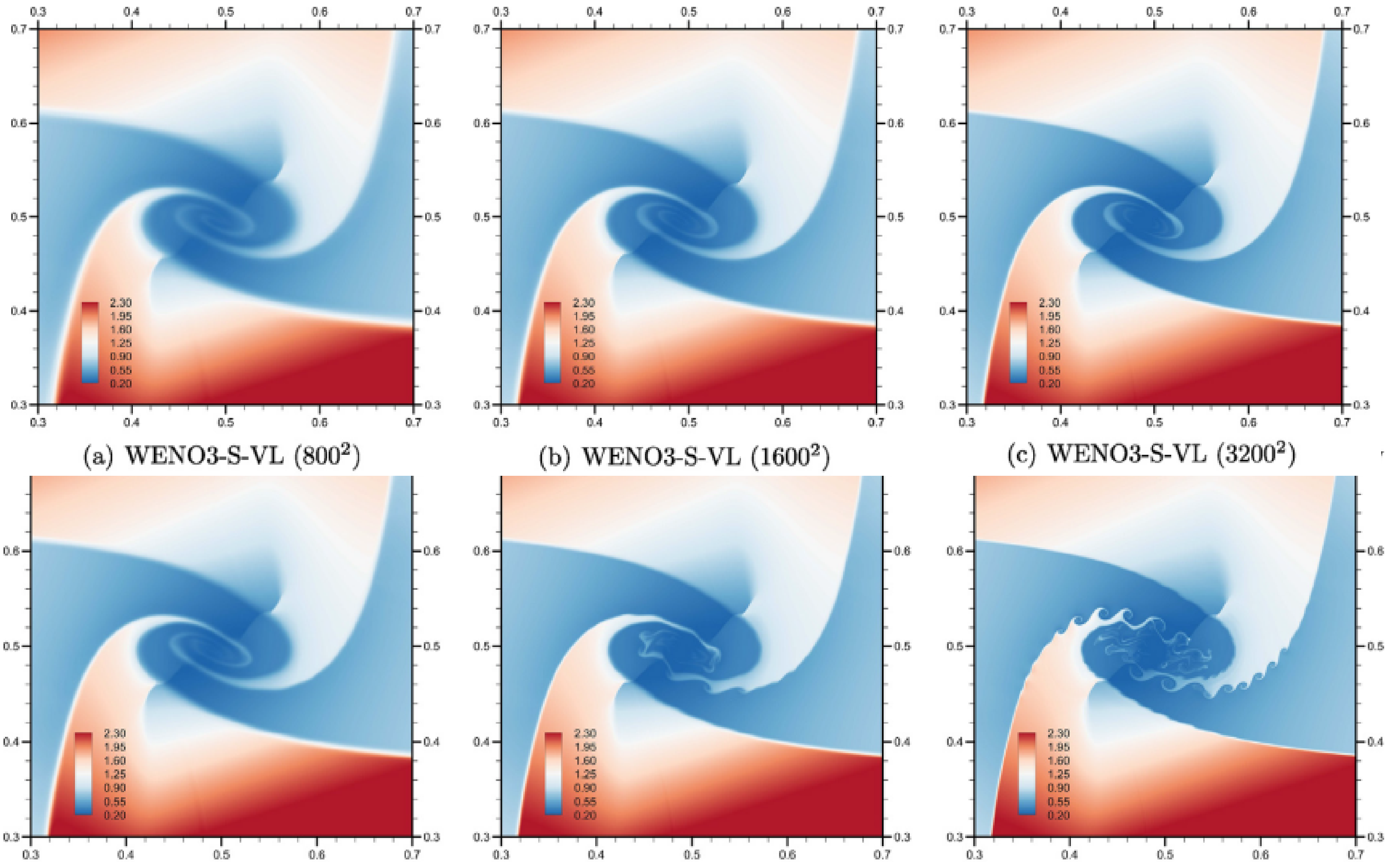
Above schemes try to solve inbuilt paradox of current high order schemes

Numerical oscillations

Numerical dissipation

The so-called high order schemes fail in some cases

2D Riemann Problem: Kelvin-Helmholtz instability



(a) WENO3-S-VL (800^2) (b) WENO3-S-VL (1600^2) (c) WENO3-S-VL (3200^2)

(d) WENO5-S-VL (800^2) (e) WENO5-S-VL (1600^2) (f) WENO5-S-VL (3200^2)

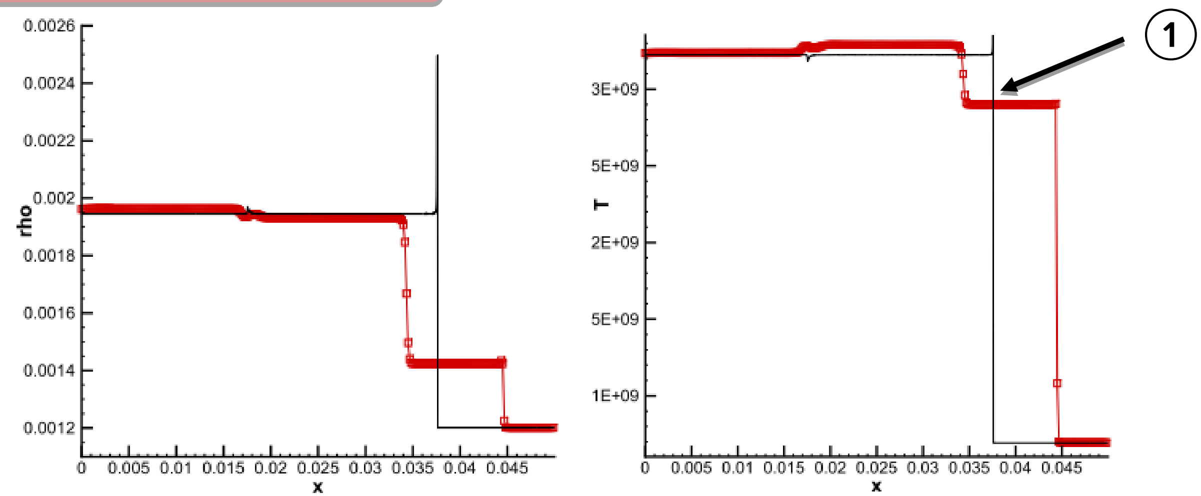
(San, *Computers & Fluids*, 2014)

1.1 Research Background

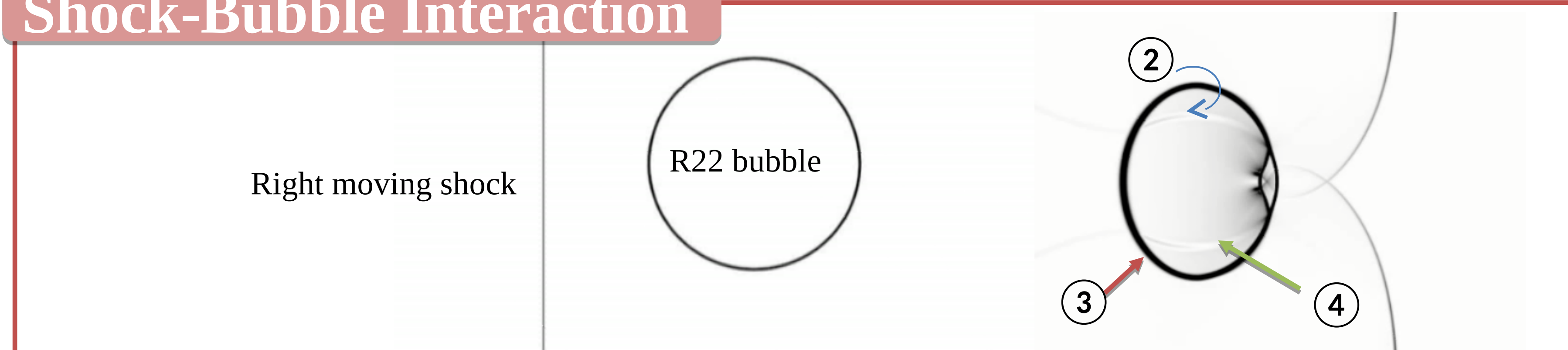
The so-called high order schemes fail in some cases

Stiff *C-J* detonation waves

5th order WENO scheme



Shock-Bubble Interaction



Issues:

① Wrong position of detonation front

② Un-resolved flow structures

③ Diffused material interface

④ Smeared waves

1.2 Research Purpose

New Schemes are Necessary

1. How to reduce numerical dissipation besides using high order schemes?

   -Boundary variation diminishing (BVD) algorithm

2. How to deal with discontinuous solutions?

   -Non-polynomial based reconstruction

Section II. Formulation of Boundary Variation Diminishing Algorithm

2.1 Formulation of Boundary Variation Diminishing Algorithm

Finite Volume Method

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$$

$$\bar{q}_i(t) \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx$$

$$\frac{\partial \bar{q}(t)}{\partial t} = -\frac{1}{\Delta x}(\tilde{f}_{i+1/2} - \tilde{f}_{i-1/2}),$$

$$\tilde{f}_{i+1/2} = f_{i+1/2}^{\mathrm{Riemann}}\boxed{(q_{i+1/2}^{L}, q_{i+1/2}^{R})}$$

Lots of schemes have been made so far to reconstruct values at cell faces

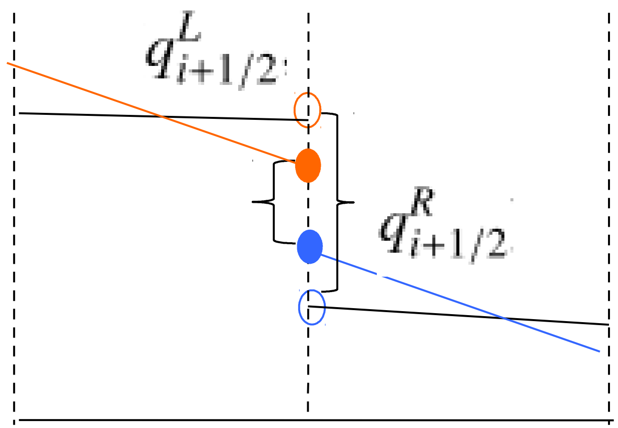
$$f_{i+1/2}^{\mathrm{Riemann}}(q_{i+1/2}^{L}, q_{i+1/2}^{R}) = \frac{1}{2}\left(f(q_{i+1/2}^{L}) + f(q_{i+1/2}^{R})\right) - \frac{|a_{i+1/2}|}{2}\left(\boxed{q_{i+1/2}^{R} - q_{i+1/2}^{L}}\right),$$

Numerical dissipation term

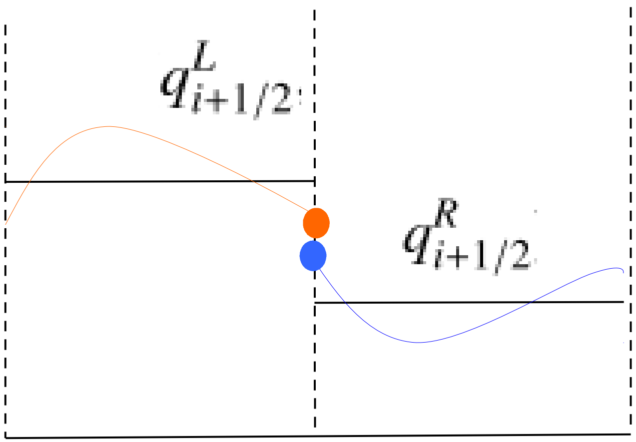
2.1 Formulation of Boundary Variation Diminishing Algorithm

Reconstruction Processes

$$f^{\text{Riemann}}_{i+1/2}(q^L_{i+1/2}, q^R_{i+1/2}) = \frac{1}{2}\left(f(q^L_{i+1/2}) + f(q^R_{i+1/2})\right) - \frac{|a_{i+1/2}|}{2}\left(q^R_{i+1/2} - q^L_{i+1/2}\right),$$



MUSCL/Piecewise Constant (PC)



High order reconstruction

Higher order solution can minimize the variations at cell boundaries if solution is smooth

However, this may not be true for discontinuities.

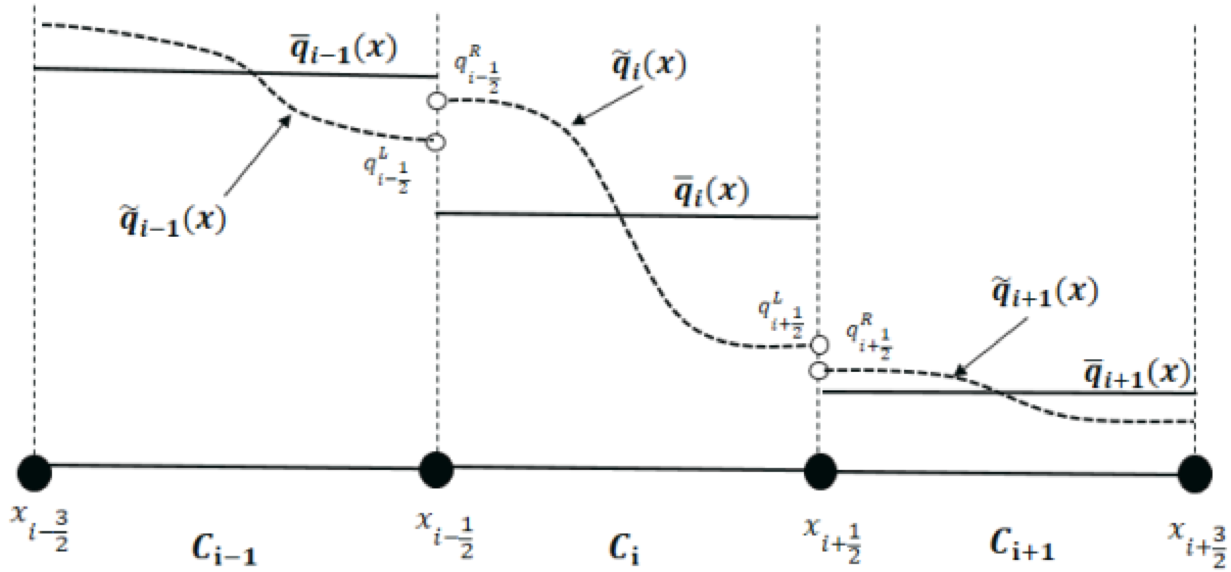
A non-polynomial reconstruction function will be considered to represent discontinuities

THINC Reconstruction

$$\tilde{q}_i(x)^{\mathrm{THINC}} = q_{min} + \frac{q_{max}}{2}\left(1+\theta\ \tanh\left(\beta\left(\frac{x-x_{i-1/2}}{x_{i+1/2}-x_{i-1/2}} - \tilde{x}_i\right)\right)\right)$$

where $q_{max} = max(q_{i-1}, q_{i+1}) - q_{min}$ and $\theta = sgn(q_{i+1} - q_{i-1})$. The jump thickness is controlled by the parameter β.

$$\bar{q}_i = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}_i(x)\, dx.$$



1.Sharpness can be controlled by beta

2.Bounded function

2.1 Formulation of Boundary Variation Diminishing Algorithm

BVD algorithm

The reconstruction function (high order or jump-like THINC function) is determined through minimizing boundary variations. Thus numerical dissipation can be reduced.

$$f_{i+1/2}^{\text{Riemann}}(q_{i+1/2}^{L}, q_{i+1/2}^{R}) = \frac{1}{2}\left(f(q_{i+1/2}^{L}) + f(q_{i+1/2}^{R})\right) - \frac{|a_{i+1/2}|}{2}\left(q_{i+1/2}^{R} - q_{i+1/2}^{L}\right),$$

2.1 Formulation of Boundary Variation Diminishing Algorithm

Examples of BVD algorithms

Using polynomial-based schemes (like TVD, or WENO) and THINC as two candidate reconstructions

1. Compute the TBVs of the target cell i using WENO and THINC and its two neighboring cells respectively,

$$TBV_i^W = |q_{i-1}^W(x_{i-\frac{1}{2}}) - q_i^W(x_{i-\frac{1}{2}})| + |q_i^W(x_{i+\frac{1}{2}}) - q_{i+1}^W(x_{i+\frac{1}{2}})|$$

$$TBV_i^T = |q_{i-1}^T(x_{i-\frac{1}{2}}) - q_i^T(x_{i-\frac{1}{2}})| + |q_i^T(x_{i+\frac{1}{2}}) - q_{i+1}^T(x_{i+\frac{1}{2}})|.$$

2. Choose the reconstruction function for cell i by

$$q_i(x) = \begin{cases} q_i^T & \text{if } TBV_i^T < TBV_i^W, \\ q_i^W & \text{otherwise} \end{cases}$$

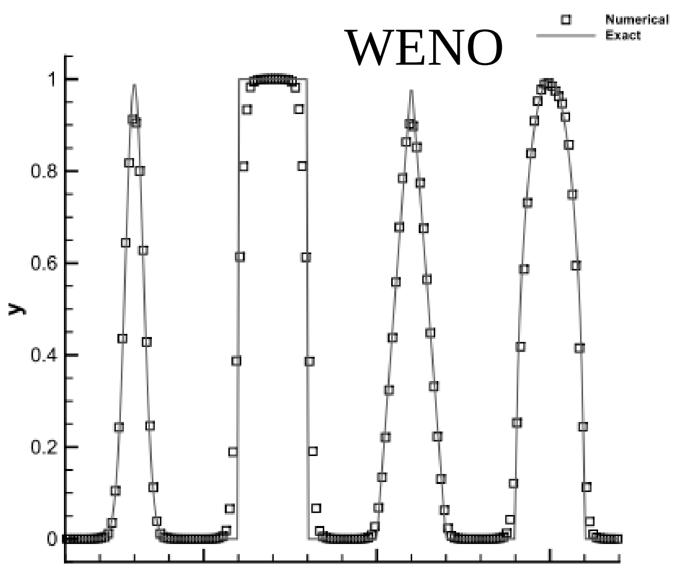
2.1 Formulation of Boundary Variation Diminishing Algorithm

