Introduction to cell-centered Lagrangian schemes

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Eulerian formalism (spatial description)

- fixed referential attached to the observer
- fixed observation zone through the fluid flows

Lagrangian formalisme (material description)

- moving referential attached to the material
- observation zone moved and deformed as the fluid flows

Lagrangian formalism advantages

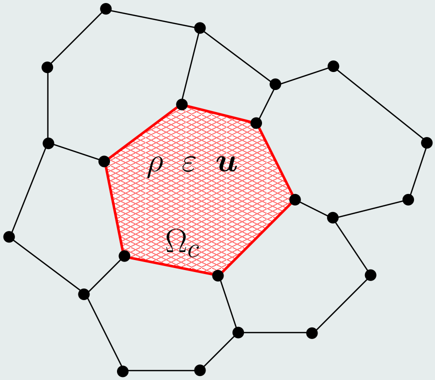
- adapted to problems undergoing large deformations
- naturally tracks interfaces in multi-material flows
- avoids the numerical diffusion of the convection terms

Lagrangian formalism drawbacks

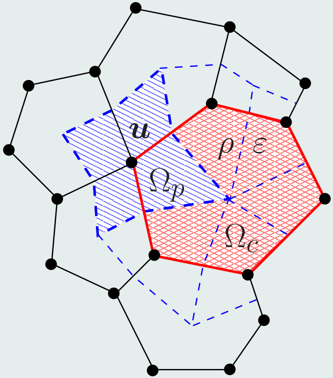
- **Robustness issue in the case of strong vorticity or shear flows**

$\Longrightarrow$ ALE method (Arbitrary Lagrangian-Eulerian)

Cell-centered formulation



Staggered formulation



Definitions

- ρ the fluid density
- u the fluid velocity
- e the fluid specific total energy
- p the fluid pressure
- $\varepsilon = e - \frac{1}{2}u^2$ the fluid specific internal energy

Euler system

- $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x} = 0$ Continuity equation
- $\dfrac{\partial \rho u}{\partial t} + \dfrac{\partial (\rho u^2 + p)}{\partial x} = 0$ Momentum conservation
- $\dfrac{\partial \rho e}{\partial t} + \dfrac{\partial (\rho u e + p u)}{\partial x} = 0$ Total energy conservation

Thermodynamical closure

- $p = p(\rho, \varepsilon)$ Equation of state

Momentum conservation

- $\dfrac{\partial \rho u}{\partial t} + \dfrac{\partial (\rho u^2 + p)}{\partial x} = 0$
- $\rho (\dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x}) + u \underbrace{(\dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x})}_{=0} + \dfrac{\partial p}{\partial x} = 0$
- $\rho (\dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x}) + \dfrac{\partial p}{\partial x} = 0$

Total energy conservation

- $\dfrac{\partial \rho e}{\partial t} + \dfrac{\partial (\rho u e + p u)}{\partial x} = 0$
- $\rho (\dfrac{\partial e}{\partial t} + u \dfrac{\partial e}{\partial x}) + e \underbrace{(\dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x})}_{=0} + \dfrac{\partial p u}{\partial x} = 0$
- $\rho (\dfrac{\partial e}{\partial t} + u \dfrac{\partial e}{\partial x}) + \dfrac{\partial p u}{\partial x} = 0$

Definitions

- $\tau = \frac{1}{\rho}$ the specific volume
- $\mathrm{U} = (\tau,\, u,\, e)^{\mathrm{t}}$ the solution vector
- $\mathrm{F}(\mathrm{U}) = (-u,\, p,\, p\,u)^{\mathrm{t}}$ the flux vector

Continuity equation

- $\dfrac{\partial\,\rho}{\partial t} + \dfrac{\partial\,\rho\,u}{\partial x} = 0$
- $\dfrac{\partial\,\rho}{\partial t} + u\,\dfrac{\partial\,\rho}{\partial x} + \rho\,\dfrac{\partial\,u}{\partial x} = 0$
- $\rho\,(\dfrac{\partial\,\tau}{\partial t} + u\,\dfrac{\partial\,\tau}{\partial x}) - \dfrac{\partial\,u}{\partial x} = 0$

Non-conservative form of the gas dynamics system

- $\rho\,(\dfrac{\partial\,\mathrm{U}}{\partial t} + u\,\dfrac{\partial\,\mathrm{U}}{\partial x}) + \dfrac{\partial\,\mathrm{F}(\mathrm{U})}{\partial x} = 0$

Moving referential

- X is the position of a point of the fluid in its initial configuration
- $x(X,t)$ is the actual position of this point, moved by the fluid flow

Trajectory equation

- $\dfrac{\partial\, x(X,t)}{\partial t} = u(x(X,t),t)$
- $x(X,0) = X$

Material derivative

- $f(x,t)$ is a smooth fluid variable
- $\dfrac{\mathrm{d}\, f}{\mathrm{d}t} \equiv \dfrac{\partial\, f(x(X,t),t)}{\partial t} = \dfrac{\partial\, f}{\partial t} + u\,\dfrac{\partial\, f}{\partial x}$

Updated Lagrangian formulation

- $\rho \dfrac{\mathrm{d}\,\mathsf{U}}{\mathrm{d}t} + \dfrac{\partial\,\mathsf{F}(\mathsf{U})}{\partial x} = 0$ Moving configuration

Definitions

- $J = \frac{\partial x}{\partial X}$ is the Jacobian associated the fluid flow
- ρ^0 is the intial fluid density

Mass conservation

- $\int_{\omega(0)} \rho^0 \,\mathrm{d}X = \int_{\omega(t)} \rho \,\mathrm{d}x$
- $\int_{\omega(t)} \rho \,\mathrm{d}x = \int_{\omega(0)} \rho \, J \,\mathrm{d}X$
- $\rho \, J = \rho^0$

Total Lagrangian formulation

- $\rho^0 \dfrac{\mathrm{d}\,\mathsf{U}}{\mathrm{d}t} + \dfrac{\partial\,\mathsf{F}(\mathsf{U})}{\partial X} = 0$ Fixed configuration

Definitions

- $\mathrm{d}m = \rho\,\mathrm{d}x = \rho^0\,\mathrm{d}X$ the mass varaiable
- $\mathsf{A}(\mathsf{U}) = \dfrac{\partial\,\mathsf{F}(\mathsf{U})}{\partial \mathsf{U}}$ the Jacobian matrix of the system
- $a = a(\rho, \varepsilon)$ the sound speed

Conservative formulation

- $\dfrac{\mathrm{d}\,\mathsf{U}}{\mathrm{d}t} + \dfrac{\partial\,\mathsf{F}(\mathsf{U})}{\partial m} = 0$

Non-conservative formulation

- $\dfrac{\mathrm{d}\,\mathsf{U}}{\mathrm{d}t} + \mathsf{A}(\mathsf{U})\,\dfrac{\partial\,\mathsf{U}}{\partial m} = 0$
- $\lambda(\mathsf{U}) = \{-\rho\,a,\, 0,\, \rho\,a\}$ the eigenvalues of $\mathsf{A}(\mathsf{U})$

Définitions

- $0 = t^0 < t^1 < \cdots < t^N = T$ a partition of the temporal domain $[0, T]$
- $\Delta t^n = t^{n+1} - t^n$ the n^{th} time step
- $\omega^0 = \bigcup_{i=1,I} \omega_i^0$ the partition of the initial domain ω^0
- $\omega_i^0 = [X_{i-\frac{1}{2}}, X_{i+\frac{1}{2}}]$ a generic cell of size ΔX_i
- $\omega_i^n = [x^n_{i-\frac{1}{2}}, x^n_{i+\frac{1}{2}}]$ the image of ω_i^0 at time t^n through the fluid flow
- $m_i = \rho_i^0 \, \Delta X_i = \rho_i^n \, \Delta x_i^n$ the constant mass of cell ω_i
- $\mathrm{U}_i^n = (\tau_i^n, \, u_i^n, \, e_i^n)^{\mathrm{t}}$ the discrete solution

First-order finite volumes scheme

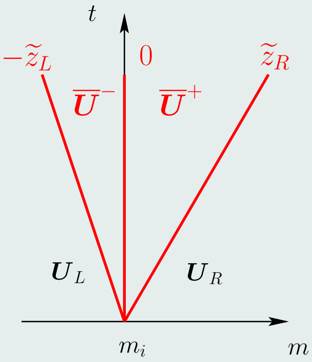
- $\mathrm{U}_i^{n+1} = \mathrm{U}_i^n - \frac{\Delta t^n}{m_i} \big(\overline{\mathrm{F}}^n_{i+\frac{1}{2}} - \overline{\mathrm{F}}^n_{i-\frac{1}{2}} \big)$
- $x^{n+1}_{i+\frac{1}{2}} = x^n_{i+\frac{1}{2}} + \Delta t^n \, \overline{u}^n_{i+\frac{1}{2}}$