Numerical Results

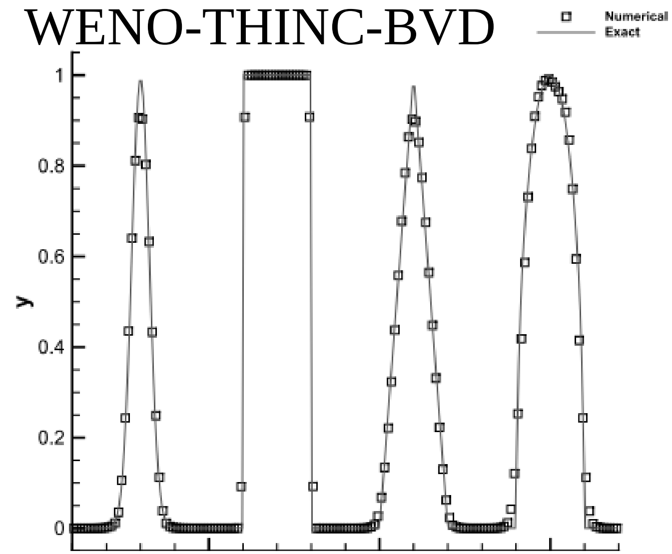
WENO WENO-THINC-BVD

L_1 error	L_1	L_1 error	L_1
2.14e-04		2.14e-04	
6.40e-06	5.07	6.40e-06	5.07
2.00e-07	5.00	2.00e-07	5.00
6.32e-09	4.99	6.32e-09	4.99
2.04e-10	4.95	2.04e-10	4.95

WENO

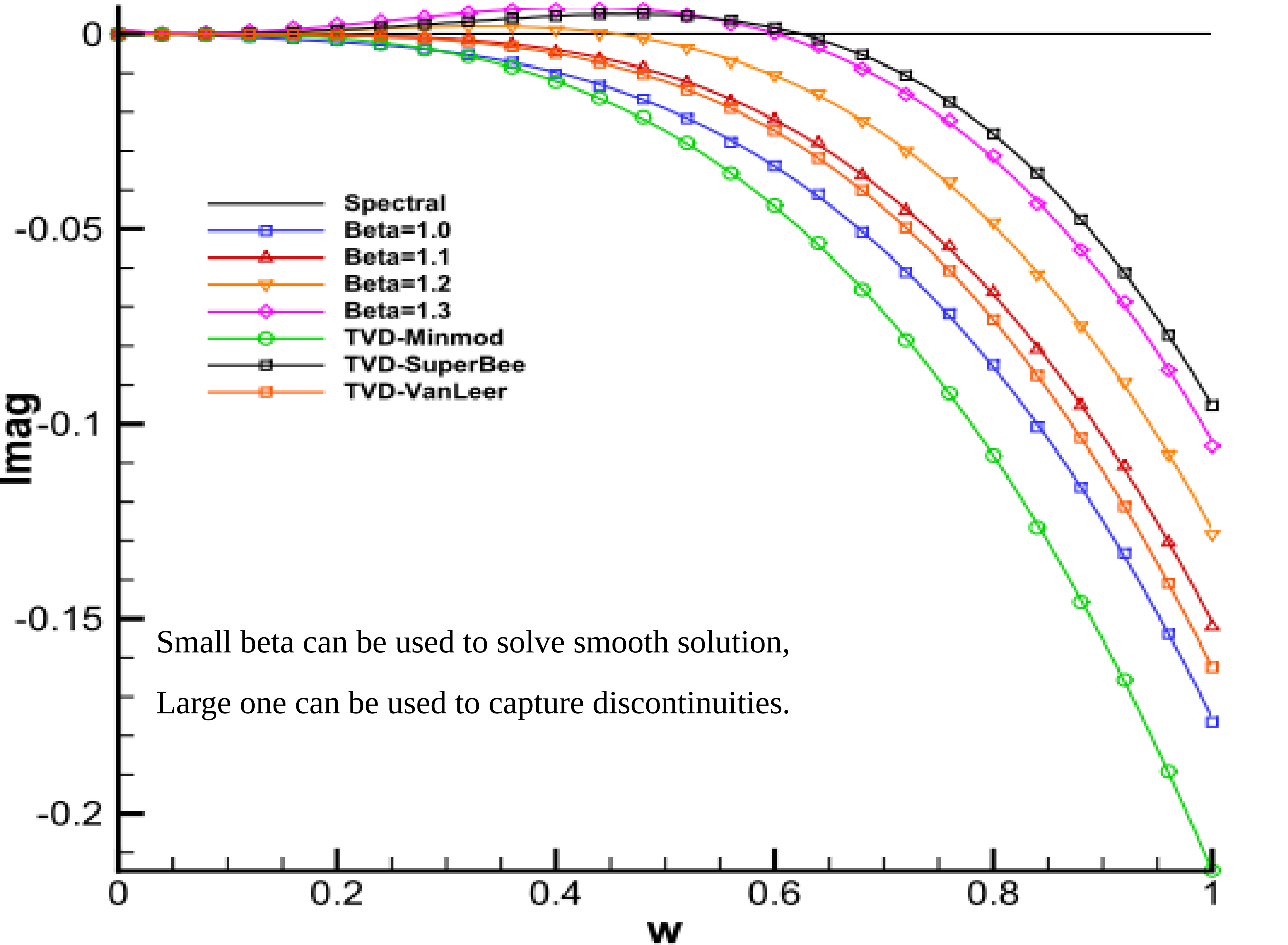


WENO-THINC-BVD



The proposed scheme achieves that for smooth solution high order reconstruction will be used while for discontinuities THINC function will be applied.

Section III. Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD


Spectral
Beta=1.0
Beta=1.1
Beta=1.2
Beta=1.3
TVD-Minmod
TVD-SuperBee
TVD-VanLeer
Imag
w
0
-0.05
-0.1
-0.15
-0.2
0
0.2
0.4
0.6
0.8
1
Small beta can be used to solve smooth solution,
Large one can be used to capture discontinuities.

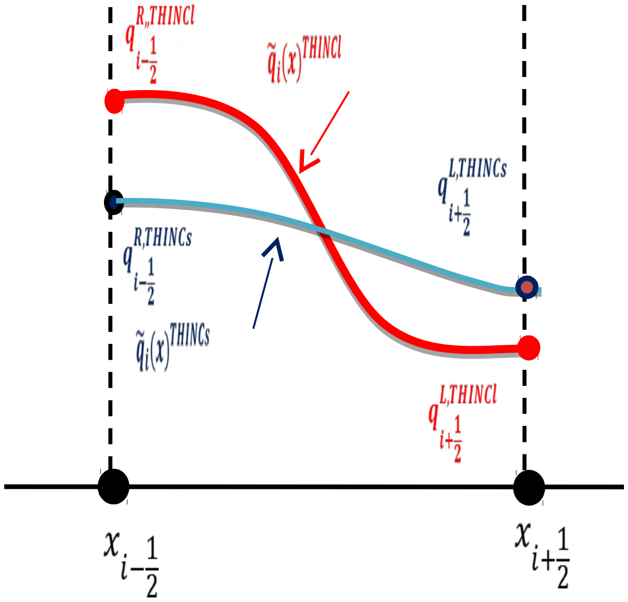
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

Adaptive THINC-BVD scheme

$$TBV_i^s = |q_{i-1/2}^{L,s} - q_{i-1/2}^{R,s}| + |q_{i+1/2}^{L,s} - q_{i+1/2}^{R,s}|,$$

$$TBV_i^l = |q_{i-1/2}^{L,l} - q_{i-1/2}^{R,l}| + |q_{i+1/2}^{L,l} - q_{i+1/2}^{R,l}|.$$

$$\tilde{q}_i^f(x) = \begin{cases} \tilde{q}_i^s(x) & \text{if } TBV_i^s < TBV_i^l \\ \tilde{q}_i^l(x) & \text{otherwise} \end{cases}$$



Very simple scheme

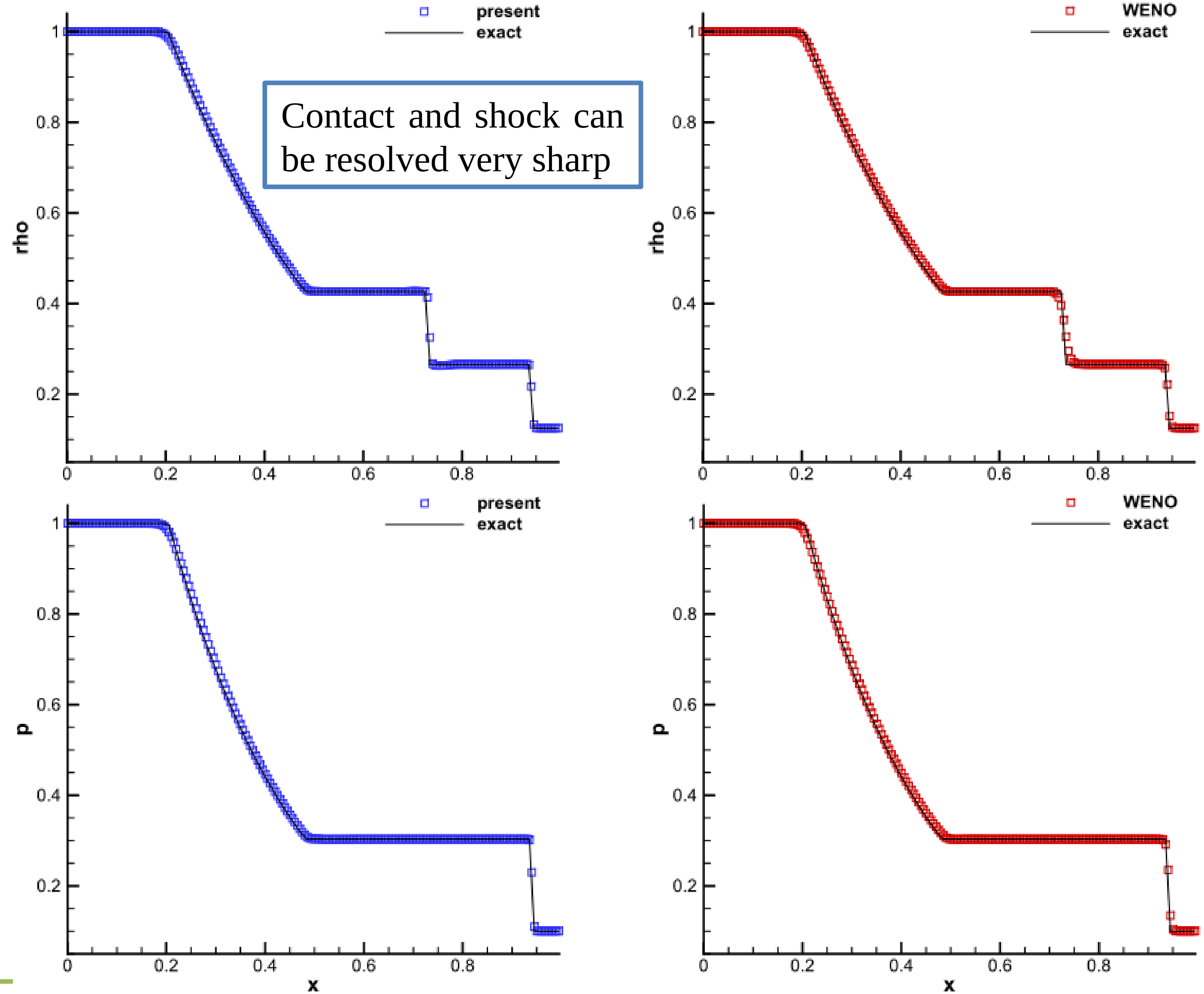
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

What's the performance of the new scheme

Schemes	Mesh	L_1 errors	L_1 order	L_∞ errors	L_∞ order
Minmod	40	4.547×10^{-2}		1.025×10^{-1}	
	80	1.337×10^{-2}	1.77	4.347×10^{-2}	1.23
	160	3.812×10^{-3}	1.81	1.795×10^{-2}	1.27
	320	1.031×10^{-3}	1.89	7.298×10^{-3}	1.30
Van Leer	40	2.101×10^{-2}		5.151×10^{-2}	
	80	5.568×10^{-3}	1.92	1.952×10^{-2}	1.40
	160	1.408×10^{-3}	1.98	7.302×10^{-3}	1.42
	320	3.423×10^{-4}	2.04	2.715×10^{-3}	1.43
Superbee	40	2.134×10^{-2}		6.087×10^{-2}	
	80	9.024×10^{-3}	1.24	3.443×10^{-2}	0.82
	160	2.642×10^{-3}	1.77	1.487×10^{-2}	1.21
	320	7.159×10^{-4}	1.88	6.651×10^{-3}	1.16
THINC-BVD	40	1.518×10^{-2}		4.721×10^{-2}	
	80	3.766×10^{-3}	2.01	1.821×10^{-2}	1.37
	160	8.969×10^{-4}	2.07	6.866×10^{-3}	1.41
	320	2.198×10^{-4}	2.03	2.545×10^{-3}	1.43
WENO	40	4.473×10^{-5}		8.799×10^{-5}	
	80	1.396×10^{-6}	5.00	2.822×10^{-6}	4.96
	160	4.361×10^{-8}	5.00	8.487×10^{-8}	5.06
	320	1.361×10^{-9}	5.00	2.544×10^{-9}	5.06

3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

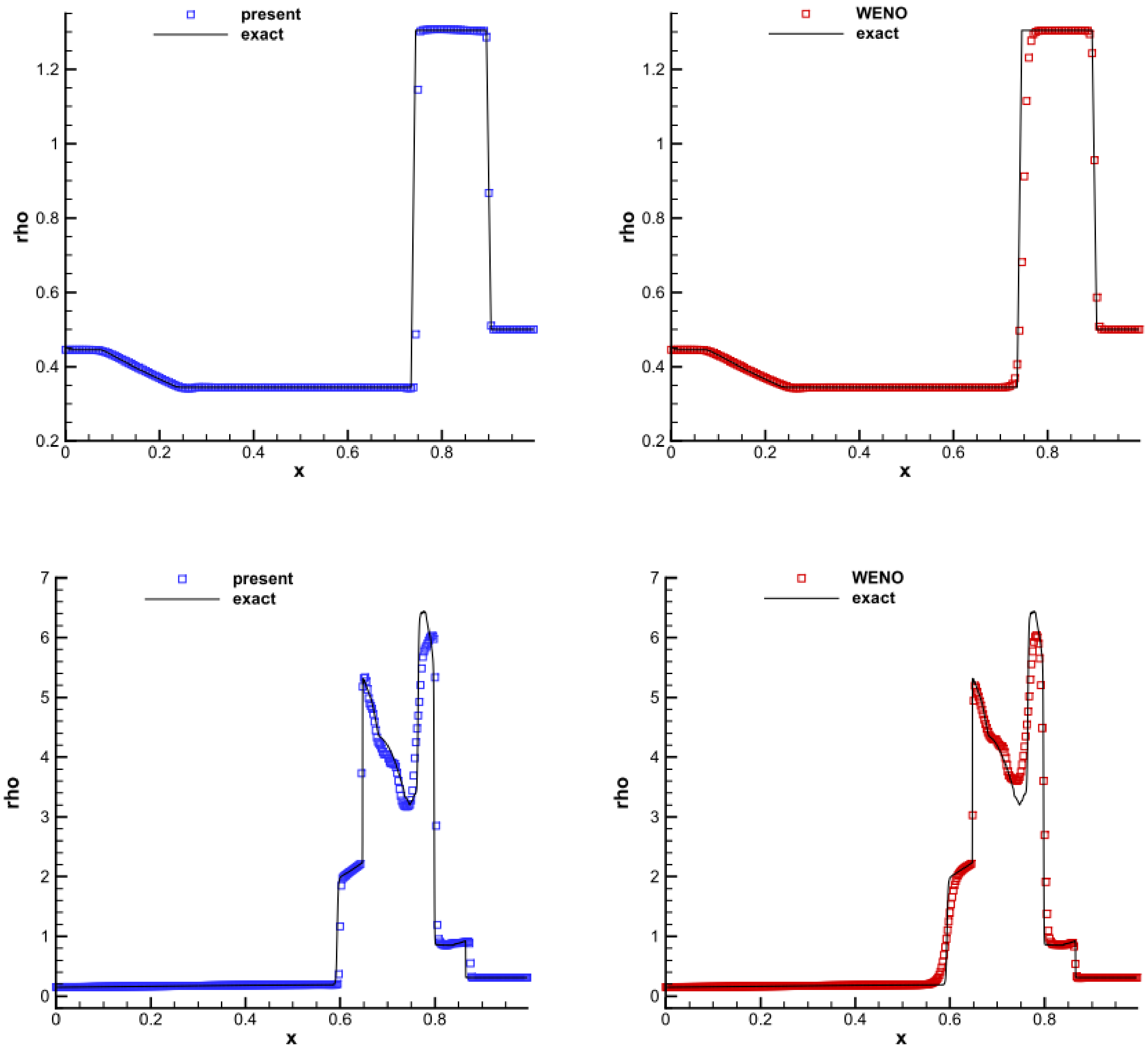
What's the performance of the new scheme



Contact and shock can be resolved very sharp

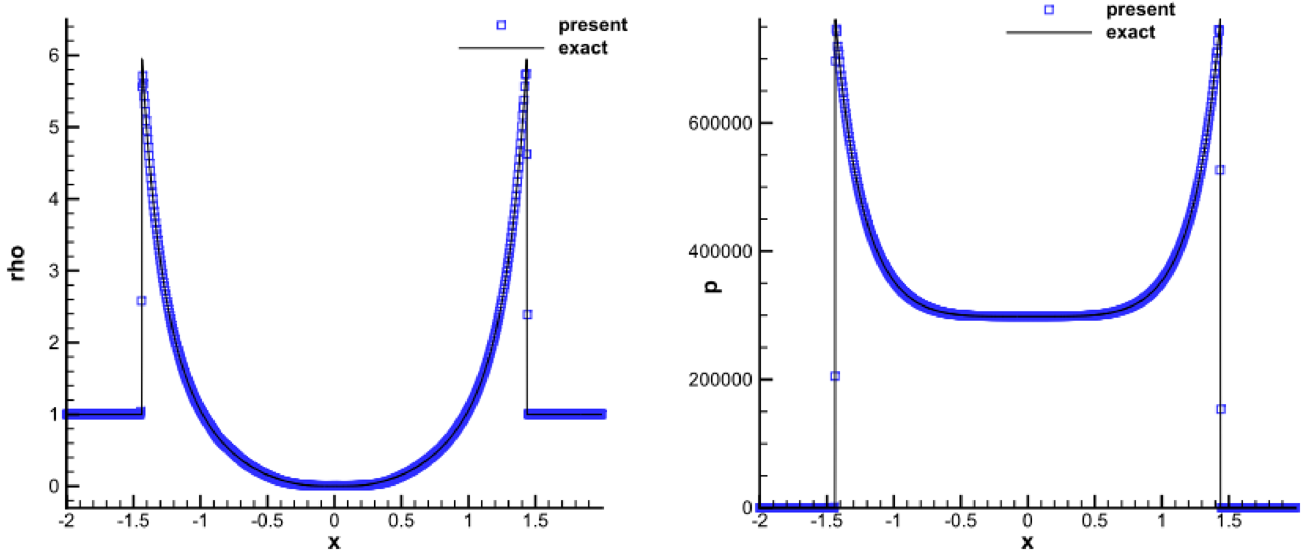
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

What’s the performance of the new scheme

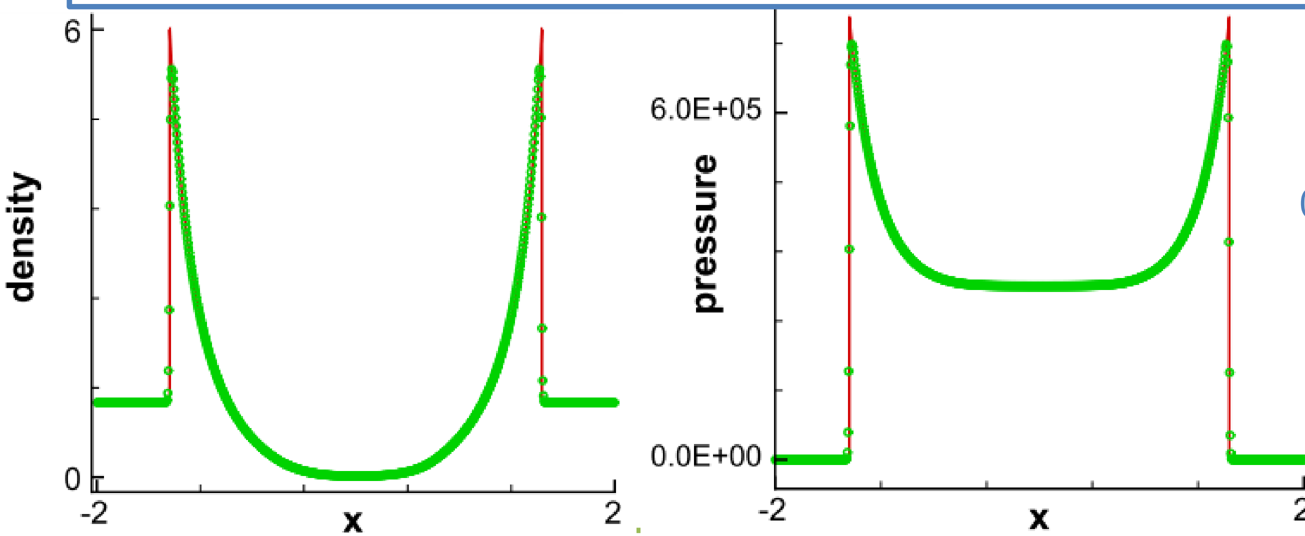


3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

What's the performance of the new scheme



High order schemes should be equipped with positivity-preserving techniques

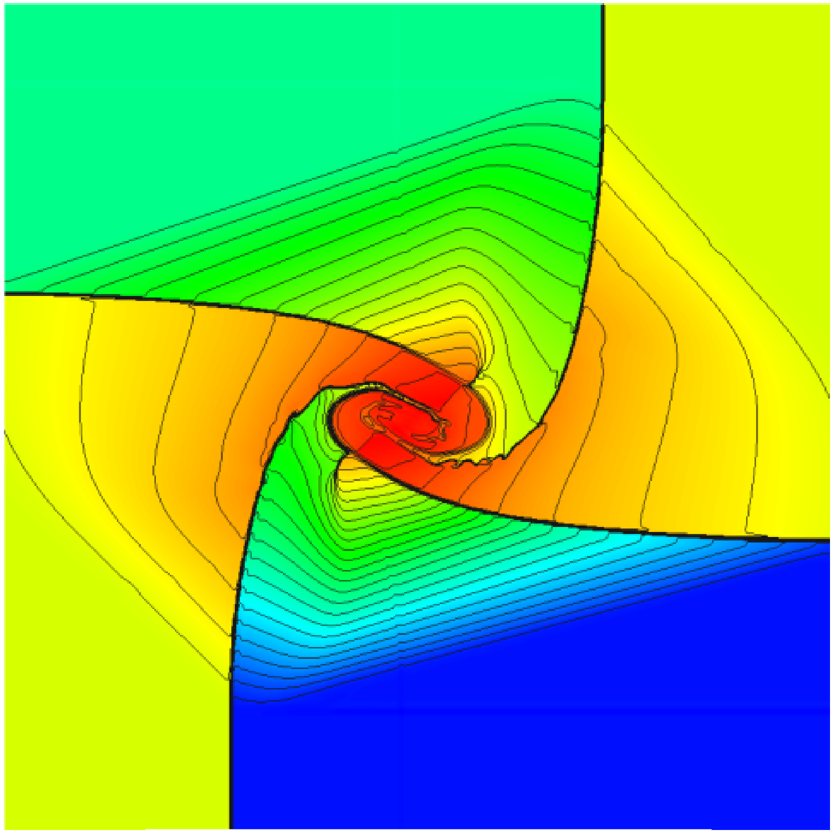


(Zhang,*JCP*,2012)

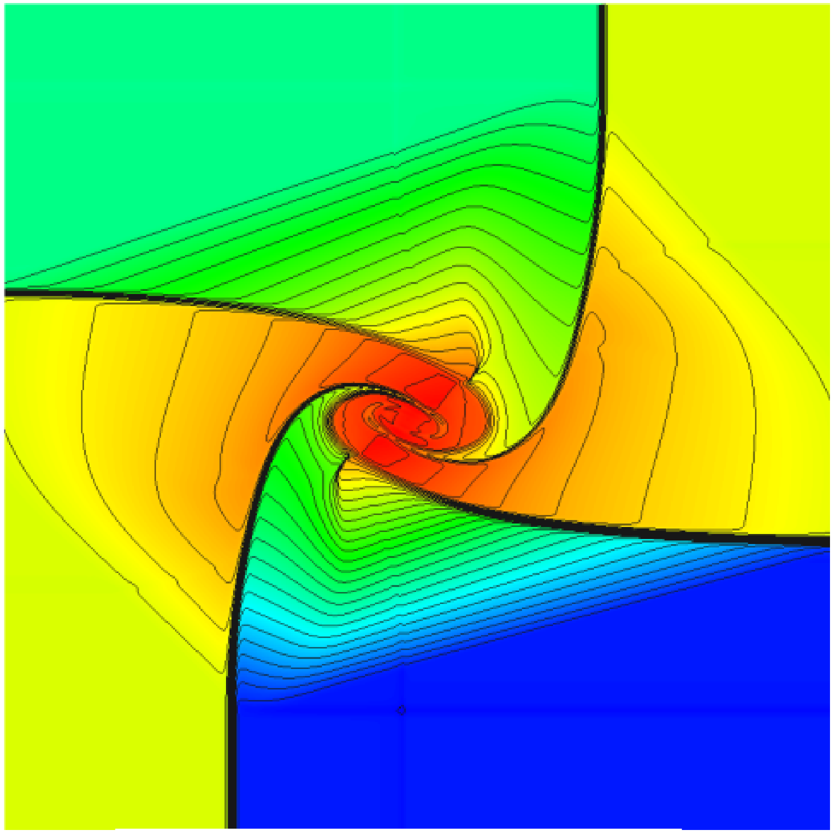
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

What's the performance of the new scheme

Numerical results: 600x600



Adaptive THINC-BVD



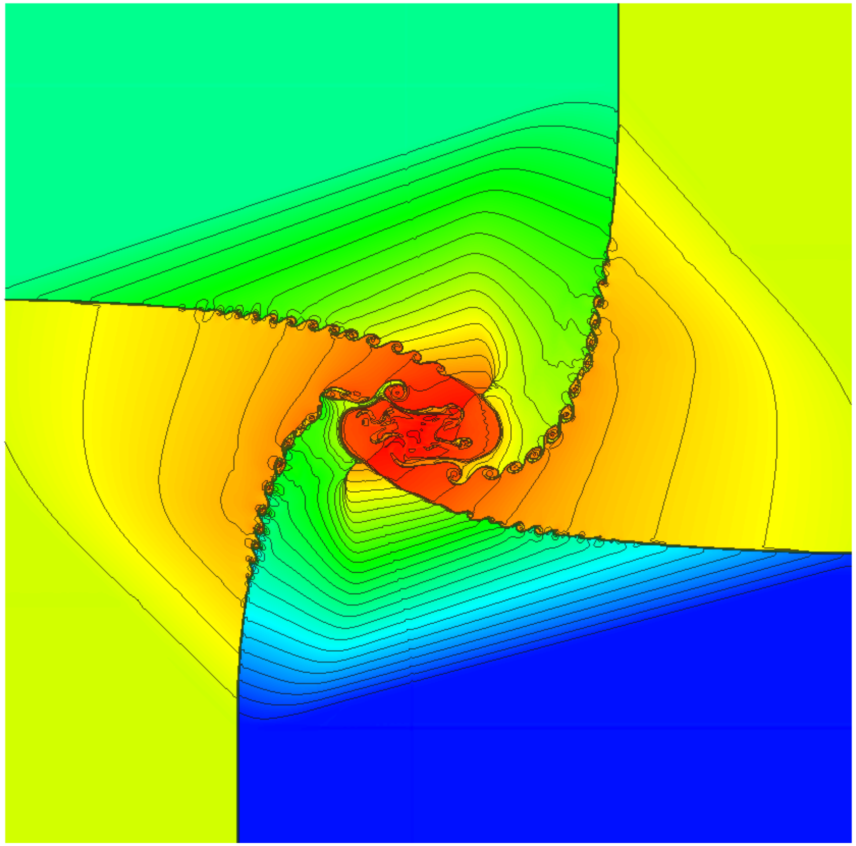
5th WENO

3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

What's the performance of the new scheme

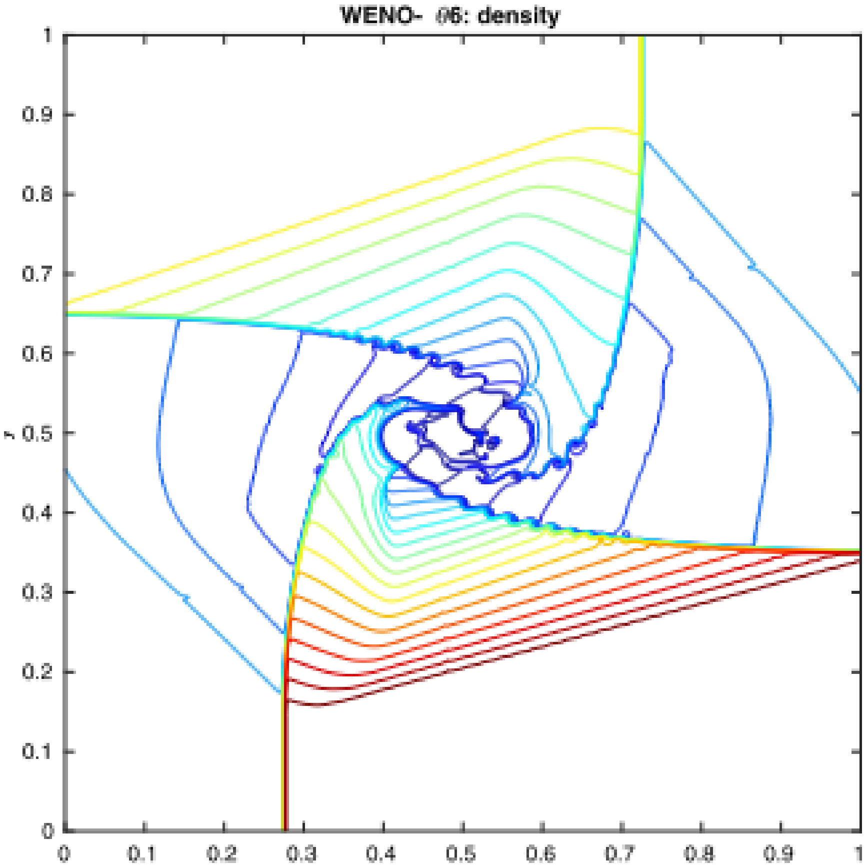
(Jung, *Adv Comput Math*, 2017)