Numerical flux

- $\overline{\mathrm{F}}^n_{i+\frac{1}{2}} = (-\overline{u}^n_{i+\frac{1}{2}}, \, \overline{p}^n_{i+\frac{1}{2}}, \, \overline{p}^n_{i+\frac{1}{2}} \, \overline{u}^n_{i+\frac{1}{2}})^{\mathrm{t}}$

Two-states linearization

- $\frac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \mathrm{A}(\mathrm{U})\,\frac{\partial\,\mathrm{U}}{\partial m} = 0 \quad \Longrightarrow \quad \begin{cases} \frac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \mathrm{A}(\widetilde{\mathrm{U}_\mathrm{L}})\,\frac{\partial\,\mathrm{U}}{\partial m} = 0 & \text{si } m\text{-}m_i < 0 \\ \frac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \mathrm{A}(\widetilde{\mathrm{U}_\mathrm{R}})\,\frac{\partial\,\mathrm{U}}{\partial m} = 0 & \text{si } m\text{-}m_i > 0 \end{cases}$



Riemann problem

Simple Riemann problem

- $\mathrm{U}(m,0) = \begin{cases} \mathrm{U}_L & \text{if } m\text{-}m_i < 0 \\ \mathrm{U}_R & \text{if } m\text{-}m_i > 0 \end{cases}$
- $\mathrm{U}(m,0) = \begin{cases} \mathrm{U}_L & \text{if } m\text{-}m_i < \text{-}\widetilde{z}_L\, t \\ \overline{\mathrm{U}}^- & \text{if } \text{-}\widetilde{z}_L\, t < m\text{-}m_i < 0 \\ \overline{\mathrm{U}}^+ & \text{if } \widetilde{z}_R\, t > m\text{-}m_i > 0 \\ \mathrm{U}_R & \text{if } m\text{-}m_i > \widetilde{z}_R\, t \end{cases}$

Relations

- $\widetilde{z}_L = \widetilde{\rho}\, a_L > 0, \quad \widetilde{z}_R = \widetilde{\rho}\, a_R > 0$
- $\overline{u}^- = \overline{u}^+ = \overline{u}, \quad \overline{p}^- = \overline{p}^+ = \overline{p}$

Numerical fluxes

- $\overline{u} = \dfrac{\widetilde{z}_L\, u_L + \widetilde{z}_R\, u_R}{\widetilde{z}_L + \widetilde{z}_R} - \dfrac{1}{\widetilde{z}_L + \widetilde{z}_R}(p_R - p_L)$
- $\overline{p} = \dfrac{\widetilde{z}_R\, p_L + \widetilde{z}_L\, p_R}{\widetilde{z}_L + \widetilde{z}_R} - \dfrac{\widetilde{z}_L\, \widetilde{z}_R}{\widetilde{z}_L + \widetilde{z}_R}(u_R - u_L)$

Intermediate states

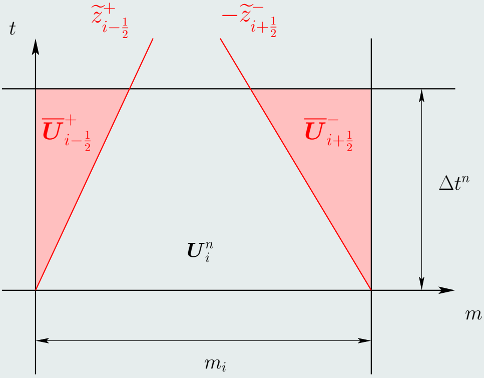
- $\overline{\tau}^- = \tau_L + \dfrac{\overline{u} - u_L}{\widetilde{z}_L}$ et $\overline{\tau}^+ = \tau_R - \dfrac{\overline{u} - u_R}{\widetilde{z}_R}$
- $\overline{e}^- = e_L - \dfrac{\overline{p}\,\overline{u} - p_L\, u_L}{\widetilde{z}_L}$ et $\overline{e}^+ = e_R + \dfrac{\overline{p}\,\overline{u} - p_R\, u_R}{\widetilde{z}_R}$

Acoustic solver

- $\widetilde{z}_L \equiv z_L = \rho_L\, a_L$ Left acoustic impedance
- $\widetilde{z}_R \equiv z_R = \rho_R\, a_R$ Right acoustic impedance

Convex combination

- $\mathrm{U}_i^{n+1} = \mathrm{U}_i^n - \frac{\Delta t^n}{m_i}(\overline{\mathrm{F}}_{i+\frac{1}{2}}^n - \overline{\mathrm{F}}_{i-\frac{1}{2}}^n) \pm \frac{\Delta t^n}{m_i}\mathrm{F}(\mathrm{U}_i^n) \pm \frac{\Delta t^n}{m_i}(\widetilde{z}_{i+\frac{1}{2}}^- + \widetilde{z}_{i-\frac{1}{2}}^+)\mathrm{U}_i^n$
- $\mathrm{U}_i^{n+1} = (1-\lambda_i)\mathrm{U}_i^n + \lambda_{i+\frac{1}{2}}^- \overline{\mathrm{U}}_{i+\frac{1}{2}}^- + \lambda_{i-\frac{1}{2}}^+ \overline{\mathrm{U}}_{i-\frac{1}{2}}^+$



Scheme illustration

Définitions

- $\lambda_{i\pm\frac{1}{2}}^{\mp} = \frac{\Delta t^n}{m_i}\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}$
- $\lambda_i = \lambda_{i+\frac{1}{2}}^- + \lambda_{i-\frac{1}{2}}^+$
- $\overline{\mathrm{U}}_{i\pm\frac{1}{2}}^{\mp} = \mathrm{U}_i^n \mp \dfrac{\overline{\mathrm{F}}_{i\pm\frac{1}{2}}^n - \mathrm{F}(\mathrm{U}_i^n)}{\widetilde{z}_{i\pm\frac{1}{2}}^{\mp}}$

CFL condition: $\lambda_i \leq 1$

- $\Delta t^n \leq \dfrac{m_i}{\widetilde{z}_{i+\frac{1}{2}}^- + \widetilde{z}_{i-\frac{1}{2}}^+}$
- $\Delta t^n \leq \frac{1}{2}\dfrac{\Delta x_i^n}{a_i^n}$ if $\widetilde{z}_{i\pm\frac{1}{2}}^{\mp} \equiv z_i^n$

Semi-discret first-order scheme

- $m_i \dfrac{\mathrm{d}\,\mathsf{U}_i}{\mathrm{d}t} = -\Big(\overline{\mathsf{F}}(\mathsf{U}_i, \mathsf{U}_{i+1}) - \overline{\mathsf{F}}(\mathsf{U}_{i-1}, \mathsf{U}_i)\Big)$

Gibbs identity

- $T\,\mathrm{d}S = \mathrm{d}\varepsilon + p\,\mathrm{d}\tau = \mathrm{d}e - u\,\mathrm{d}u + p\,\mathrm{d}\tau$

Semi-discret production of entropy

- $m_i\,T_i \dfrac{\mathrm{d}\,S_i}{\mathrm{d}t} = m_i \dfrac{\mathrm{d}\,e_i}{\mathrm{d}t} + u_i\,m_i \dfrac{\mathrm{d}\,u_i}{\mathrm{d}t} + p_i\,m_i \dfrac{\mathrm{d}\,\tau_i}{\mathrm{d}t}$
- $m_i\,T_i \dfrac{\mathrm{d}\,S_i}{\mathrm{d}t} = \widetilde{z}^{-}_{i+\frac{1}{2}}\,(\overline{u}_{i+\frac{1}{2}} - u_i)^2 + \widetilde{z}^{+}_{i-\frac{1}{2}}\,(\overline{u}_{i-\frac{1}{2}} - u_i)^2 \geq 0$

Positivity of the discrete scheme

F. VILAR, C.-W. SHU AND P.-H. MAIRE, *Positivity-preserving cell-centered Lagrangian schemes for multi-material compressible flows: Form first-order to high-orders. Part I: The 1D case.* JCP, 2016.

High-order extension of the finite-volume scheme

- MUSCL, (W)ENO, DG, ...