1200x1200



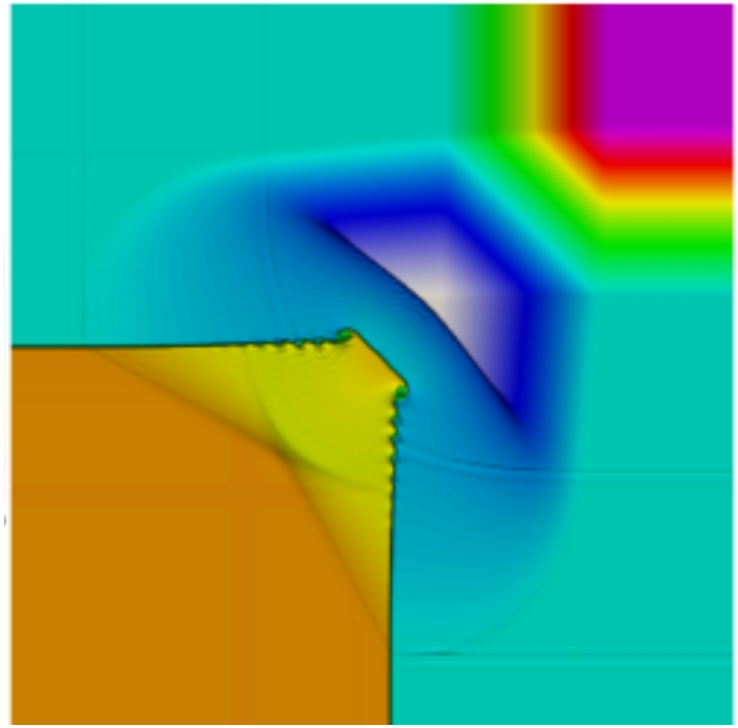
THINC-BVD

1200x1200

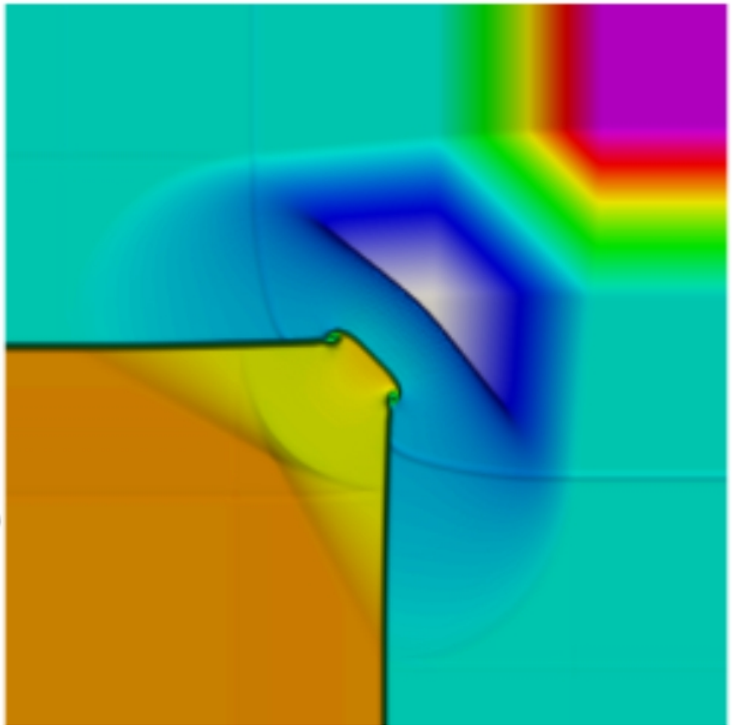


3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

What's the performance of the new scheme



THINC-BVD

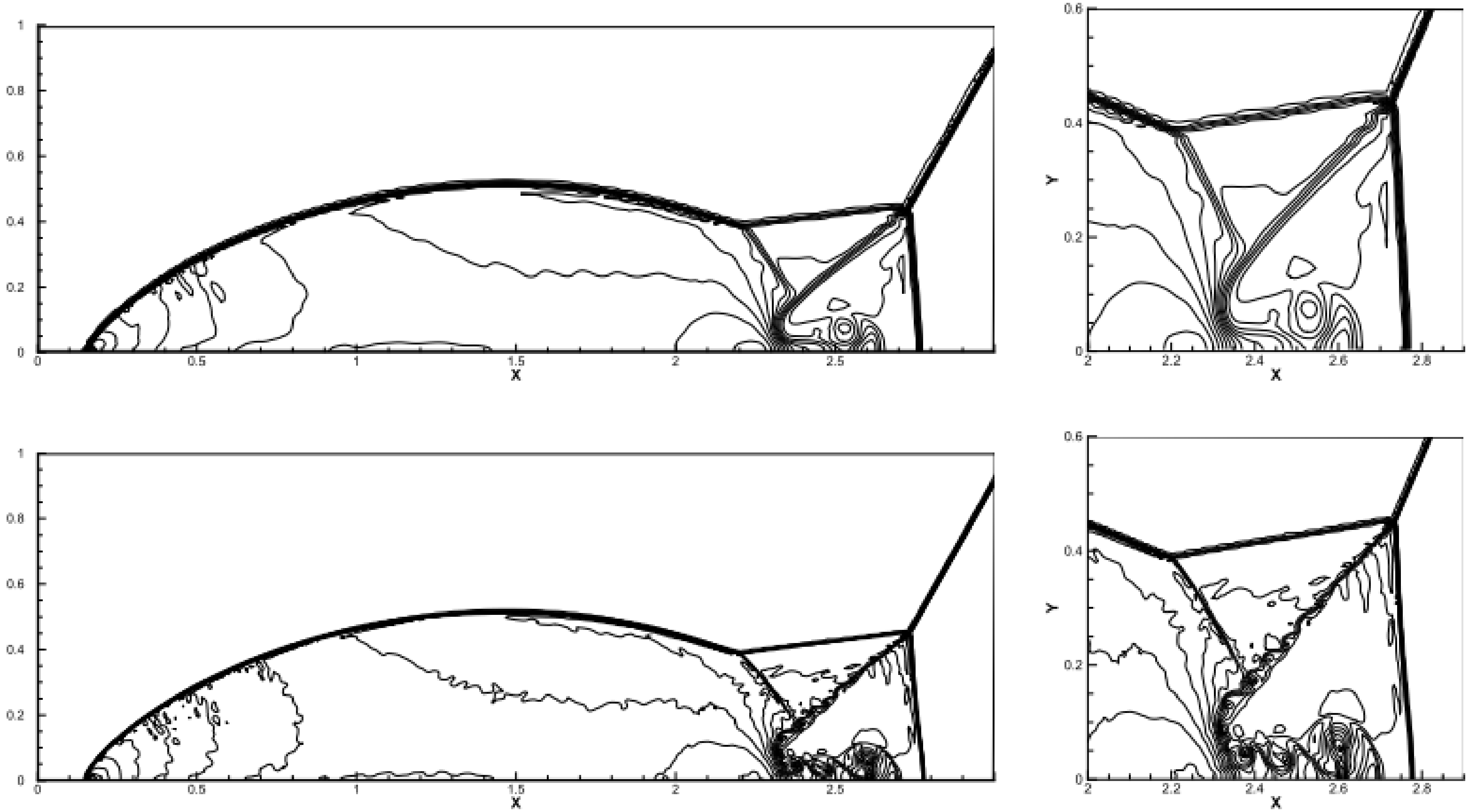


5^{th} WENO

3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

What's the performance of the new scheme

5th WENO

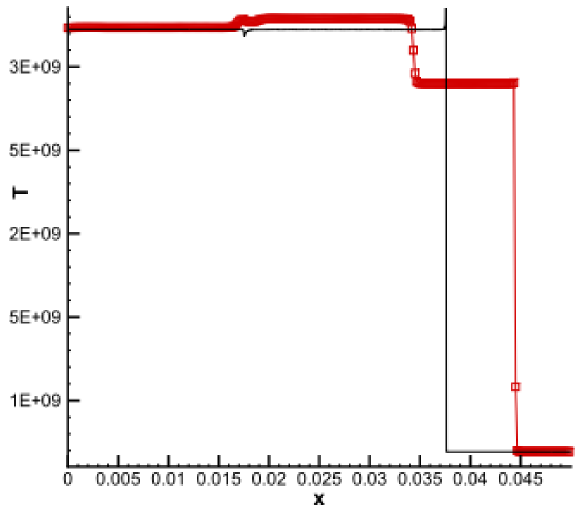


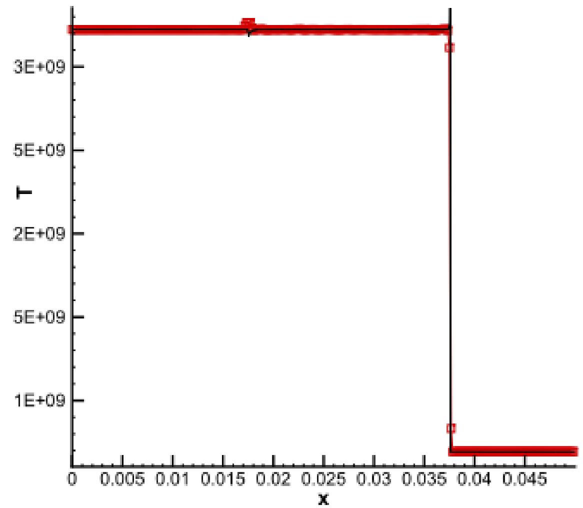
THINC-BVD

3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

C-J detonation wave with the Heaviside model

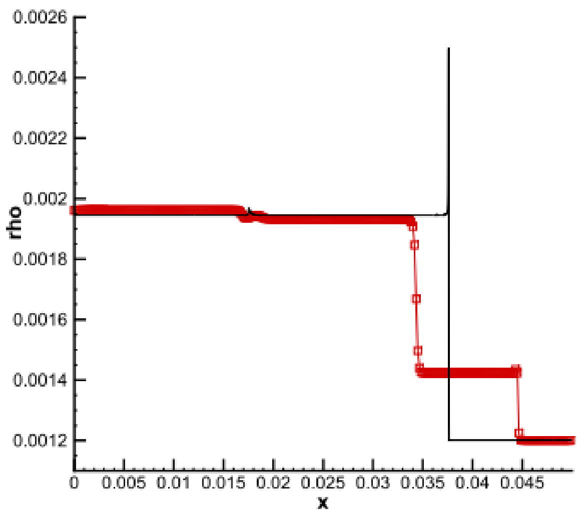
$$K(T) = -\frac{1}{\xi} H(T - T_{ign}),$$

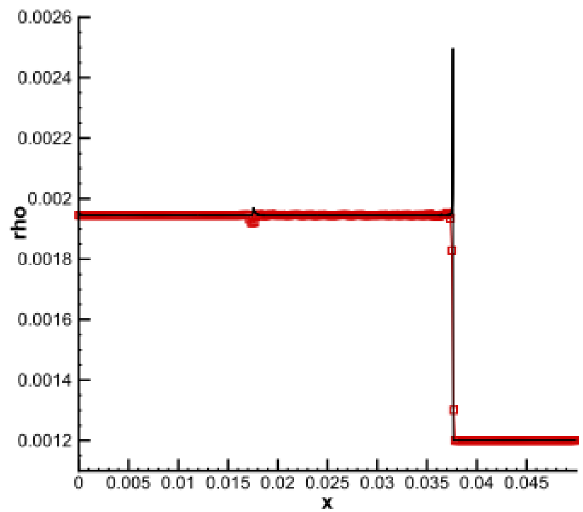




5th WENO

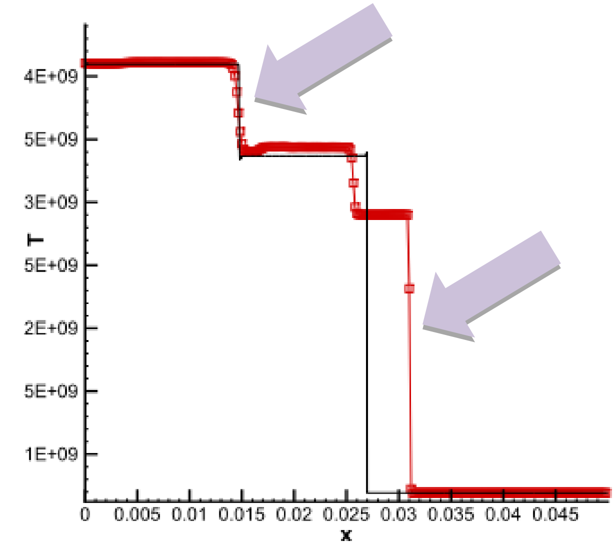
Adaptive THINC-BVD

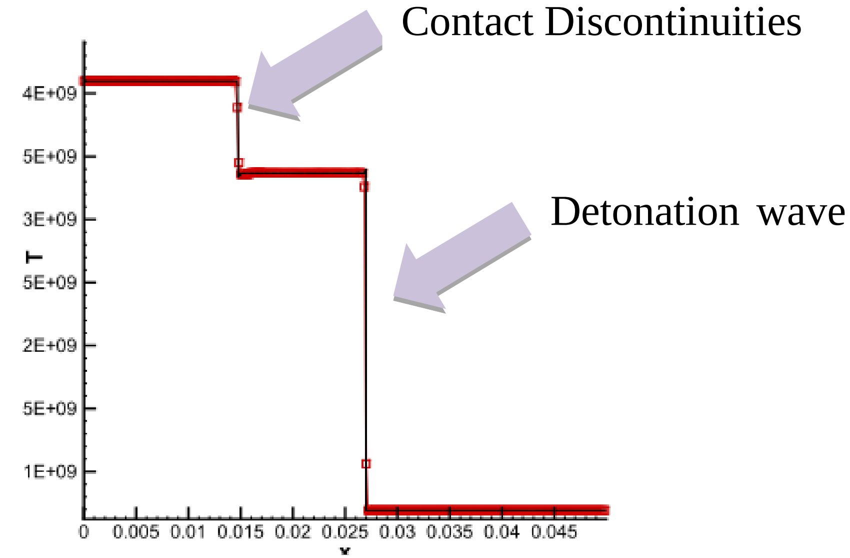




3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

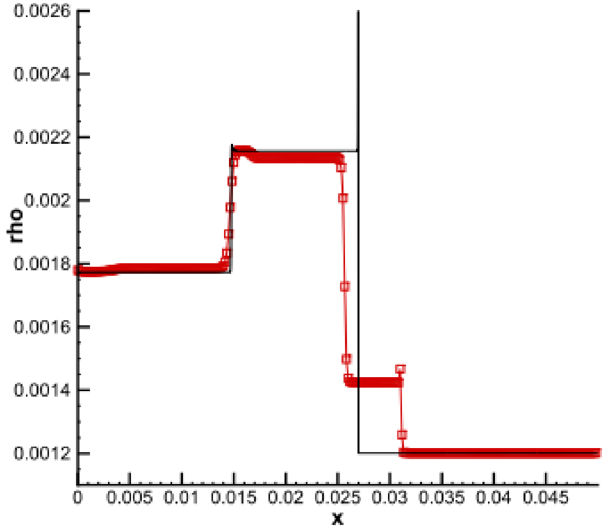
A strong detonation

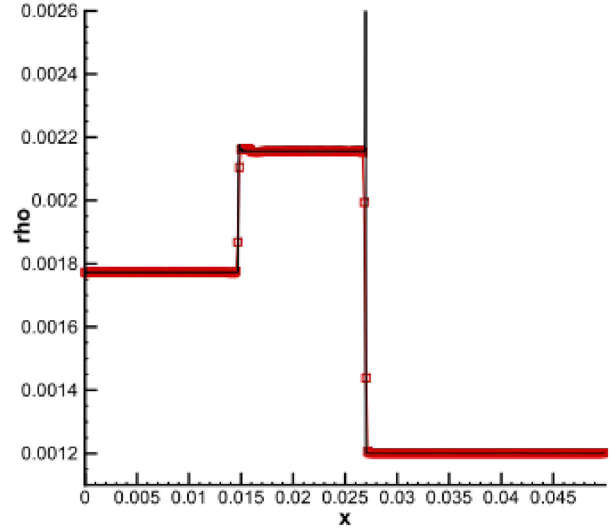




5th WENO

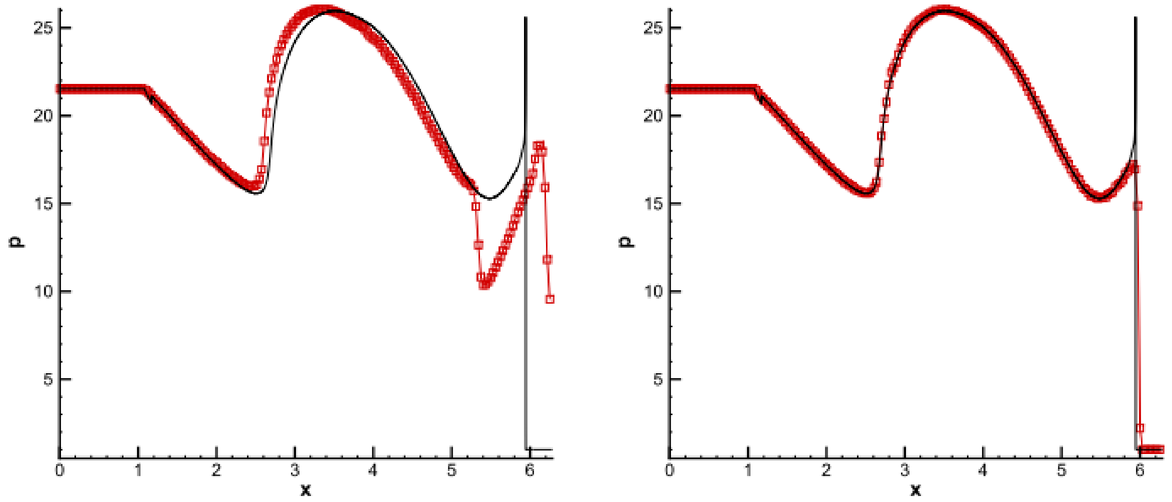
Adaptive THINC-BVD





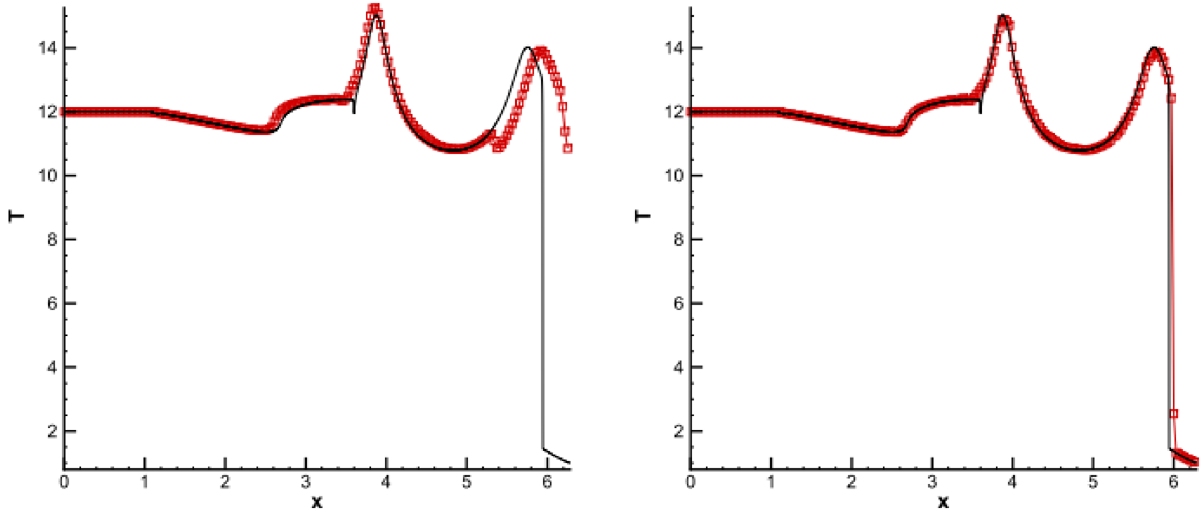
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

Interaction between a detonation wave and an oscillatory profile



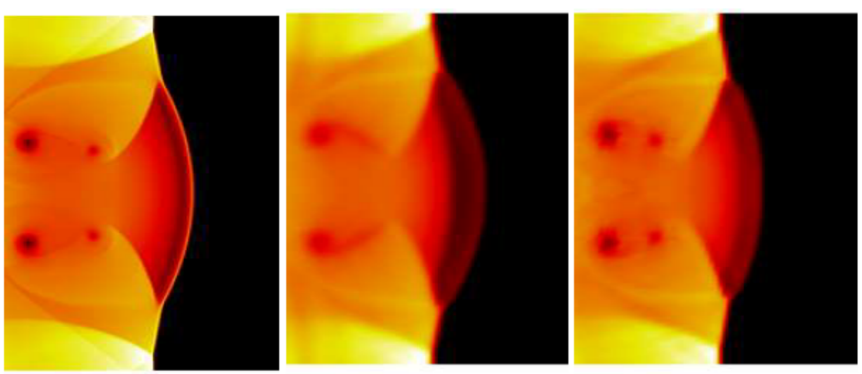
5th WENO

Adaptive THINC-BVD



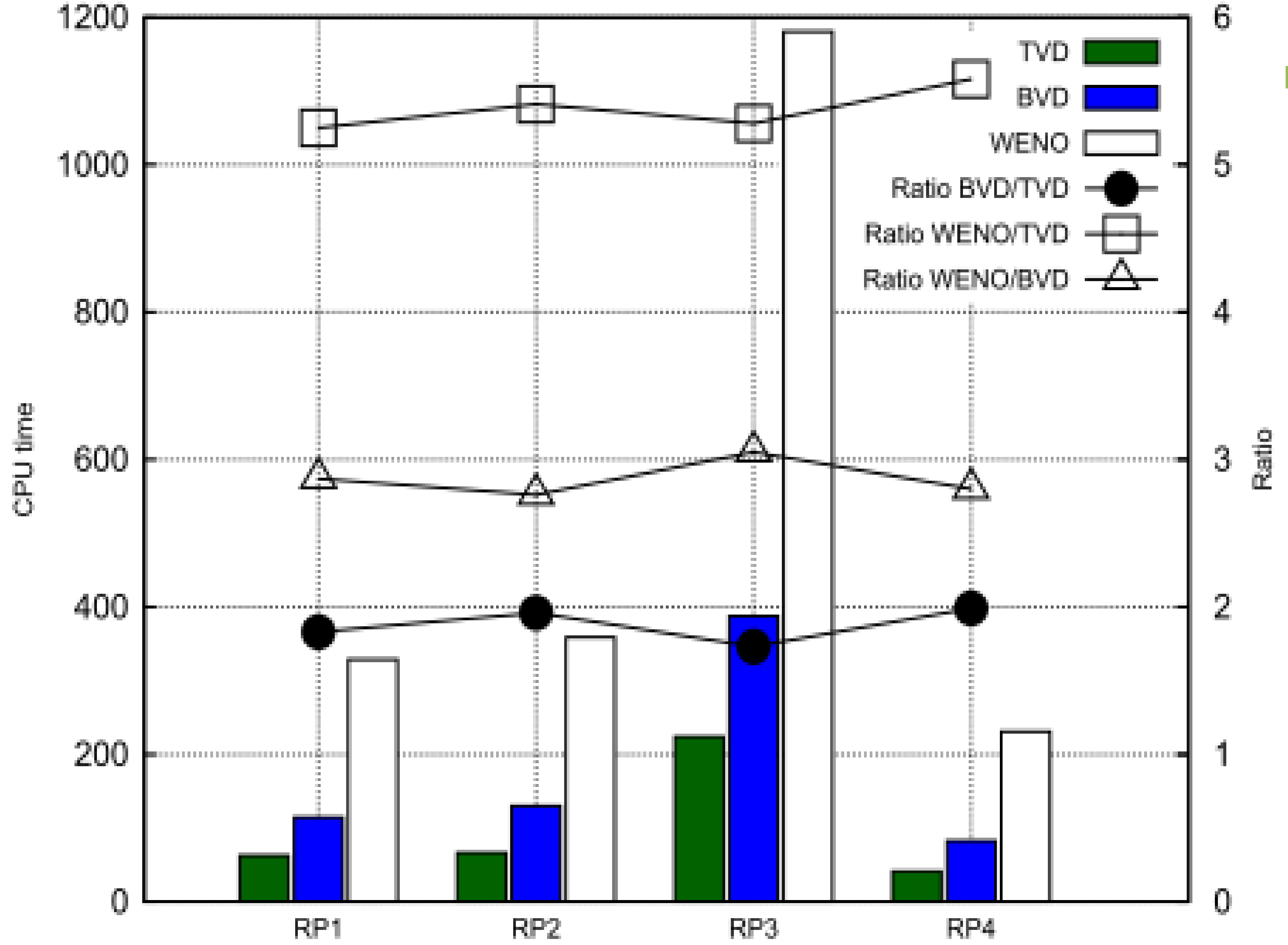
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

2D detonation waves



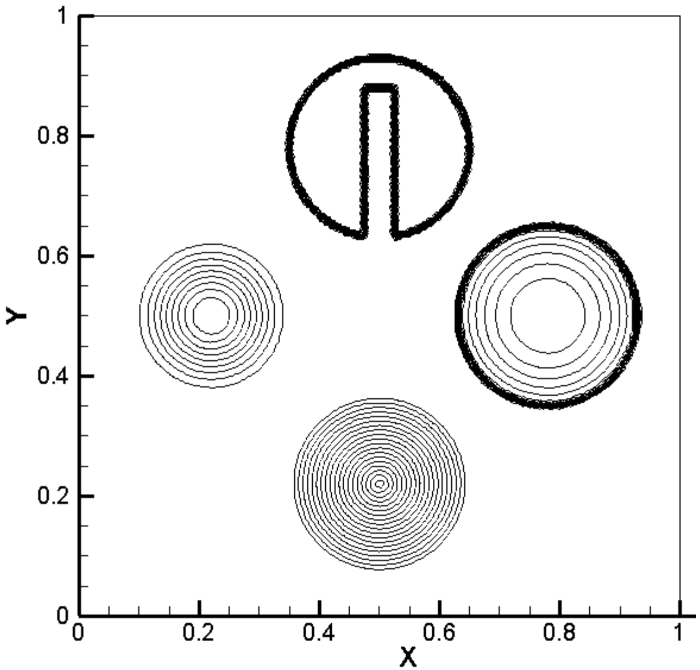
Reference 5th WENO ATHINC-RBVD

Reference 5th WENO ATHINC-RBVD


CPU time
Ratio
RP1
RP2
RP3
RP4
0
200
400
600
800
1000
1200
0
1
2
3
4
5
6
TVD
BVD
WENO
Ratio BVD/TVD
Ratio WENO/TVD
Ratio WENO/BVD

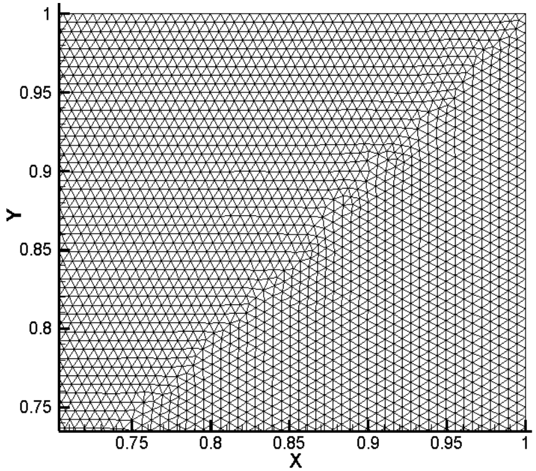
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

Extension to unstructured grids

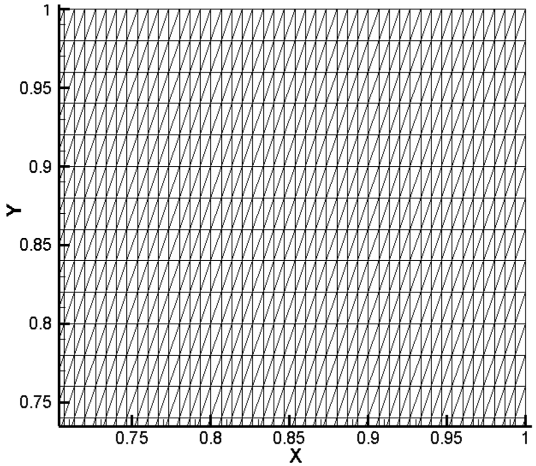


Initial condition

Grid One



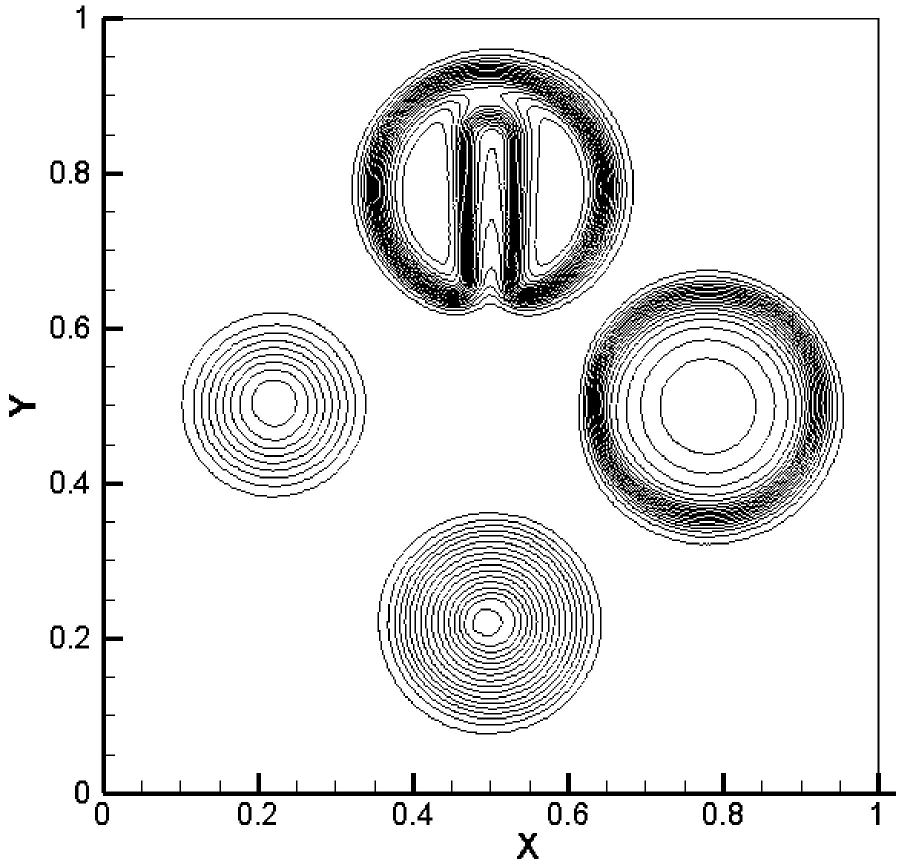
Grid Two



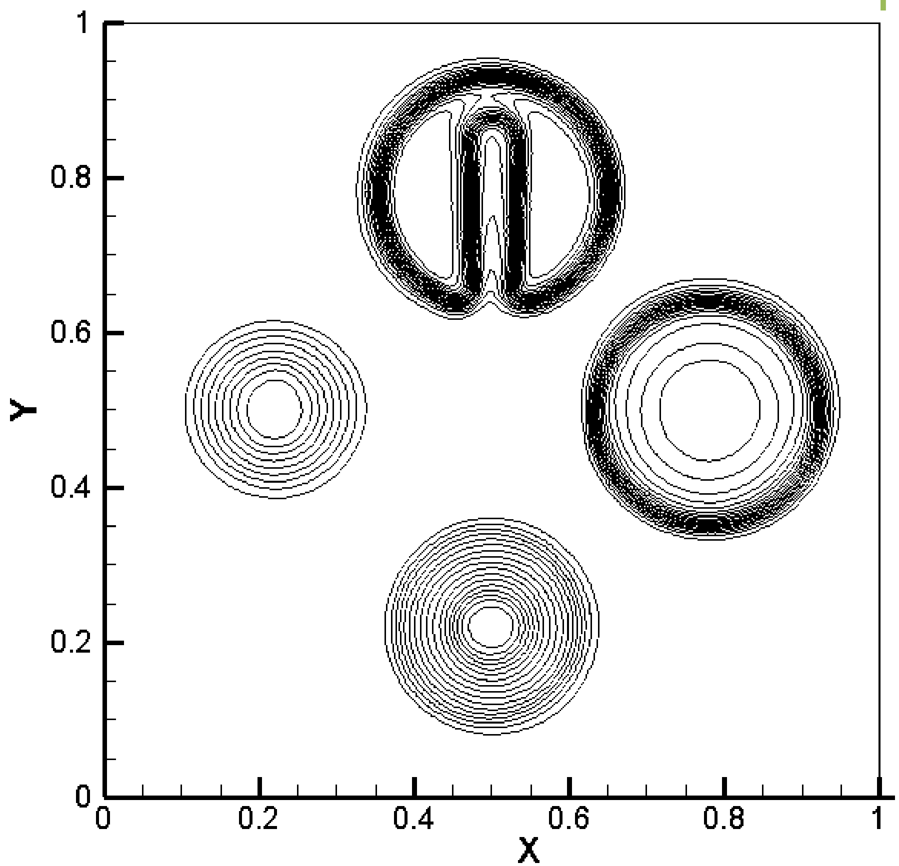
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

Extension to unstructured grids

Grid One



MLP (J.S. Park, *JCP*, 2010)

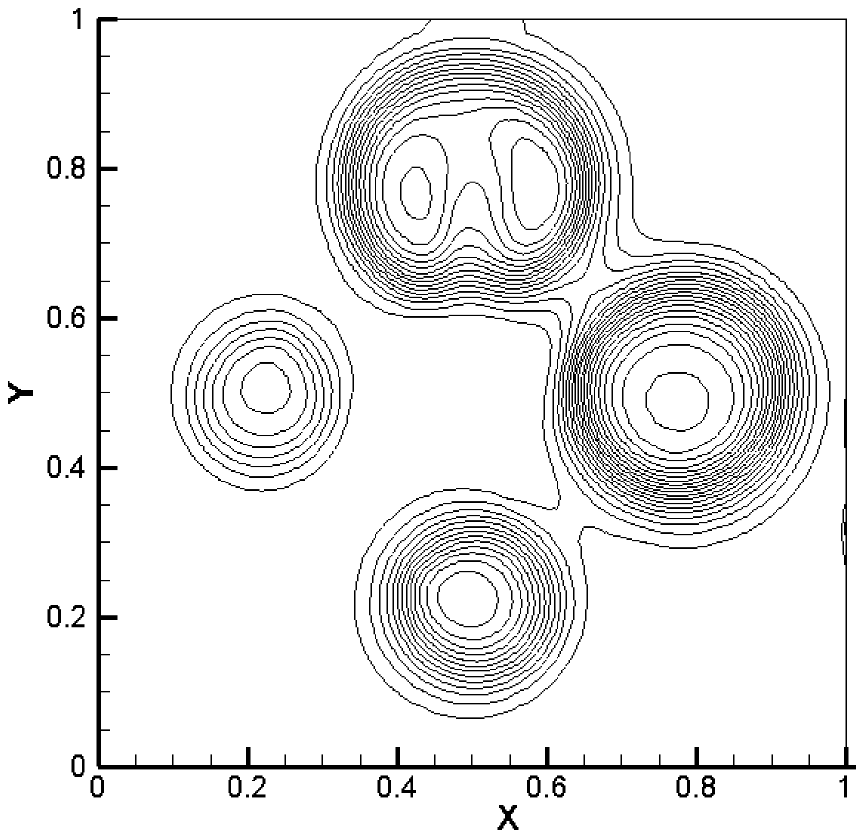


THINC with small beta

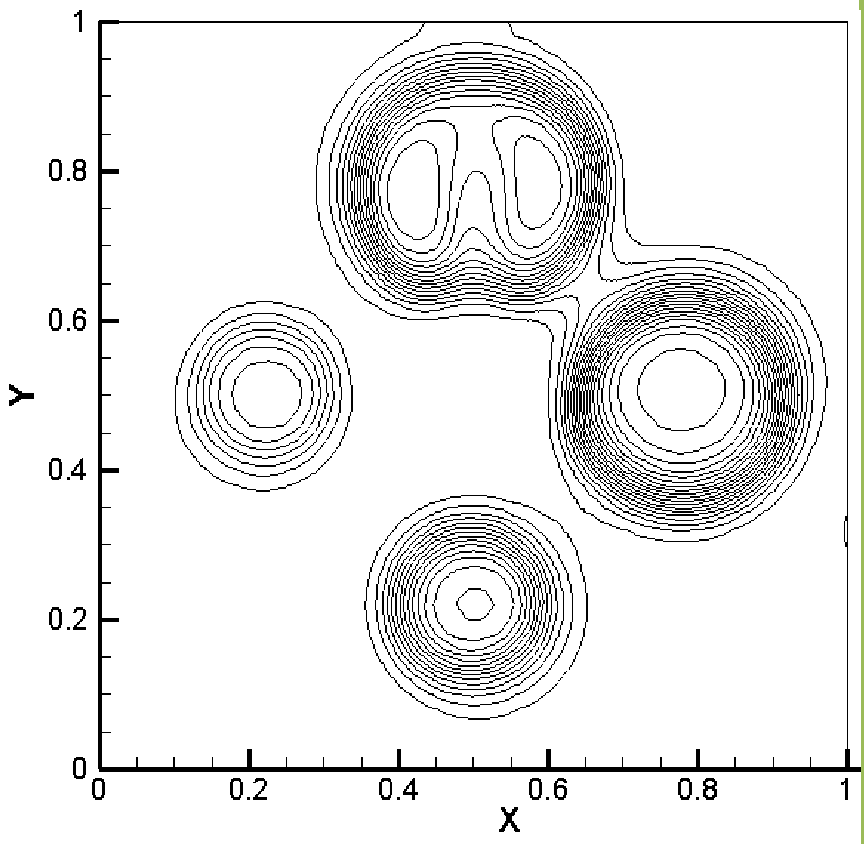
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

Extension to unstructured grids

Grid Two



MLP (J.S. Park, *JCP*, 2010)



THINC with small beta

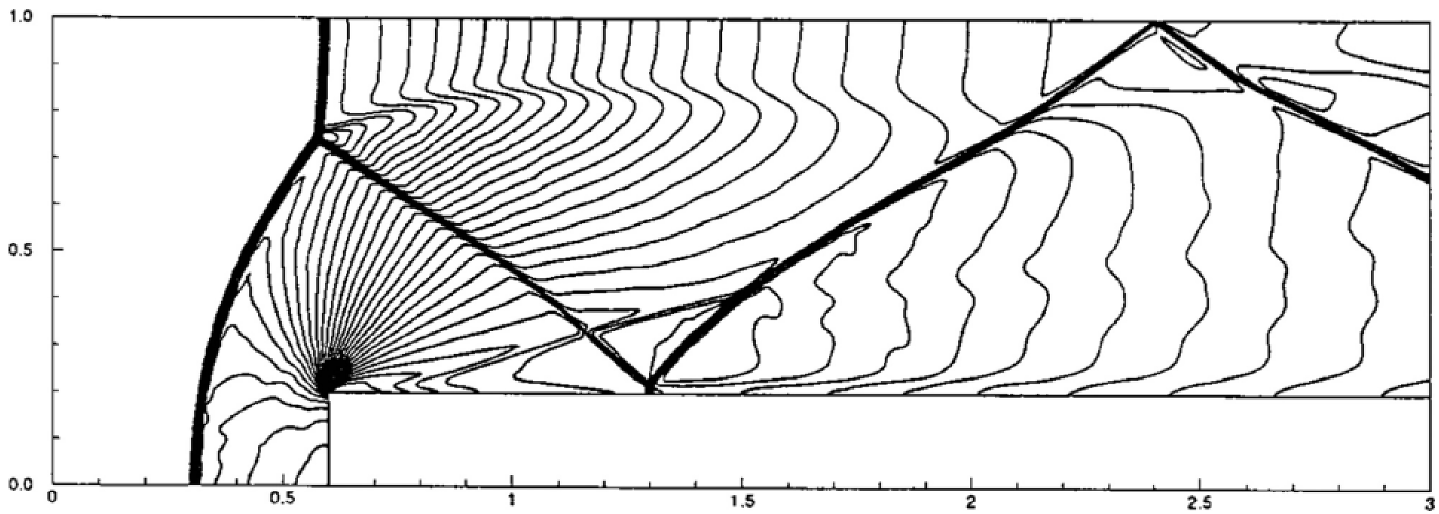
3.1 Limiting-Free Discontinuity-capturing schemes: Adaptive THINC-BVD

Extension to unstructured grids

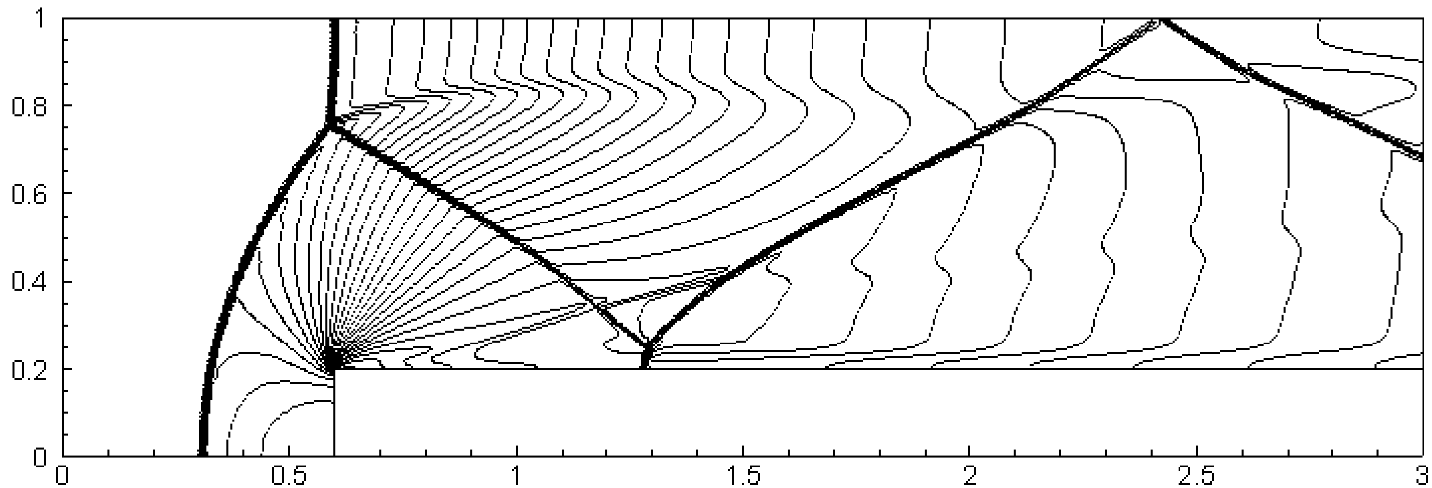
Triangular Mesh:
h=1/160

3nd WENO
(C. Hu, *JCP*, 1999)