Equation on the mean values

- $\mathrm{U}_i^{n+1} = \mathrm{U}_i^n - \dfrac{\Delta t^n}{m_i}\left[\overline{\mathrm{F}}(\mathrm{U}^-_{i+\frac{1}{2}}, \mathrm{U}^+_{i+\frac{1}{2}}) - \overline{\mathrm{F}}(\mathrm{U}^-_{i-\frac{1}{2}}, \mathrm{U}^+_{i-\frac{1}{2}})\right]$
- $\mathrm{U}^+_{i-\frac{1}{2}}$ and $\mathrm{U}^-_{i+\frac{1}{2}}$ are the high-order values in ω_i at points $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$

Moving or total formulation

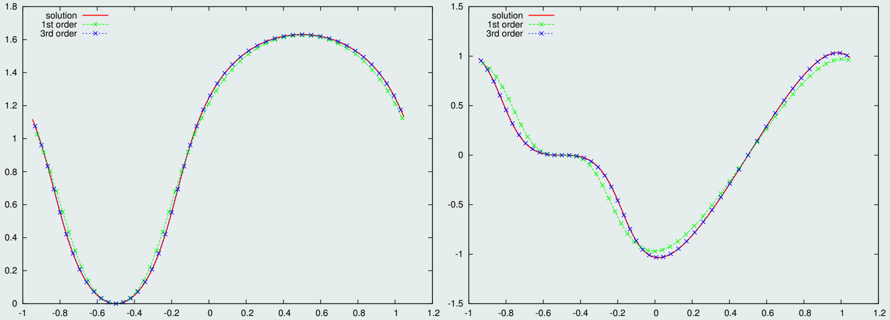
- $\rho \dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \dfrac{\partial\,\mathrm{F}(\mathrm{U})}{\partial x} = 0$ ou $\rho^0 \dfrac{\partial\,\mathrm{U}}{\partial t} + \dfrac{\partial\,\mathrm{F}(\mathrm{U})}{\partial X} = 0$

Piecewise polynomial approximation

- $\mathrm{U}^n_{h,i}(x)$ the polynomial approximation of the solution on ω_i^n
- $\mathrm{U}^n_{h,i}(X)$ the polynomial approximation of the solution on ω_i^0
- $\mathrm{U}^{\mp}_{i\pm\frac{1}{2}} = \mathrm{U}^n_{h,i}(x_{i\pm\frac{1}{2}})$ (moving config.) or $\mathrm{U}^{\mp}_{i\pm\frac{1}{2}} = \mathrm{U}^n_{h,i}(X_{i\pm\frac{1}{2}})$ (fixed config.)

Initial solution on $X \in [0, 1]$

- $\rho^0(X) = 1 + 0.9999995\,\sin(2\pi X), \quad u^0(X) = 0, \quad p^0(X) = \rho^0(X)^\gamma$
- Periodic boundary conditions



(a) Density profiles (b) Velocity profiles

Figure: Solutions at time $t = 0.1$ on 50 cells for a smooth problem

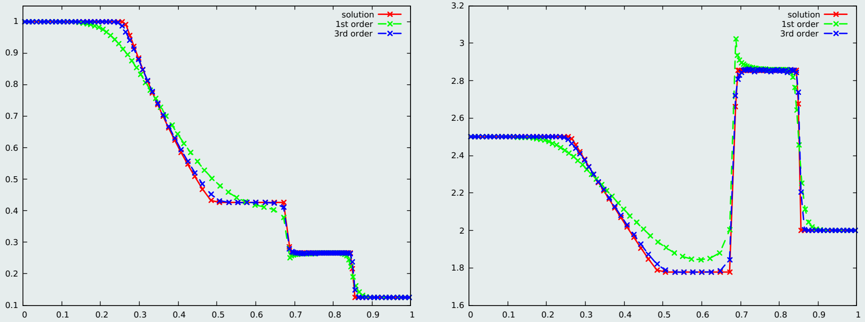
Convergence rates

	L_1		L_2		L_∞	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	$E^h_{L_\infty}$	$q^h_{L_\infty}$
$\frac{1}{50}$	9.69E-5	3.02	9.31E-5	3.01	2.75E-4	3.01
$\frac{1}{100}$	1.19E-5	3.01	1.16E-5	3.00	3.40E-5	3.01
$\frac{1}{200}$	1.48E-6	3.00	1.44E-6	3.00	4.923E-6	3.00
$\frac{1}{400}$	1.85E-7	3.00	1.80E-7	3.00	5.26E-7	3.00
$\frac{1}{800}$	2.30E-8	-	2.25E-8	-	6.56E-8	-

Table: Convergence rates on the pressure for a 3rd order DG scheme

Initial solution on $X \in [0, 1]$

- $(\rho^0, u^0, p^0) = \begin{cases} (1, 0, 1), & 0 < X < 0.5, \\ (0.125, 0, 0.1), & 0.5 < X < 1. \end{cases}$



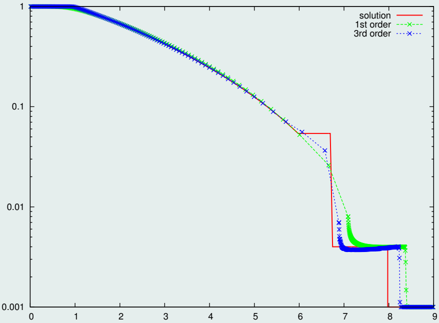
(a) Density profiles

(b) Internal energy profiles

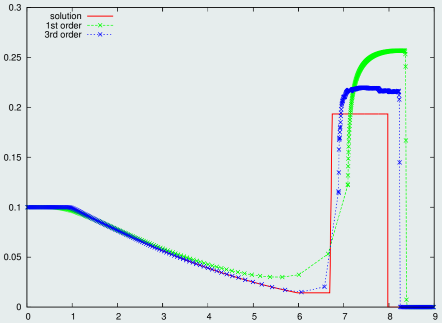
Figure: Solutions at time $t = 0.2$ on 100 cells for a Sod shock tube problem

Initial solution on $X \in [0, 9]$

- $(\rho^0,\, u^0,\, e^0) = \begin{cases} (1,\, 0,\, 0.1), & 0 < X < 3, \\ (0.001,\, 0,\, 10^{-7}), & 3 < X < 9. \end{cases}$



(a) Density profiles



(b) Internal energy profiles

Figure: Solutions at time $t = 6$ on 400 cells for a Leblanc shock tube problem

Convergence

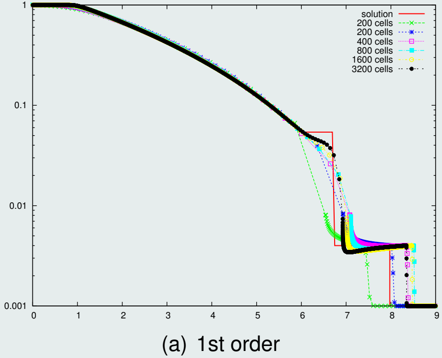
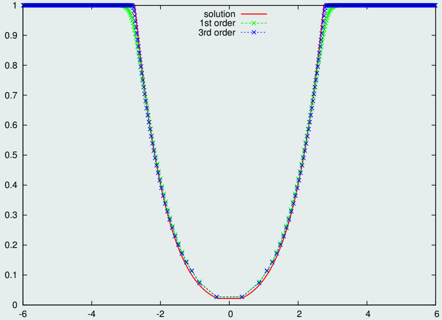


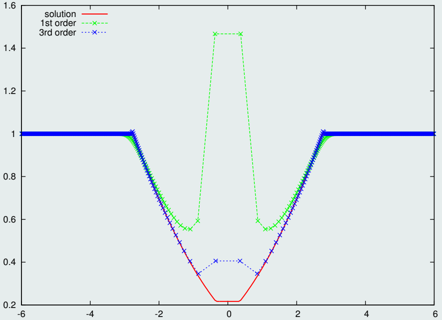
Figure: Convergence at time $t = 6$ for a Leblanc shock tube problem

Initial solution on $X \in [-4, 4]$

- $$(\rho^0,\, u^0,\, p^0) = \begin{cases} (1,\, -2,\, 0.4), & -4 < X < 0, \\ (1,\, 2,\, 0.4), & 0 < X < 4. \end{cases}$$



(a) Density profiles

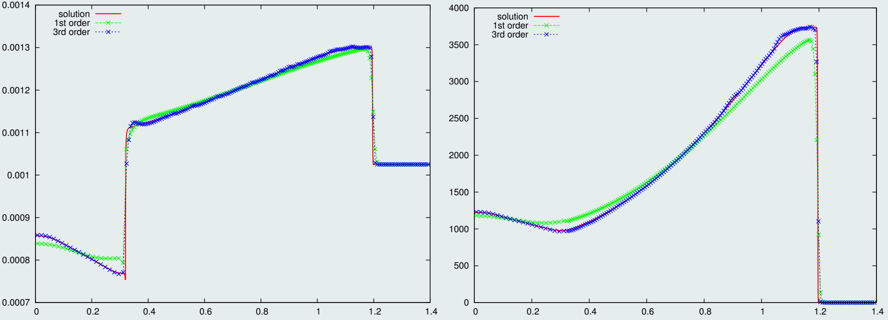


(b) Internal energy profiles

Figure: Solutions at time $t = 1$ on 400 cells for a 123 problem

Initial solution on $X \in [0, 1.4]$

- $(\rho^0,\, u^0,\, p^0) = \begin{cases} (1.63 \times 10^{-3},\, 0,\, 8.381 \times 10^3), & 0 < X < 0.16, \\ (1.025 \times 10^{-3},\, 0,\, 1), & 0.16 < X < 3.0. \end{cases}$
- On $[0, 0.3]$, gaseous product of the explosion (JWL EOS)
- On $[0.3, 1.4]$, water (stiffened gas EOS)