Mesh is fined to h=1/160 around corner



THINC



Section IV. Implementation on Compressible-multiphase flow

4.1 Simulations of Compressible Multiphase Flow

Background

Single-Equivalent-Fluid (SEF) Model

e.g. Five equations model (Allaire, *JCP*, 2002)

$$\frac{\partial}{\partial t}(\alpha_1\rho_1) + \nabla\cdot(\alpha_1\rho_1\mathbf{u}) = 0,$$

$$\frac{\partial}{\partial t}(\alpha_2\rho_2) + \nabla\cdot(\alpha_2\rho_2\mathbf{u}) = 0,$$

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = 0,$$

$$\frac{\partial E}{\partial t} + \nabla\cdot(E\mathbf{u} + p\mathbf{u}) = 0,$$

$$\frac{\partial\alpha_1}{\partial t} + \mathbf{u}\cdot\nabla\alpha_1 = 0,$$

Can be discretized as single phase flow

4.1 Simulations of Compressible Multiphase Flow

Background

1. High Order Shock Capturing Scheme

High order WENO scheme, e.g. (Johnsen, *JCP*, 2006), (Coralic, *JCP*, 2014)

1. Numerical oscillation may lead to unstable and unbounded. (Coralic, *JCP*, 2014)
2. Complicated characteristic decomposition to deal with complicated EOS. (He, *JCP*, 2017)
3. Contact discontinuities are still diffusive for long time evolution

2. Interface sharpening Scheme

- Artificial compression

  (Shukla, *JCP*, 2010), (Shukla, *JCP*, 2014)

- Anti-diffusion

  (Kokh, *JCP*, 2010), (So, *JCP*, 2012)

Issues

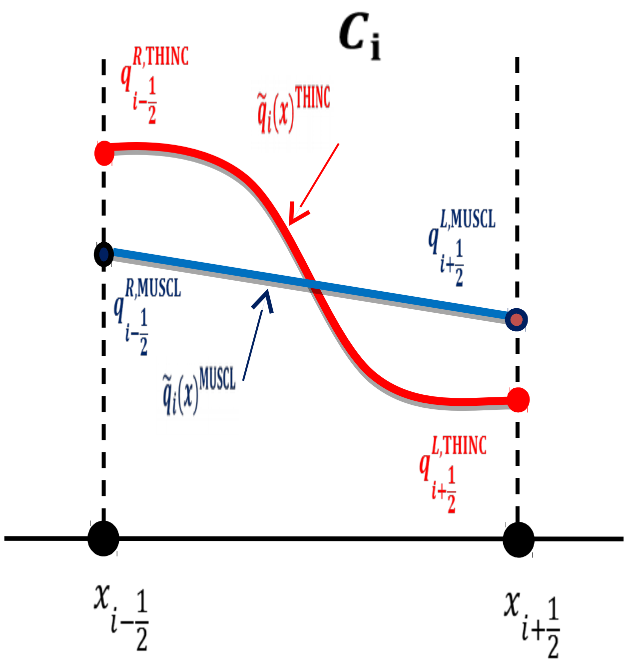
Un-consistent problem

$$a_1 \Longleftrightarrow \begin{array}{c} a_2\rho_2 \\ a_1\rho_1 \\ E \end{array}$$

4.1 Simulations of Compressible Multiphase Flow

The benefits from BVD algorithm

BVD can be applied to all state variables, which leads to a consistent reconstruction scheme

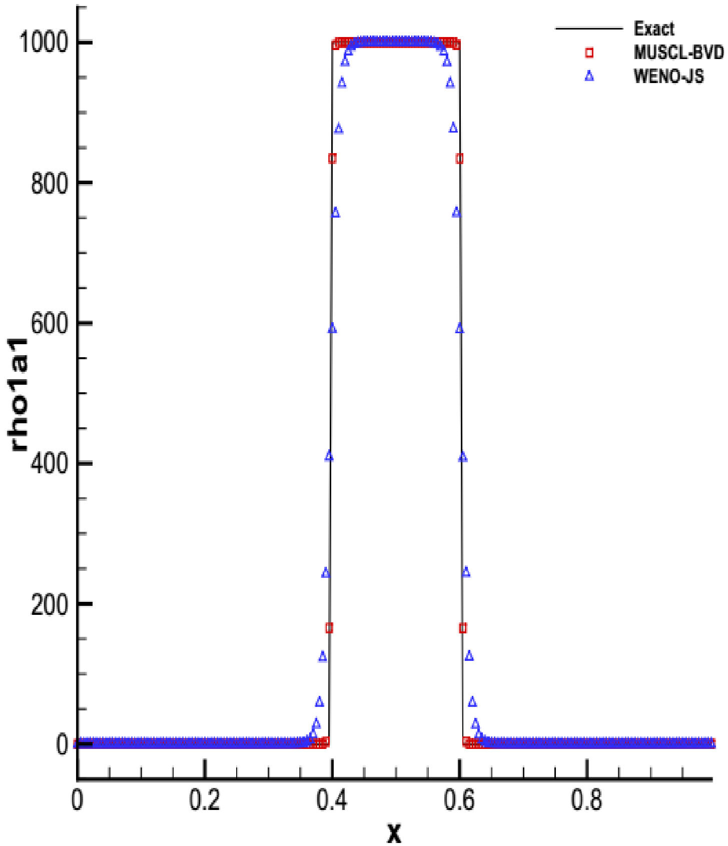


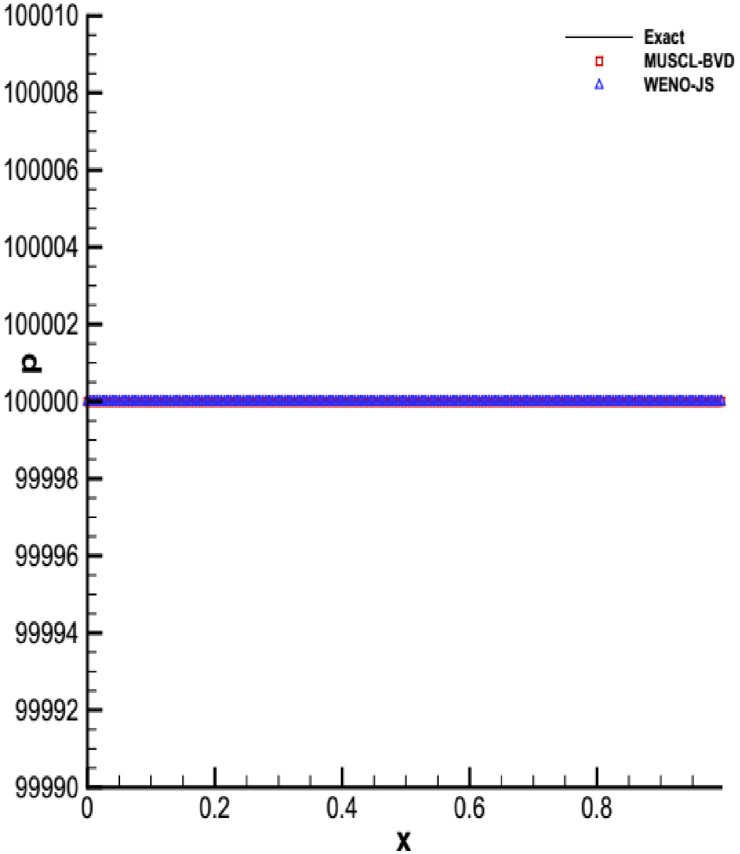
Numerical results will be shown with MUSCL-THINC-BVD scheme

4.1 Simulations of Compressible Multiphase Flow

Gas-Liquid advection

Passive advection of a square liquid column with constant pressure and velocity while there is a jump about volume fraction and density



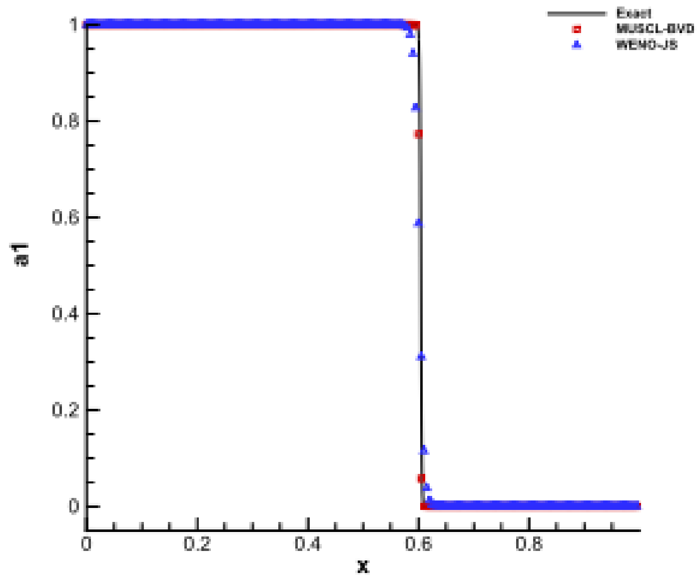


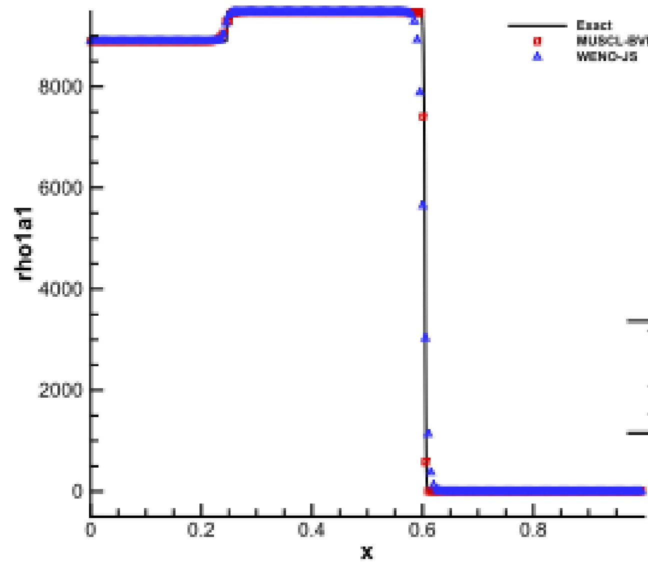
Time cost

WENO-JS	9.90s
MUSCL-BVD	4.23s

Copper explosive

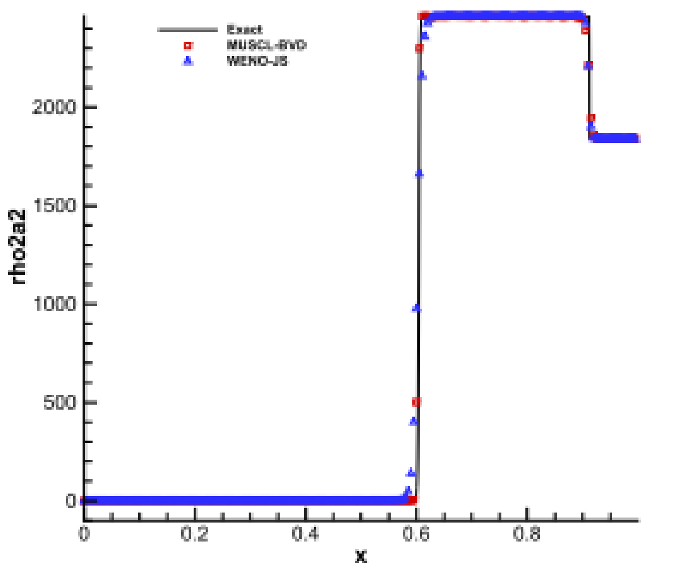
Right-moving copper plate interact with a solid explosive. Involving Complex equation of state

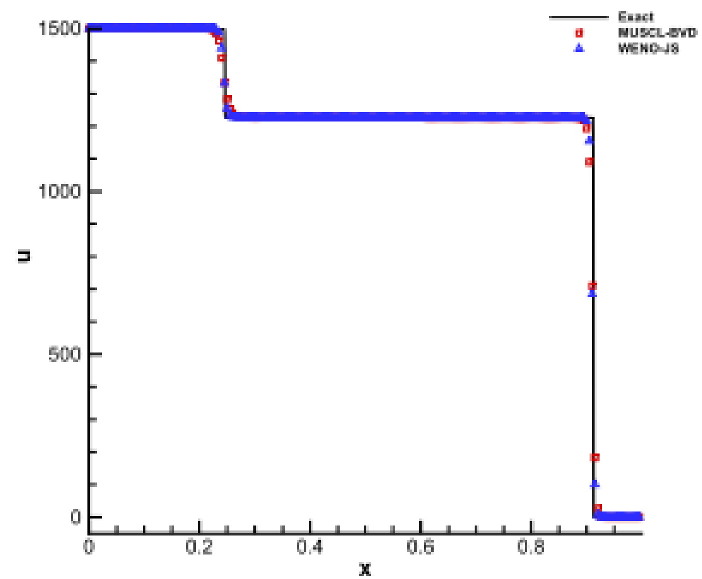




Time cost

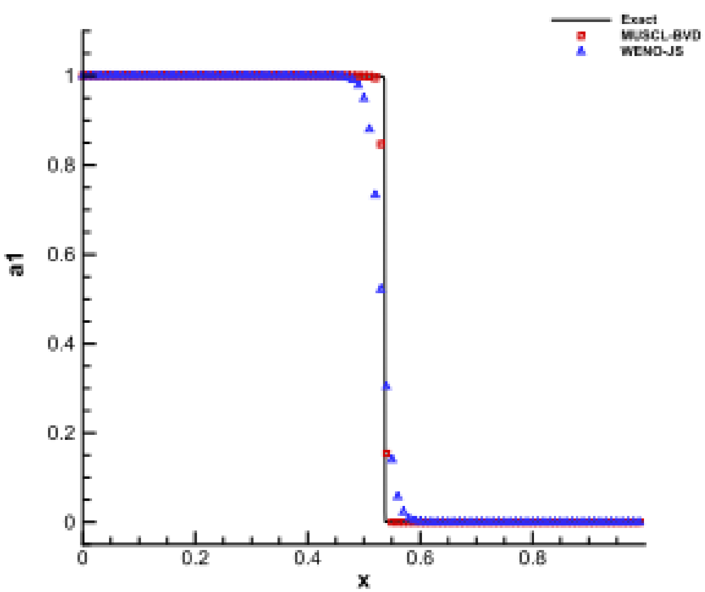
WENO-JS	3.02s
MUSCL-BVD	1.74s

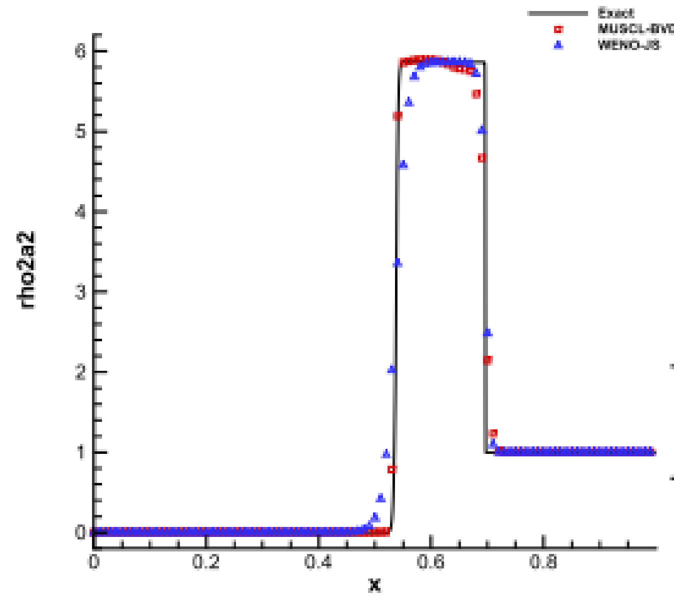




Shock-interface interaction

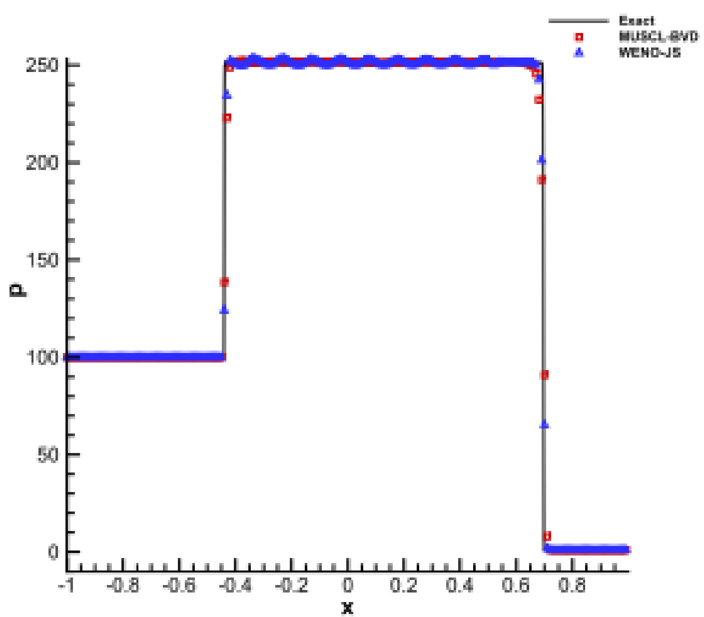
Less oscillation

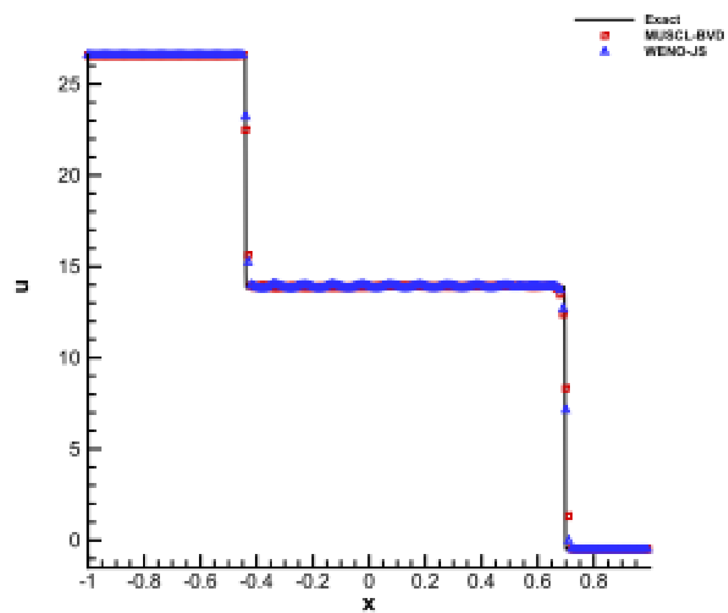




Time cost

WENO-JS	8.90s
MUSCL-BVD	3.72s

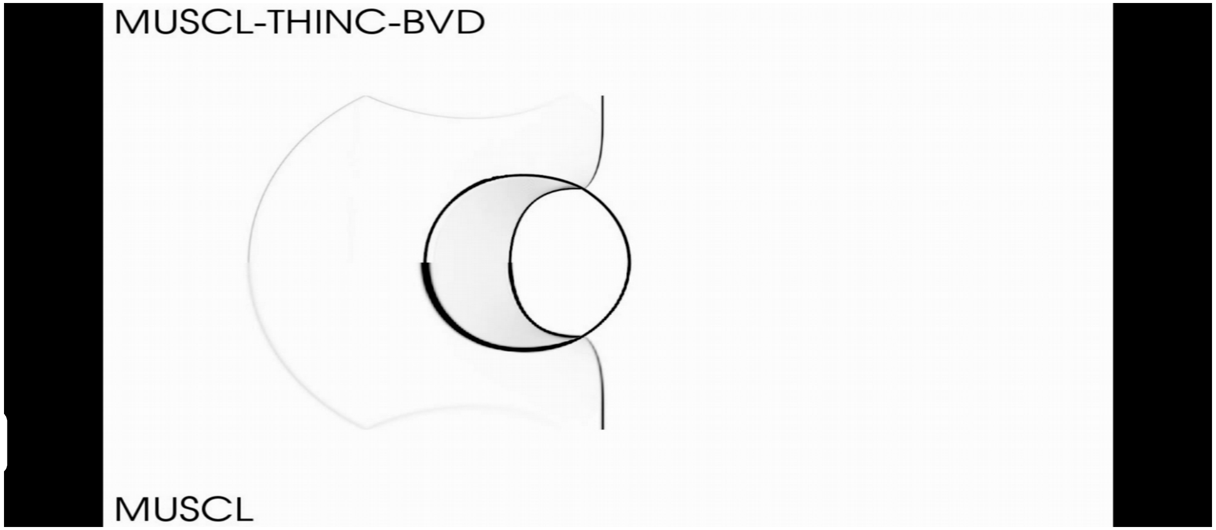




Shock Bubble Interaction

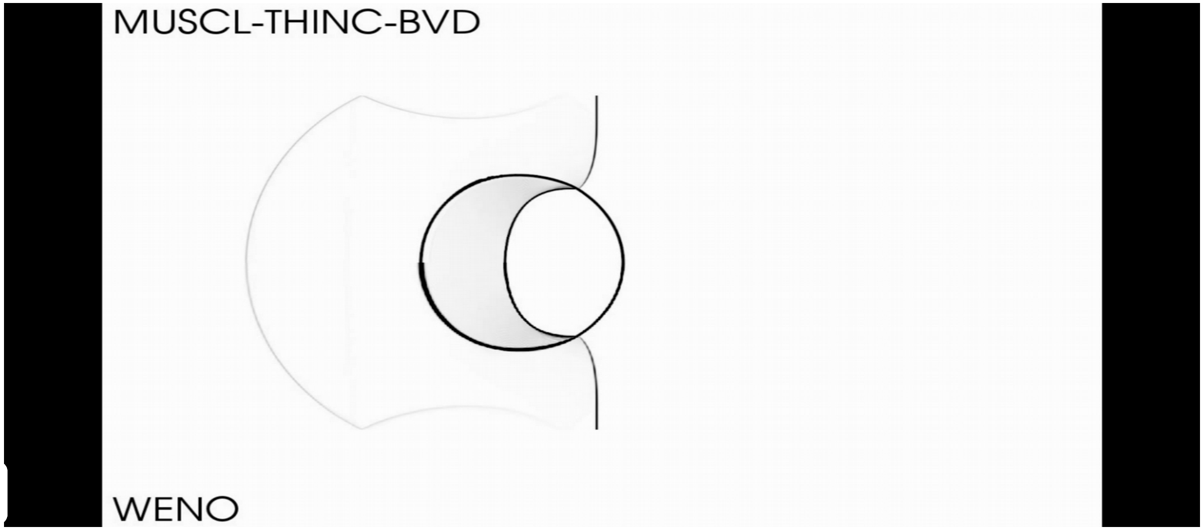
MUSCL-THINC-BVD

MUSCL



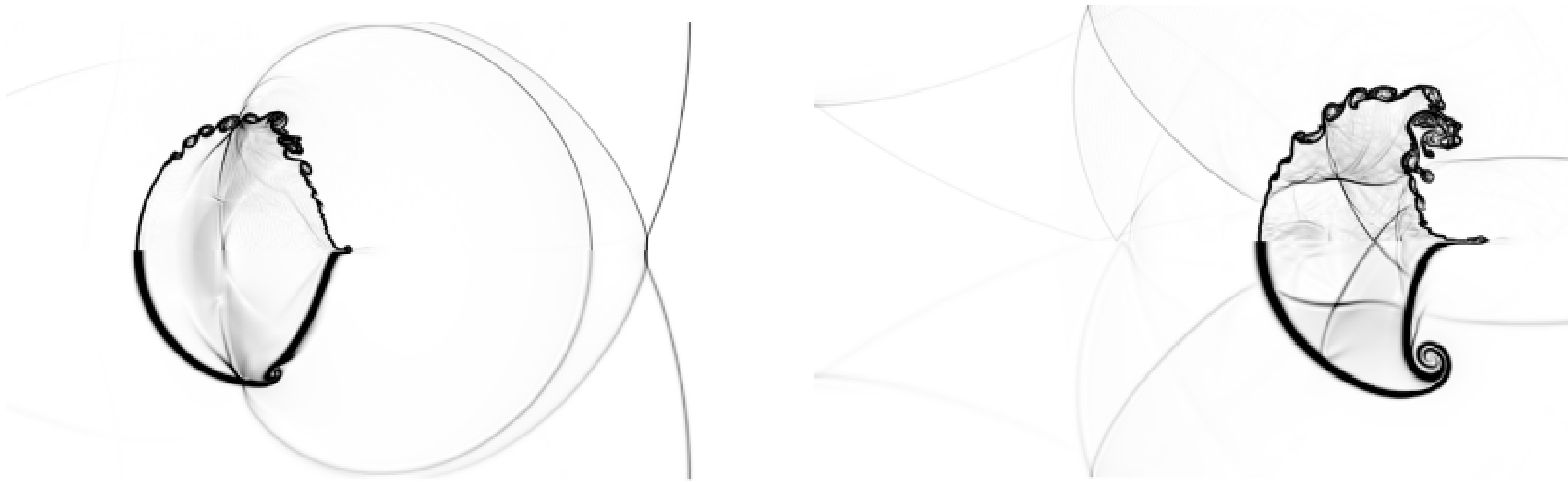
MUSCL-THINC-BVD

WENO



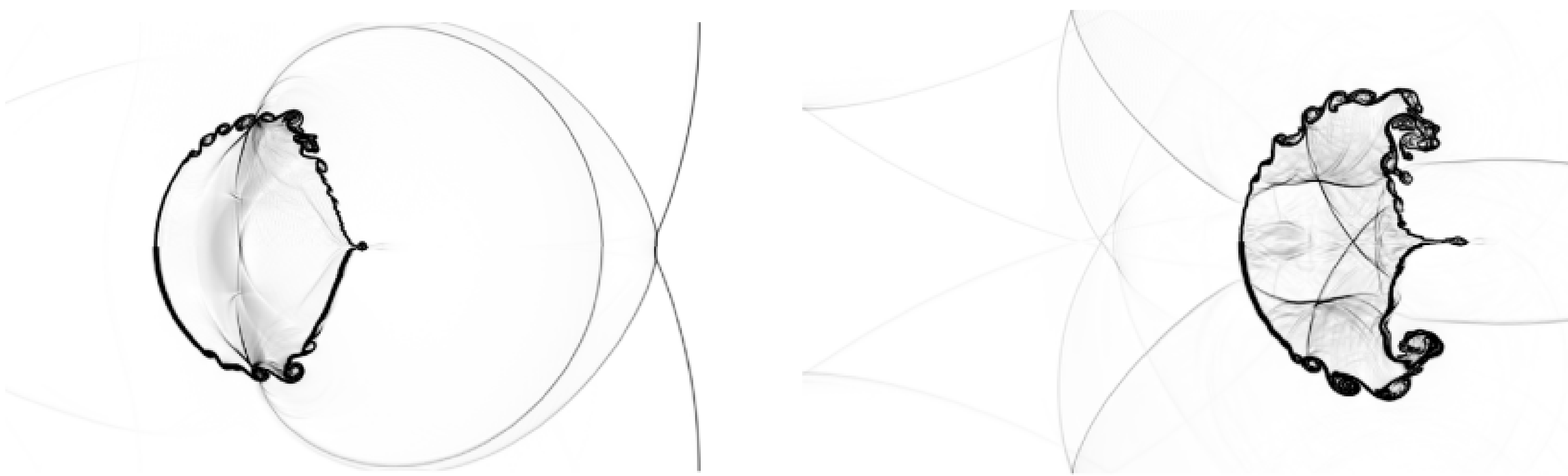
Shock Bubble Interaction

MUSCL-THINC-BVD



MUSCL

MUSCL-THINC-BVD



WENO

Shock Bubble Interaction

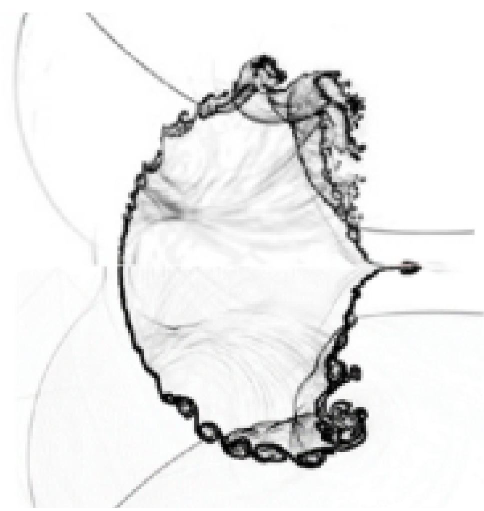
Anti-diffusion
(So,*JCP*,2012)

Same Grids
Resolution

MUSCL-
THINC-
BVD

Multi-scale
(Luo,*JCP*,2016)