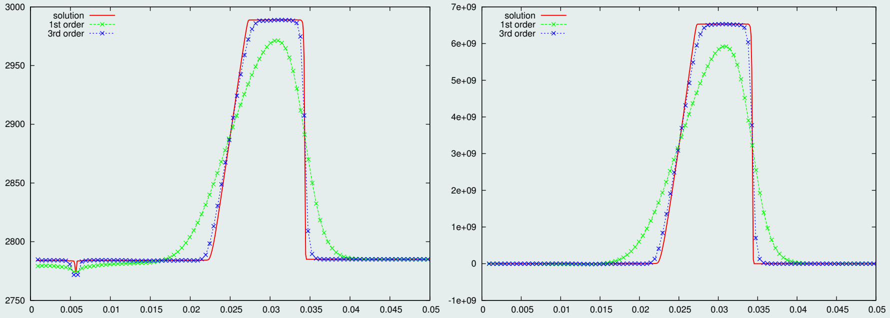


(a) Density profiles   (b) Pressure profiles

Figure: Solutions at time $t = 0.00025$ on 400 cells for a underwater TNT explosion

Initial solution on $X \in [0, 0.05]$

- $\rho^0(X) = 2785, \quad p^0(X) = 10^{-6}, \quad u^0(X) = \begin{cases} 800, & 0 < X < 0.005, \\ 0, & 0.005 < X < 0.05. \end{cases}$
- Aluminium (Mie-Grüneisen EOS)



(a) Density profiles (b) Pressure profiles

Figure: Solutions at time $t = 5 \times 10^{-6}$ on 100 cells for a flying plate impact

Euler equations

- $$\frac{\partial \rho}{\partial t} + \nabla_x \cdot \rho\, \boldsymbol{u} = 0$$
- $$\frac{\partial \rho\, \boldsymbol{u}}{\partial t} + \nabla_x \cdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u} + p\, \mathrm{I}_d) = \boldsymbol{0}$$
- $$\frac{\partial \rho\, e}{\partial t} + \nabla_x \cdot (\rho\, \boldsymbol{u}\, e + p\, \boldsymbol{u}) = 0$$

Trajectory equation

- $$\frac{\mathrm{d}\, \boldsymbol{x}(\boldsymbol{X}, t)}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X}, t), t), \qquad \boldsymbol{x}(\boldsymbol{X}, 0) = \boldsymbol{X}$$

Material derivative

- $$\frac{\mathrm{d}\, f(\boldsymbol{x}, t)}{\mathrm{d}t} = \frac{\partial\, f(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u} \cdot \nabla_x f(\boldsymbol{x}, t)$$

Definitions

- $\mathrm{U} = (\tau, \boldsymbol{u}, e)^{\mathrm{t}}$
- $\mathrm{F(U)} = (-\boldsymbol{u},\, \mathbb{1}(1)\, p,\, \mathbb{1}(2)\, p,\, p\, \boldsymbol{u})^{\mathrm{t}}$ where $\mathbb{1}(i) = (\delta_{i1},\, \delta_{i2})^{\mathrm{t}}$

Updated Lagrangian formulation

- $\rho \dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \nabla_x \cdot \mathrm{F(U)} = 0$ Moving configuration

Deformation gradient tensor

- $\mathrm{J} = \nabla_X \boldsymbol{x}$ with $|\mathrm{J}| = \det \mathrm{J} > 0$
- $\nabla_X \cdot \left(|\mathrm{J}|\mathrm{J}^{-\mathrm{t}}\right) = \mathbf{0}$ Piola compatibility condition

Mass conservation

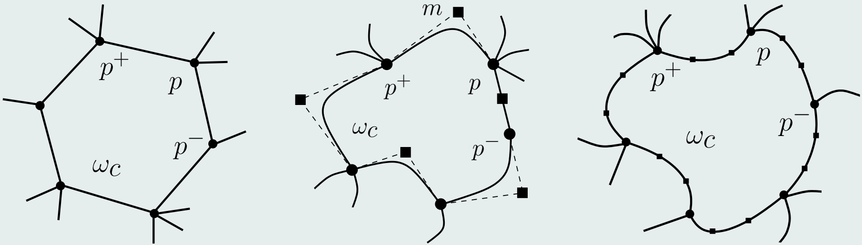
- $\rho\, |\mathrm{J}| = \rho^0$

Total Lagrangian formulation

- $\rho^0 \dfrac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \nabla_X \cdot \left(|\mathrm{J}|\mathrm{J}^{-1}\mathrm{F(U)}\right) = 0$ Fixed configuration

Définitions

- $0 = t^0 < t^1 < \cdots < t^N = T$ a partition of the time domain $[0, T]$
- $\omega^0 = \bigcup_{c=1,I} \omega_c^0$ a partition of the initial domain ω^0
- ω_c^n the image of ω_c^0 at time t^n through the fluid flow
- m_c the constant mass of cell ω_c
- $\mathrm{U}_c^n = (\tau_c^n, u_c^n, e_c^n)^{\mathrm{t}}$ the discrete solution



(a) Straight line edges (b) Conical edges (c) Polynomial edges

Figure: Generic polygonal cell

Integration

- $\mathrm{U}_c^{n+1} = \mathrm{U}_c^n - \dfrac{\Delta t^n}{m_c} \displaystyle\int_{\partial \omega_c} \overline{\mathrm{F}} \,\boldsymbol{.}\, \boldsymbol{n} \,\mathrm{d}s$
- Integration of the cell boundary term (analytically, quadrature, . . .)

General first-order finite volumes scheme

- $\mathrm{U}_c^{n+1} = \mathrm{U}_c^n - \dfrac{\Delta t^n}{m_c} \displaystyle\sum_{q \in \mathcal{Q}_c} \overline{\mathrm{F}}_{qc} \,\boldsymbol{.}\, l_{qc} \boldsymbol{n}_{qc}$
- $\overline{\mathrm{F}}_{qc} = (-\overline{\boldsymbol{u}}_q,\ \mathbb{1}(1)\,\overline{p}_{qc},\ \mathbb{1}(2)\,\overline{p}_{qc},\ \overline{p}_{qc}\,\overline{\boldsymbol{u}}_q)^{\mathrm{t}}$ numarical flux at point q
- $\boldsymbol{x}_q^{n+1} = \boldsymbol{x}_q^n + \Delta t^n\, \overline{\boldsymbol{u}}_q$

Definitions

- $\mathcal{Q}_c$ the chosen control point set of cell ω_c
- $l_{qc} \boldsymbol{n}_{qc}$ some normals to be defined

Remark

- $\overline{\mathsf{F}}_{qc}$ is local to the cell ω_c
- Only $\overline{\boldsymbol{u}}_{qc} = \overline{\boldsymbol{u}}_q$ needs to be continuous, to advect the mesh
- Loss of the scheme conservation?

(a) Face control point (b) Grid node

Figure: Points neighboring cell sets

1D numerical fluxes

- $\overline{p}_{qc} = p_c^n - \widetilde{z}_{qc}\,(\overline{\boldsymbol{u}}_q - \boldsymbol{u}_c^n)\,.\,\boldsymbol{n}_{qc}$
- $\widetilde{z}_{qc} > 0$ local approximation of the acoustic impedance

Conservation

- $\displaystyle\sum_c m_c \, \mathrm{U}_c^{n+1} = \sum_c m_c \, \mathrm{U}_c^n + \mathrm{BC}$?
- For sake of simplicity, we consider $\mathrm{BC} = 0$
- Necessary condition: $\displaystyle\sum_c \sum_{q \in \mathcal{Q}_c} \overline{p}_{qc} \, l_{qc} \boldsymbol{n}_{qc} = \mathbf{0}$

Example of a solver: LCCDG schemes

- Conditions suffisantes
- $\forall p \in \mathcal{P}(\omega), \quad \displaystyle\sum_{c \in \mathcal{C}_p} \left[\overline{p}_{pc}^- \, l_{pc}^- \boldsymbol{n}_{pc}^- + \overline{p}_{pc}^+ \, l_{pc}^+ \boldsymbol{n}_{pc}^+ \right] = \mathbf{0}$

$$\Longrightarrow \quad \overline{\boldsymbol{u}}_p = \left(\sum_{c \in \mathcal{C}_p} \mathrm{M}_{pc} \right)^{-1} \sum_{c \in \mathcal{C}_p} \left(\mathrm{M}_{pc} \boldsymbol{u}_c^n + p_c^n \, l_{pc} \boldsymbol{n}_{pc} \right)$$

- $\forall q \in \mathcal{Q}(\omega) \setminus \mathcal{P}(\omega), \quad (\overline{p}_{qL} - \overline{p}_{qR}) \, l_{qL} \boldsymbol{n}_{qL} = \mathbf{0} \quad \Longleftrightarrow \quad \overline{p}_{qL} = \overline{p}_{qR}$

$$\Longrightarrow \quad \overline{\boldsymbol{u}}_q = \left(\frac{\widetilde{z}_{qL} \, \boldsymbol{u}_L^n + \widetilde{z}_{qR} \, \boldsymbol{u}_R^n}{\widetilde{z}_{qL} + \widetilde{z}_{qR}} \right) - \frac{p_R^n - p_L^n}{\widetilde{z}_{qL} + \widetilde{z}_{qR}} \, \boldsymbol{n}_{qf_{pp^+}}$$

Convex combinaison

- $$\mathrm{U}_c^{n+1} = \mathrm{U}_c^n - \frac{\Delta t^n}{m_c} \sum_{q\in\mathcal{Q}_c} \overline{\mathrm{F}}_{qc} \cdot l_{qc}\boldsymbol{n}_{qc} + \frac{\Delta t^n}{m_c} \mathrm{F}(\mathrm{U}_c^n) \cdot \underbrace{\sum_{q\in\mathcal{Q}_c} l_{qc}\boldsymbol{n}_{qc}}_{=\mathbf{0}}$$
- $$\mathrm{U}_c^{n+1} = (1-\lambda_c)\,\mathrm{U}_c^n + \sum_{q\in\mathcal{Q}_c} \lambda_{qc}\,\overline{\mathrm{U}}_{qc}$$

Definitions

- $\lambda_{qc} = \frac{\Delta t^n}{m_c}\widetilde{z}_{qc}\, l_{qc}$ and $\lambda_c = \sum_{q\in\mathcal{Q}_c} \lambda_{qc}$
- $$\overline{\mathrm{U}}_{qc} = \mathrm{U}_c^n - \frac{\left(\overline{\mathrm{F}}_{qc} - \mathrm{F}(\mathrm{U}_c^n)\right)}{\widetilde{z}_{qc}} \cdot \boldsymbol{n}_{qc}$$

CFL condition

- $$\Delta t^n \leq \frac{m_c}{\sum_{q\in\mathcal{Q}_c} \widetilde{z}_{qc}\, l_{qc}} \quad \left(= \frac{|\omega_c^n|}{a_c^n \sum_{q\in\mathcal{Q}_c} l_{qc}} \qquad \text{if} \quad \widetilde{z}_{qc} \equiv z_c^n = \rho_c^n\, a_c^n \right)$$

Semi-discret first-order scheme

- $m_c \dfrac{\mathrm{d}\,\mathsf{U}_c}{\mathrm{d}t} = - \sum\limits_{q \in \mathcal{Q}_c} \overline{\mathsf{F}}_{qc} \,.\, l_{qc} \boldsymbol{n}_{qc}$

Gibbs identity

- $T\,\mathrm{d}S = \mathrm{d}\varepsilon + p\,\mathrm{d}\tau = \mathrm{d}e - \boldsymbol{u} \,.\, \mathrm{d}\boldsymbol{u} + p\,\mathrm{d}\tau$

Semi-discret production of entropy

- $m_c\, T_c \dfrac{\mathrm{d}\,S_c}{\mathrm{d}t} = m_c \dfrac{\mathrm{d}\,e_c}{\mathrm{d}t} + \boldsymbol{u}_c \,.\, m_c \dfrac{\mathrm{d}\,\boldsymbol{u}_c}{\mathrm{d}t} + p_c\, m_c \dfrac{\mathrm{d}\,\tau_c}{\mathrm{d}t}$
- $m_c\, T_c \dfrac{\mathrm{d}\,S_c}{\mathrm{d}t} = \sum\limits_{q \in \mathcal{Q}_c} \widetilde{z}_{qc}\, l_{qc} \left[(\overline{\boldsymbol{u}}_q - \boldsymbol{u}_c) \,.\, \boldsymbol{n}_{qc} \right]^2 \geq 0$

Positivity of the discrete scheme

F. Vilar, C.-W. Shu and P.-H. Maire, *Positivity-preserving cell-centered Lagrangian schemes for multi-material compressible flows: Form first-order to high-orders. Part II: The 2D case.* JCP, 2016.

Mean values equation

- $\mathrm{U}_c^{n+1} = \mathrm{U}_c^n - \dfrac{\Delta t^n}{m_c} \displaystyle\sum_{q \in \mathcal{Q}_c} \overline{\mathrm{F}}_{qc} \,.\, l_{qc} \boldsymbol{n}_{qc}$
- In $\overline{\mathrm{F}}_{qc}$, the mean values are substituted by the high-order values U_{qc} in ω_c at points q

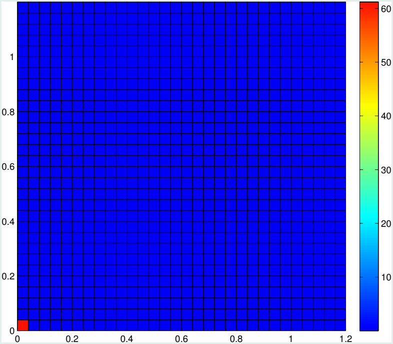
Updated or total Lagrangian formulation

- $$\rho \frac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \nabla_x \,.\, \mathrm{F}(\mathrm{U}) = 0 \qquad \text{ou} \qquad \rho^0 \frac{\mathrm{d}\,\mathrm{U}}{\mathrm{d}t} + \nabla_X \,.\, \left(|\mathrm{J}|\mathrm{J}^{-1}\mathrm{F}(\mathrm{U})\right) = 0$$

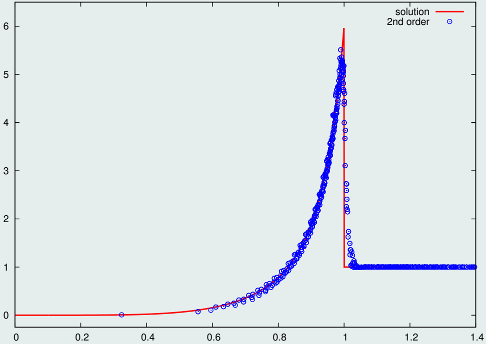
Piecewise polynomial approximation

- $\mathrm{U}_{h,c}^n(\boldsymbol{x})$ the polynomial approximation of the solution on ω_c^n
- $\mathrm{U}_{h,c}^n(\boldsymbol{X})$ the polynomial approximation of the solution on ω_c^0
- $\mathrm{U}_{qc} = \mathrm{U}_{h,c}^n(\boldsymbol{x}_q)$ (moving config.) or $\mathrm{U}_{qc} = \mathrm{U}_{h,c}^n(\boldsymbol{X}_q)$ (fixed config.)