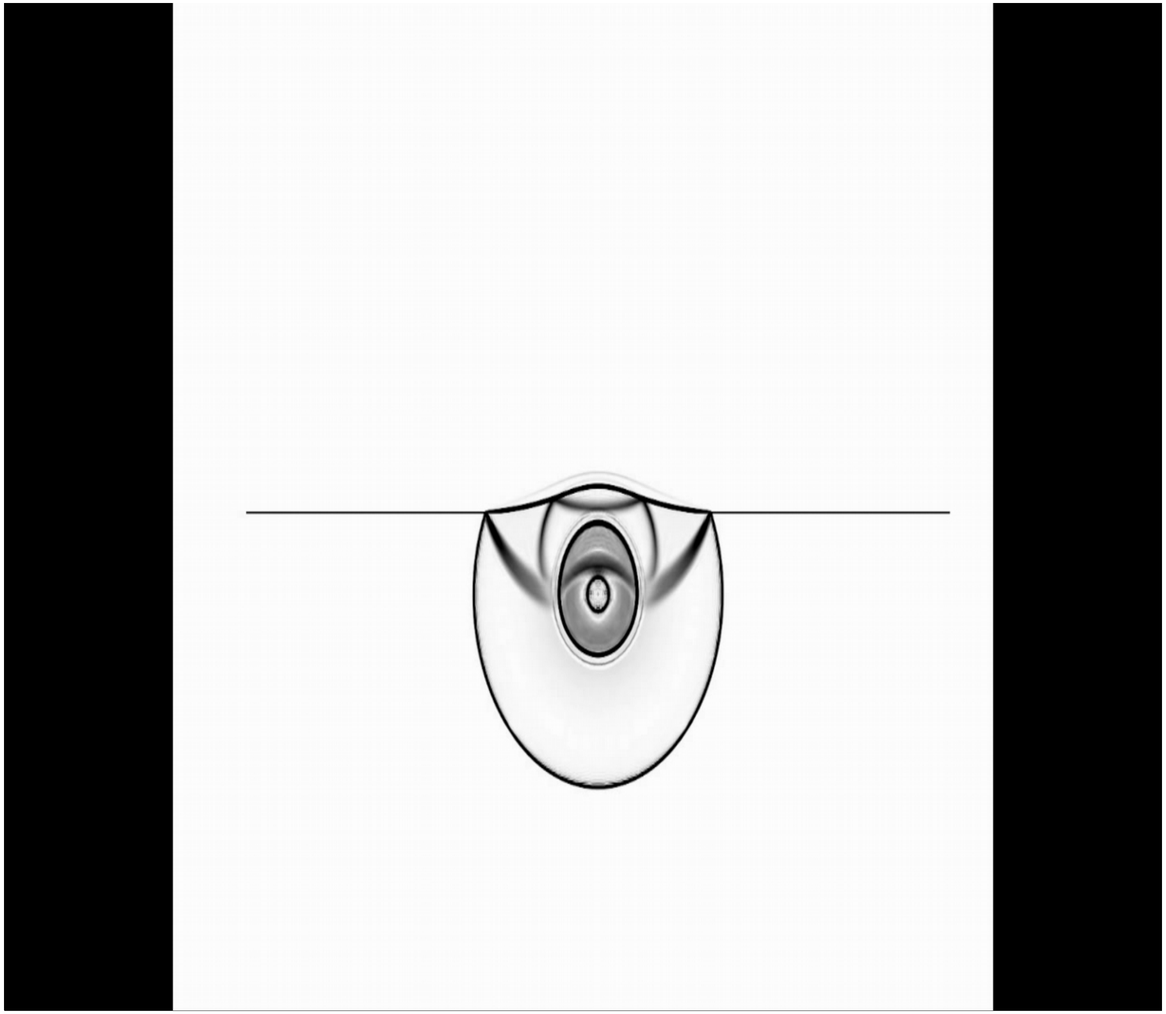
1150 along diameters



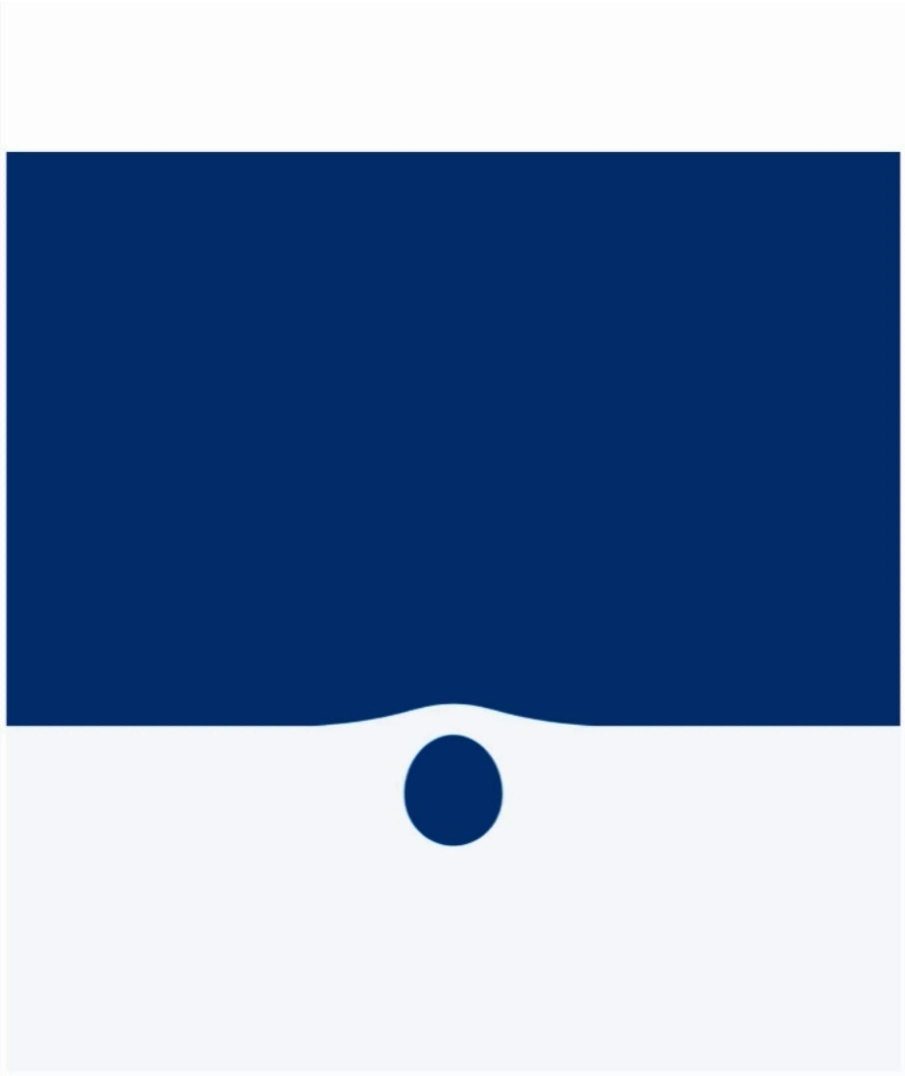
400 along diameters

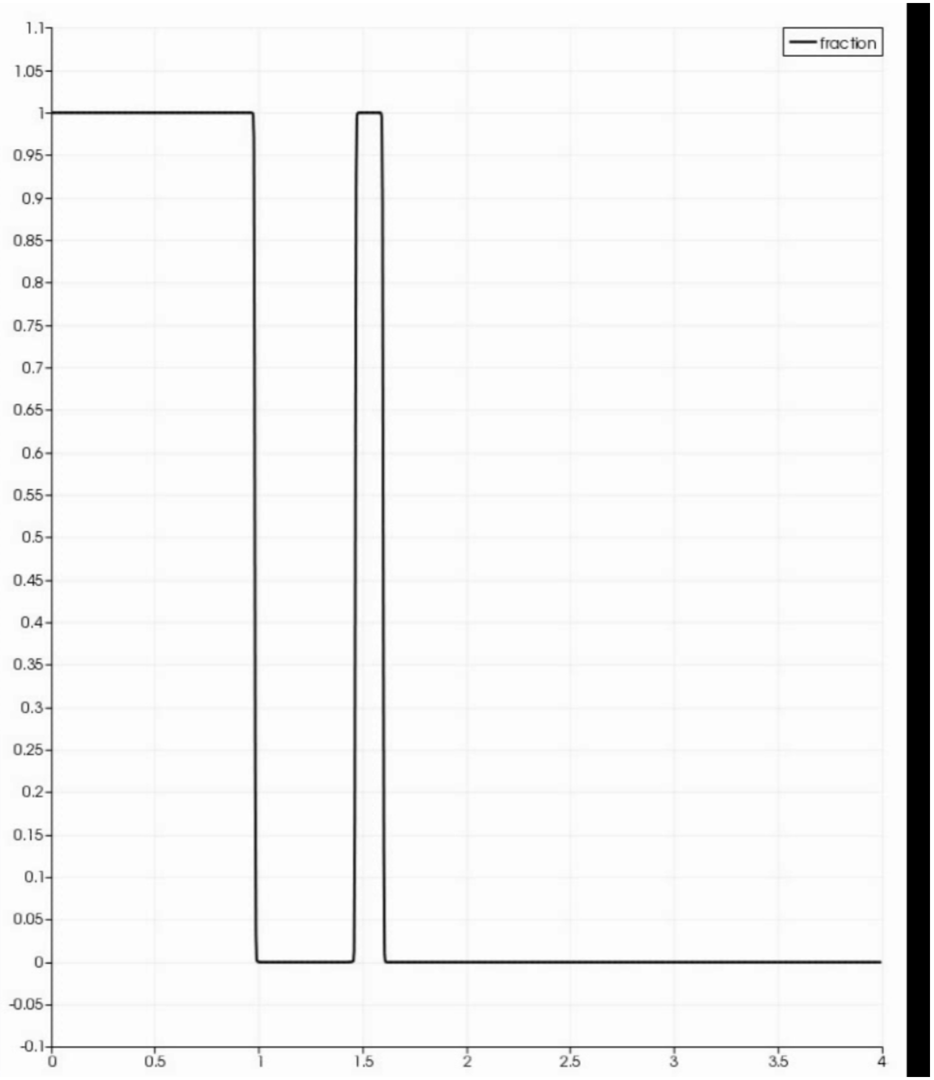
MUSCL-
THINC-
BVD

Under water explosion



Under Water Explosion





Under Water Explosion

MUSCL-THINC-BVD

MUSCL

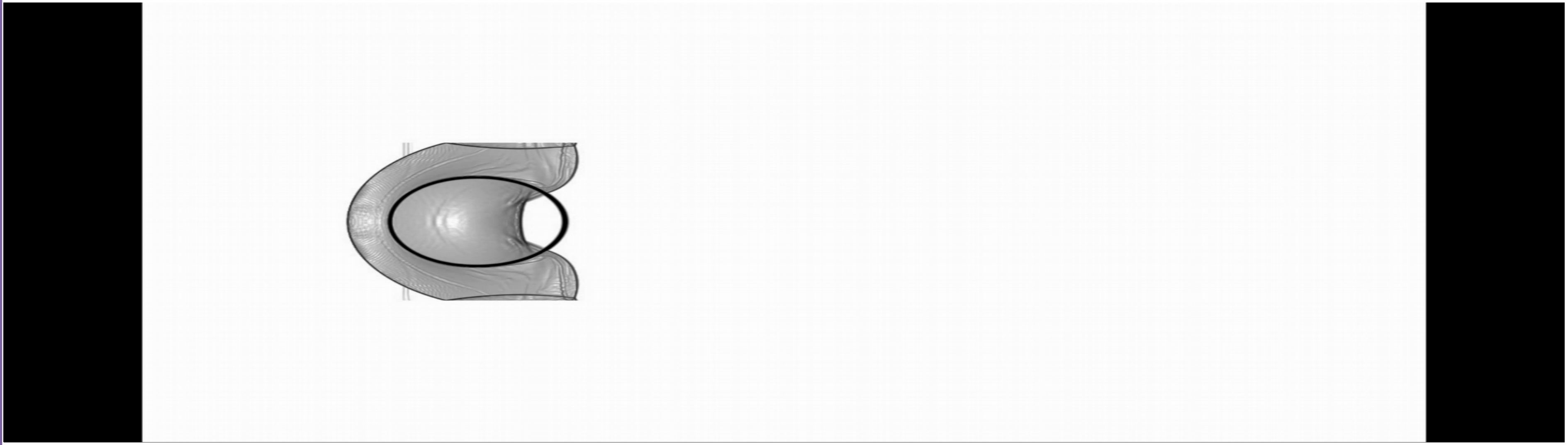
WENO

MUSCL-THINC-BVD

MUSCL

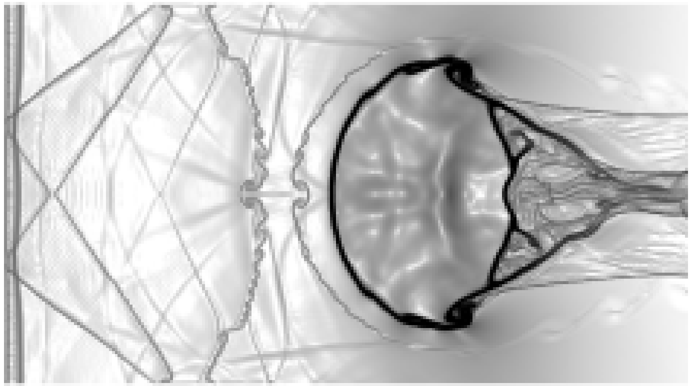
WENO

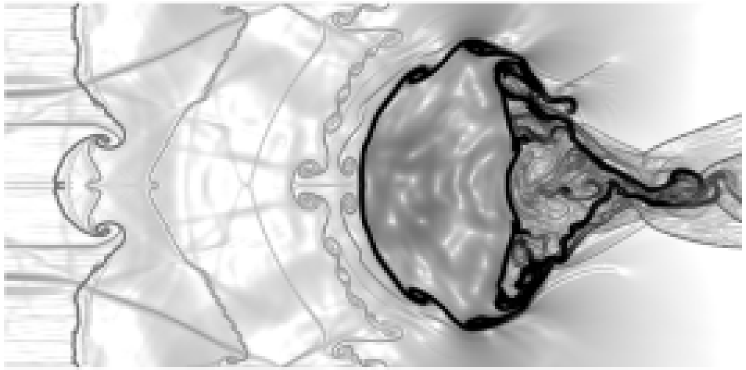
Shock Water Interaction



Shock Water Interaction

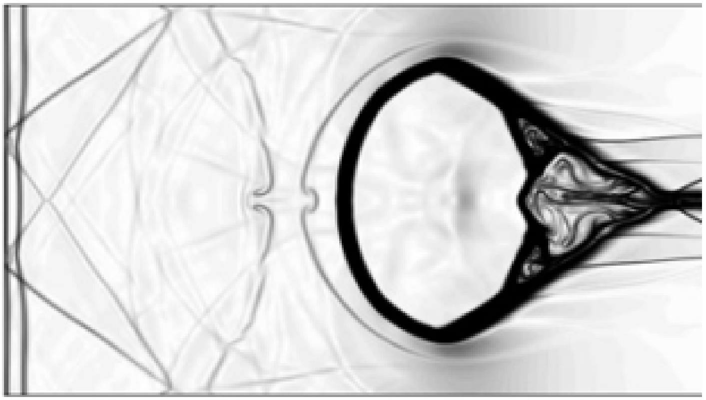
Top: MUSCL-THINC-BVD



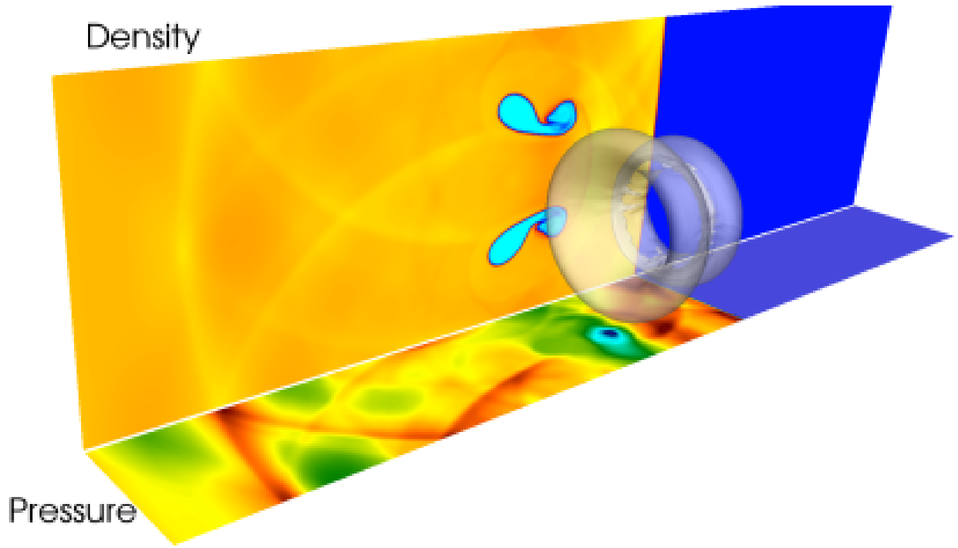


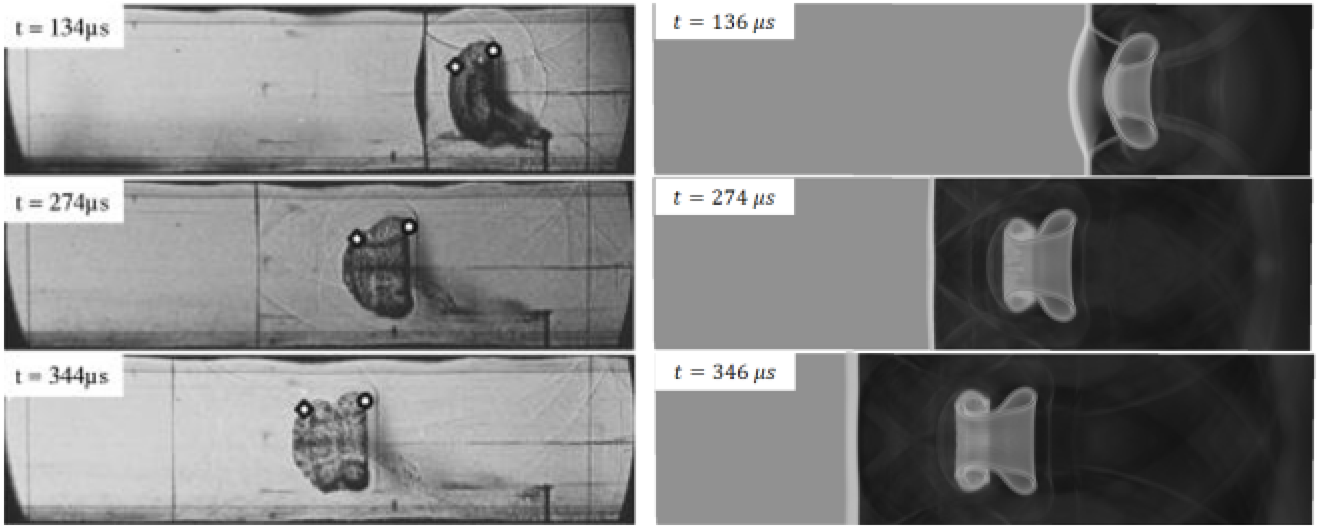
Under the same grids number

Bottom: 5^{th} WENO + artificial interface compression (Shukla,*JCP*,2010)



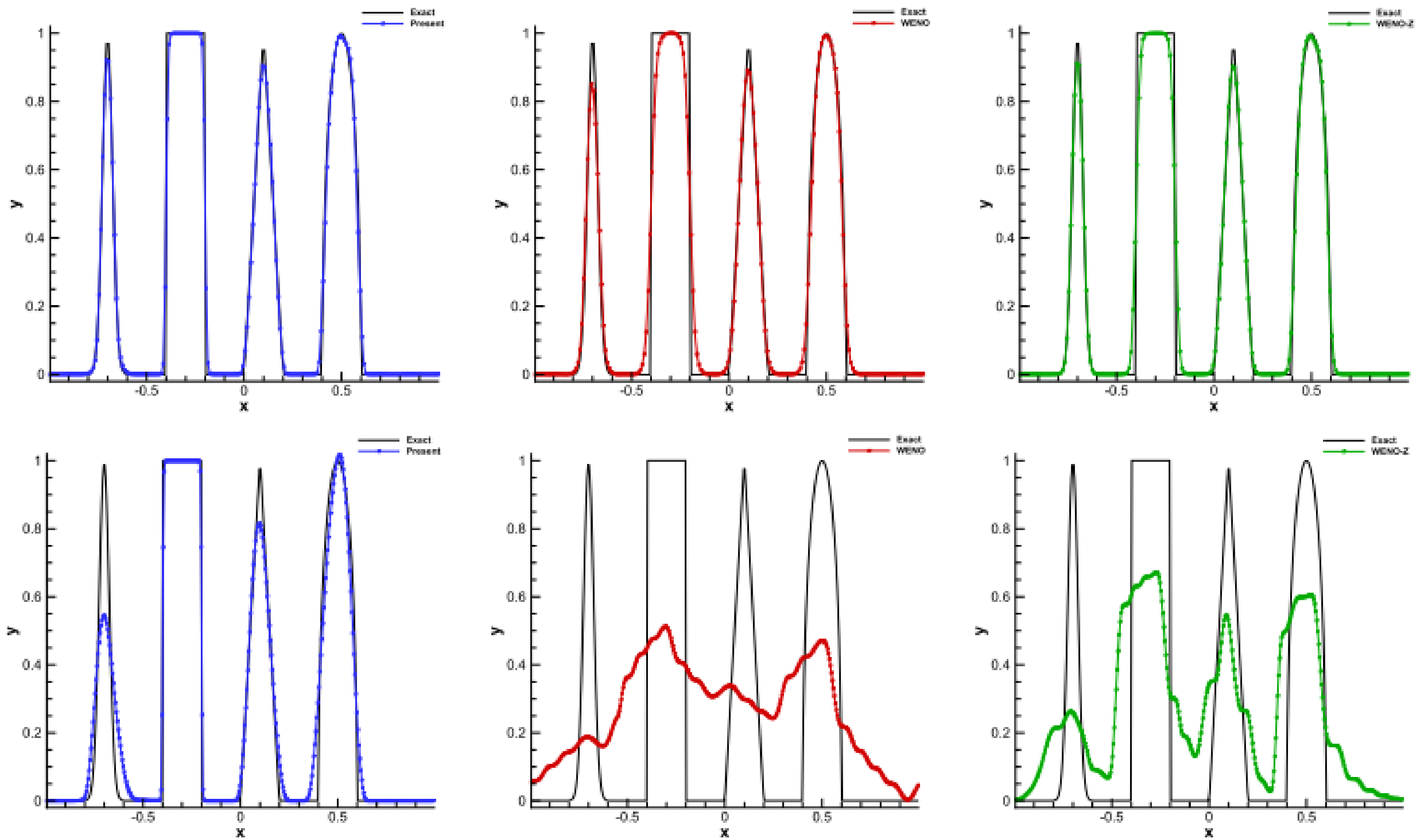
3D Shock Helium Interaction





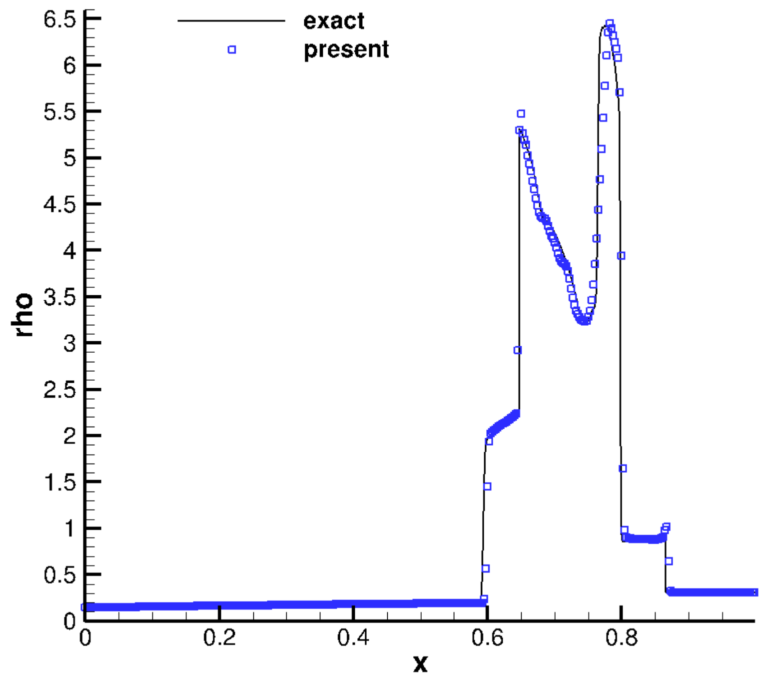
Higher order scheme is still necessary to resolve wide-spectral waves including turbulence

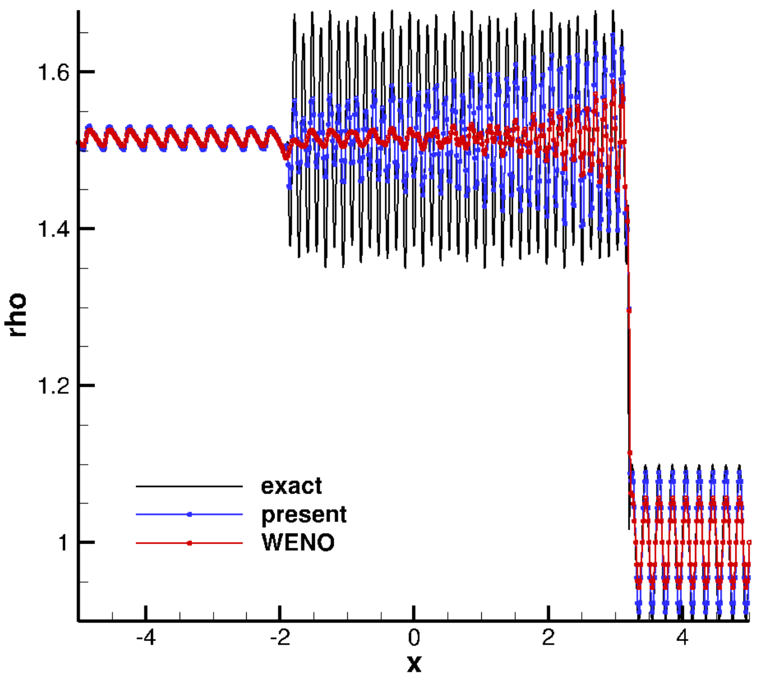
Future work: combine high order polynomials interpolation with BVD algorithm



Higher order scheme is still necessary to resolve wide-spectral waves including turbulence

Future work: combine linear high order polynomials interpolation with BVD algorithm





Thank you for your time and advices