Sedov point blast problem



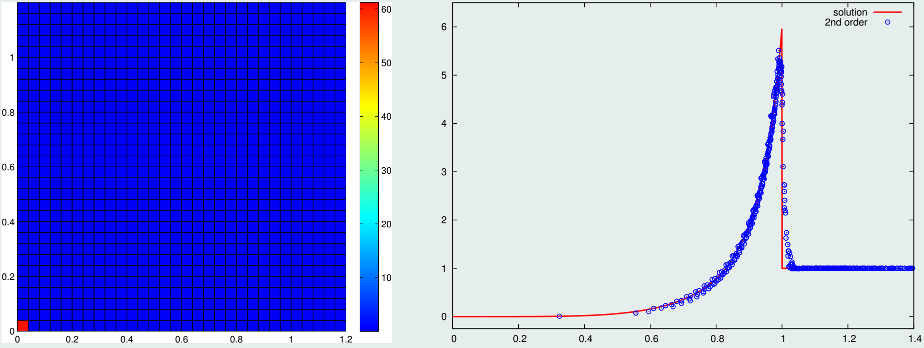
(a) Pressure field



(b) Density profiles

Figure : Solution at time $t = 1$ for a Sedov problem on a 30×30 Cartesian mesh

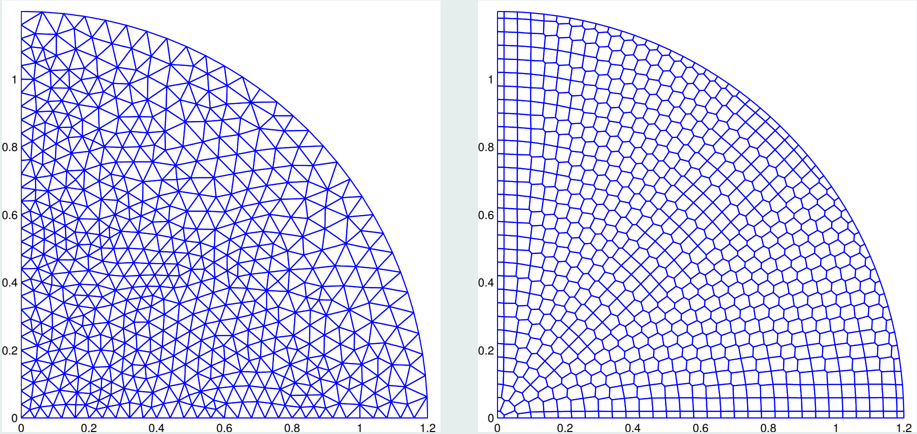
Sedov point blast problem



(a) Pressure field (b) Density profiles

Figure : Solution at time $t = 1$ for a Sedov problem on a 30×30 Cartesian mesh

Sedov point blast problem

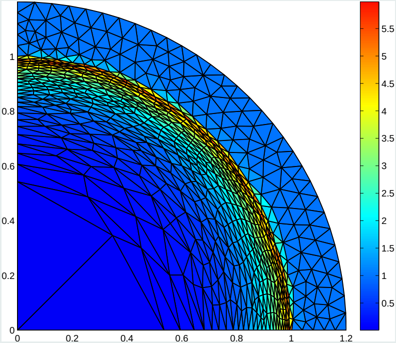


(c) Triangular grid - 1110 cells

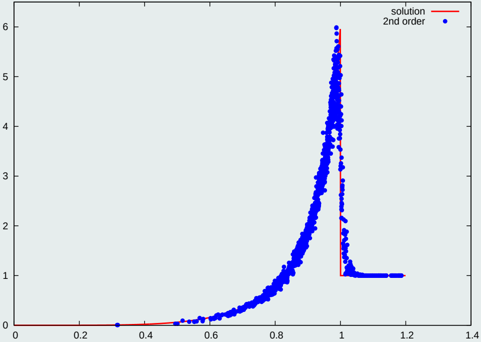
(d) Polygonal grid - 775 cells

Figure : Initial unstructured grids for Sedov point blast problem

Sedov point blast problem



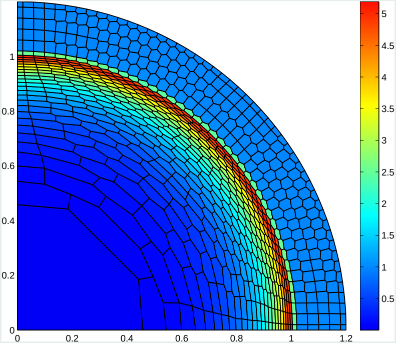
(e) Density field



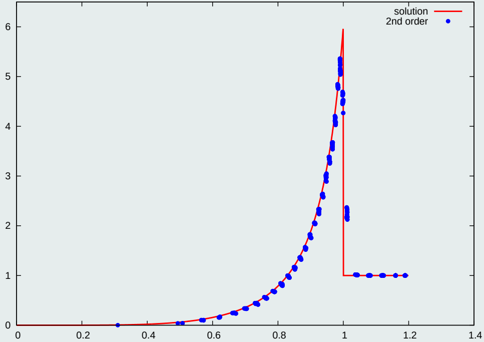
(f) Density profiles

Figure : Solution at time $t = 1$ for a Sedov problem on a grid made of 1110 triangular cells

Sedov point blast problem



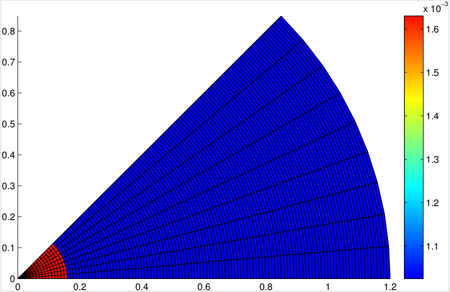
(g) Density field



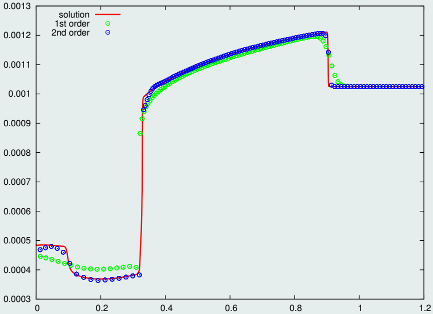
(h) Density profiles

Figure : Solution at time $t = 1$ for a Sedov problem on a grid made of 775 polygonal cells

Underater TNT explosion



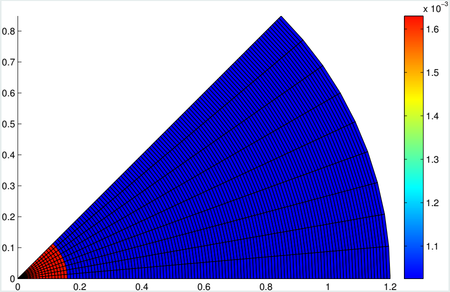
(i) Density field - 2nd order



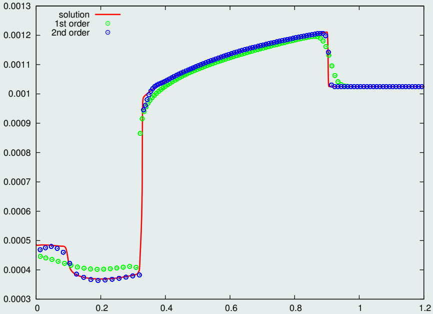
(j) Density profiles

Figure : Solution at time $t = 2.5 \times 10^{-4}$ for a underwater TNT explosion on a 120×9 polar mesh

Underater TNT explosion



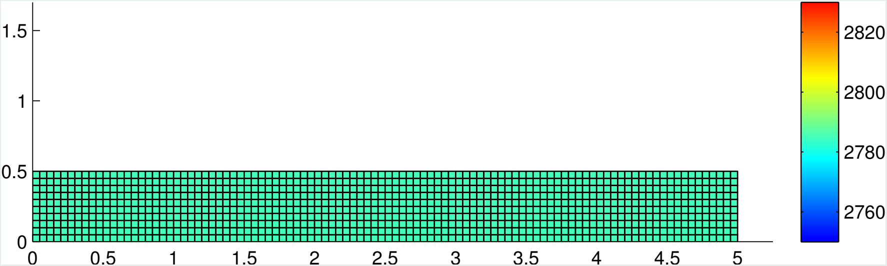
(i) Density field - 2nd order



(j) Density profiles

Figure : Solution at time $t = 2.5 \times 10^{-4}$ for a underwater TNT explosion on a 120×9 polar mesh

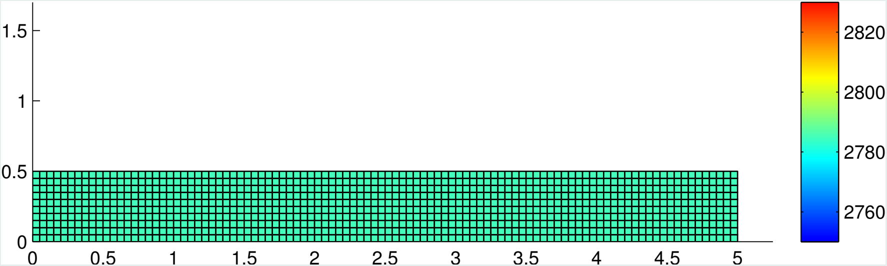
Aluminium projectile impact problem



(k) Density field

Figure : Solution at time $t = 0.05$ for a projectile impact problem on a 100×10 Cartesian mesh

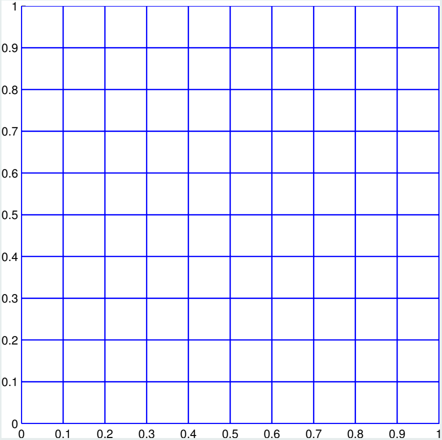
Aluminium projectile impact problem



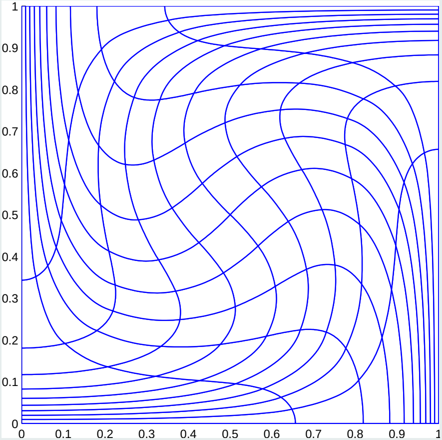
(k) Density field

Figure : Solution at time $t = 0.05$ for a projectile impact problem on a 100×10 Cartesian mesh

Taylor-Green vortex



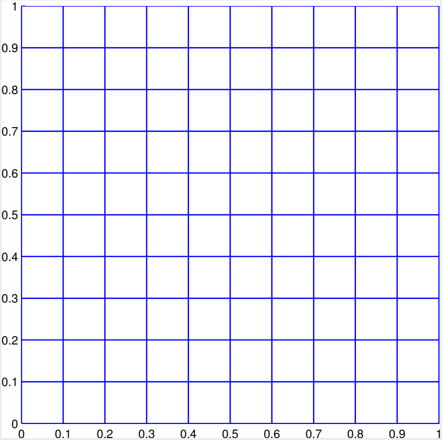
(l) 2nd order



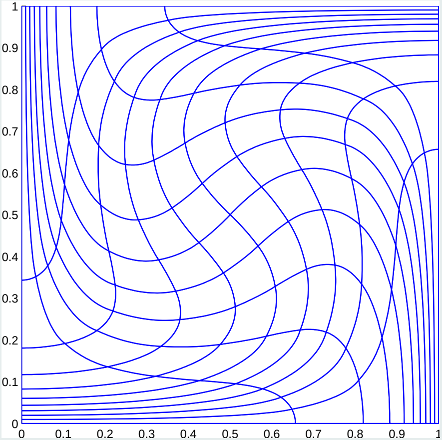
(m) Exact solution

Figure : Final deformed grids at time $t = 0.75$, on a 10×10 Cartesian mesh

Taylor-Green vortex



(l) 2nd order



(m) Exact solution

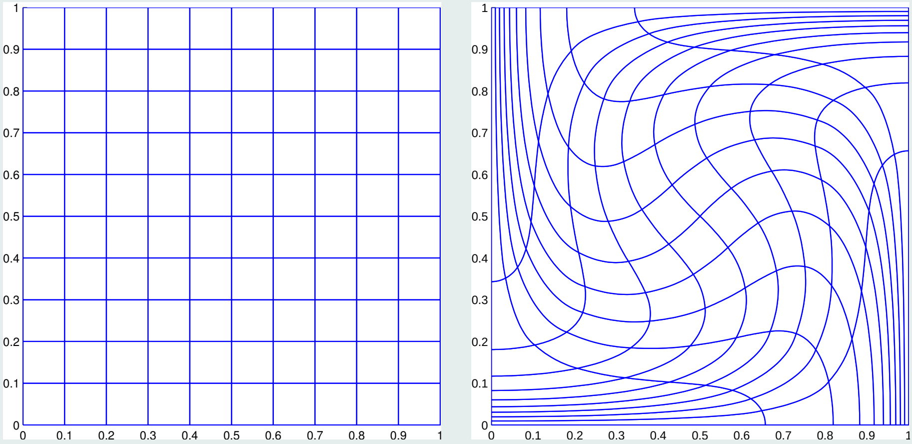
Figure : Final deformed grids at time $t = 0.75$, on a 10×10 Cartesian mesh

Convergence rates

	L_1		L_2		L_∞	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	$E^h_{L_\infty}$	$q^h_{L_\infty}$
$\frac{1}{10}$	5.06E-3	1.94	6.16E-3	1.93	2.20E-2	1.84
$\frac{1}{20}$	1.32E-3	1.98	1.62E-3	1.97	5.91E-3	1.95
$\frac{1}{40}$	3.33E-4	1.99	4.12E-4	1.99	1.53E-3	1.98
$\frac{1}{80}$	8.35E-5	2.00	1.04E-4	2.00	3.86E-4	1.99
$\frac{1}{160}$	2.09E-5	-	2.60E-5	-	9.69E-5	-

Table: Convergence rates on the pressure for a 2nd order DG scheme

Taylor-Green vortex

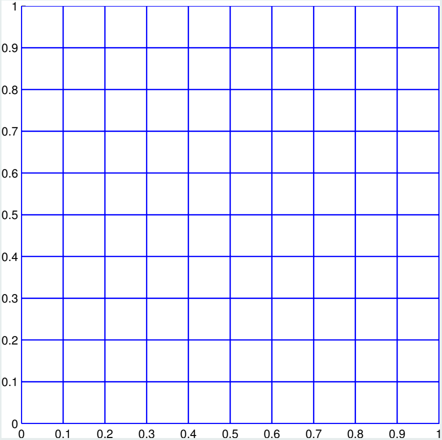


(n) 3rd order

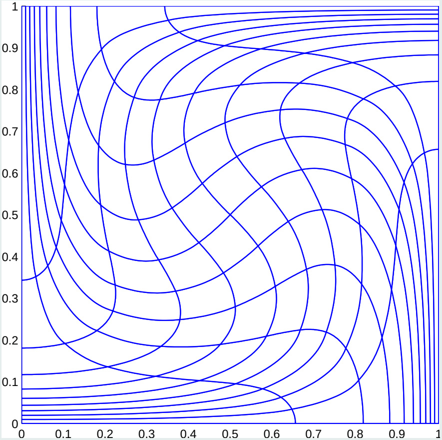
(o) Exact solution

Figure : Final deformed grids at time $t = 0.75$, on a 10×10 Cartesian mesh

Taylor-Green vortex



(n) 3rd order



(o) Exact solution

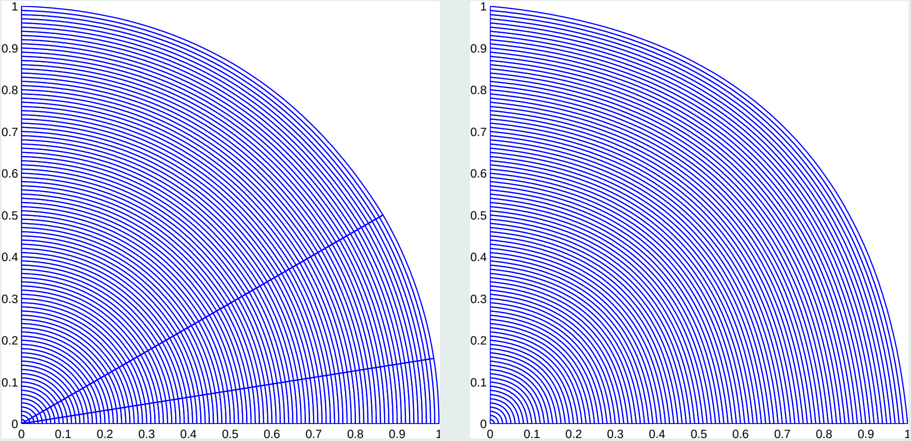
Figure : Final deformed grids at time $t = 0.75$, on a 10×10 Cartesian mesh

Convergence rates

	L_1		L_2		L_∞	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	$E^h_{L_\infty}$	$q^h_{L_\infty}$
$\frac{1}{10}$	2.67E-4	2.96	3.36E-7	2.94	1.21E-3	2.86
$\frac{1}{20}$	3.43E-5	2.97	4.36E-5	2.96	1.66E-4	2.93
$\frac{1}{40}$	4.37E-6	2.99	5.59E-6	2.98	2.18E-5	2.96
$\frac{1}{80}$	5.50E-7	2.99	7.06E-7	2.99	2.80E-6	2.99
$\frac{1}{160}$	6.91E-8	-	8.87E-8	-	3.53E-7	-

Table: Convergence rates on the pressure for a 3rd order DG scheme

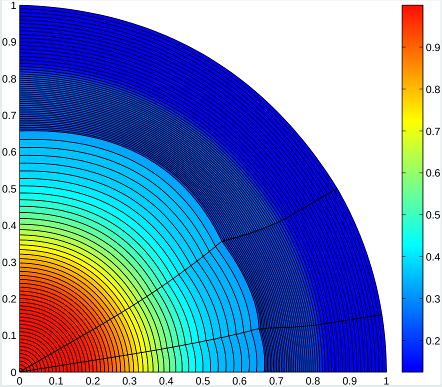
Polar meshes - symmetry preservation



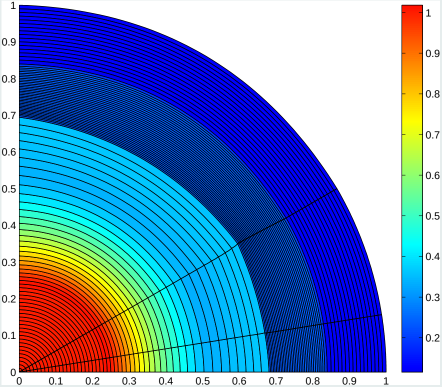
(p) 100×3 (q) 100×1

Figure : Curvilinear grids defined in polar coordinates

Sod shock tube problem - symmetry preservation



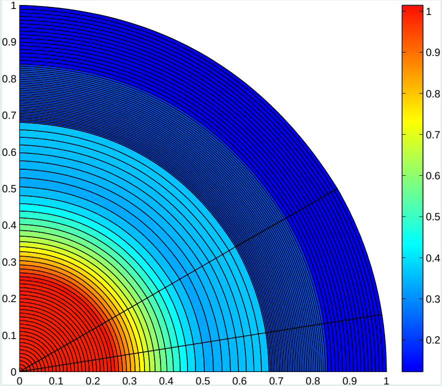
(r) 1st order



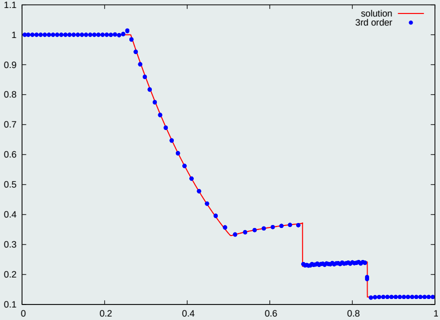
(s) 2nd order

Figure : Density fields with 1st and 2nd order schemes on a 3rd mesh

Sod shock tube problem - symmetry preservation



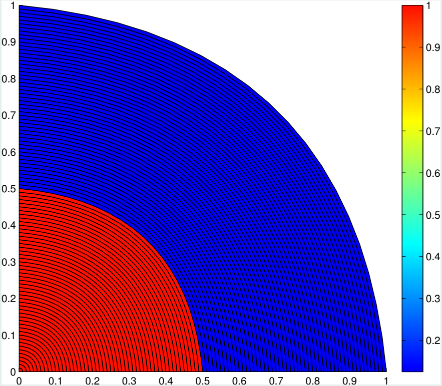
(t) Density field



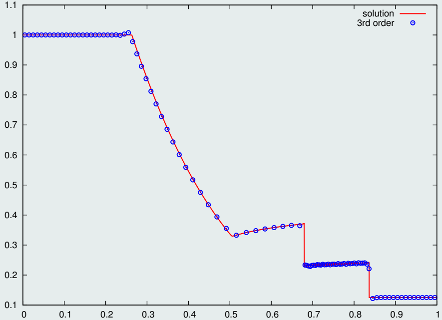
(u) Density profiles

Figure : 3rd order solution for a Sod shock tube problem on a 100×3 polar grid

Sod shock tube problem - symmetry preservation



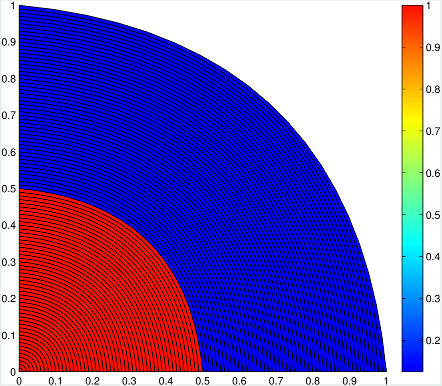
(v) Density field



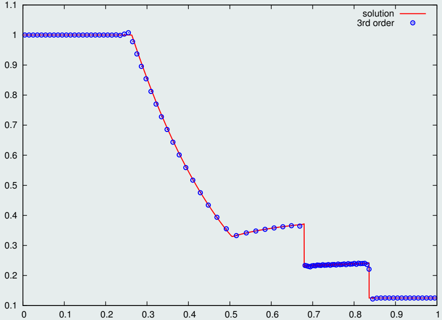
(w) Density profiles

Figure : 3rd order solution for a Sod shock tube problem on a 100 $\times$ 1 polar grid

Sod shock tube problem - symmetry preservation



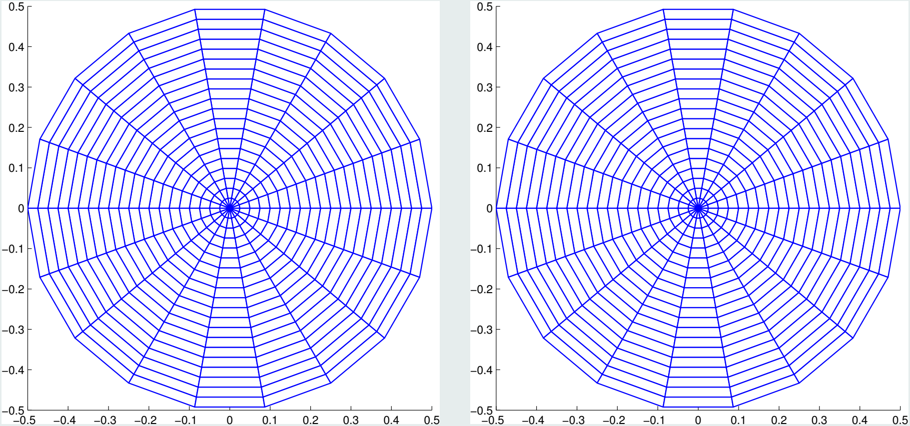
(v) Density field



(w) Density profiles

Figure : 3rd order solution for a Sod shock tube problem on a 100×1 polar grid

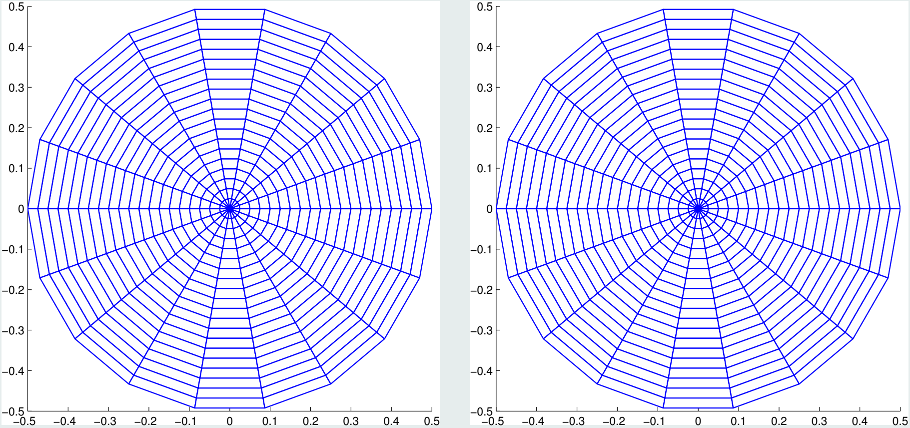
Gresho-like vortex problem



(a) 1st order (b) 2nd order

Figure : Final deformed grids at time $t = 1$, on a 20×18 polar mesh

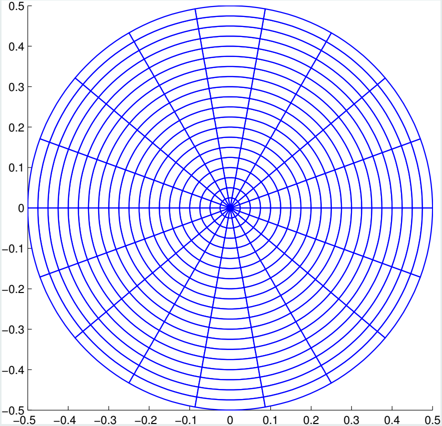
Gresho-like vortex problem



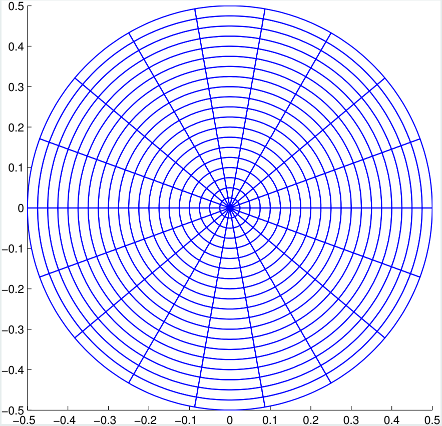
(a) 1st order (b) 2nd order

Figure : Final deformed grids at time $t = 1$, on a 20×18 polar mesh

Gresho-like vortex problem



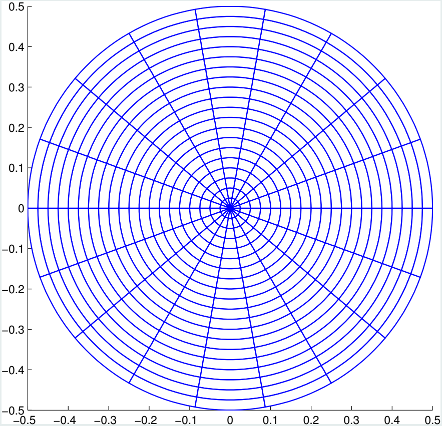
(c) 3rd order



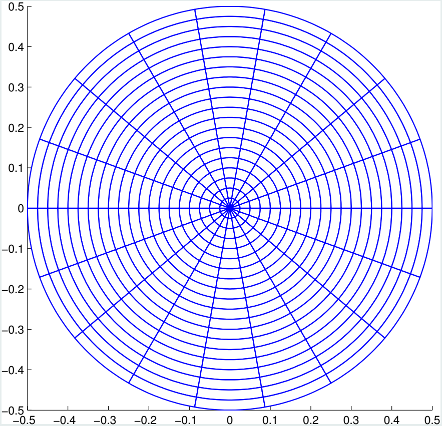
(d) Exact solution

Figure : Final deformed grids at time $t = 1$, on a 20×18 polar mesh

Gresho-like vortex problem



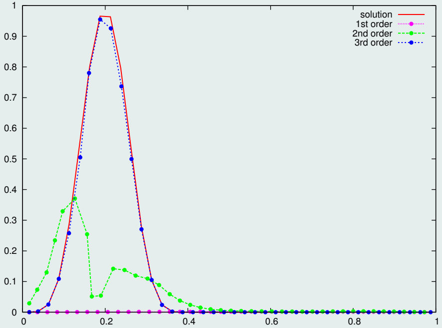
(c) 3rd order



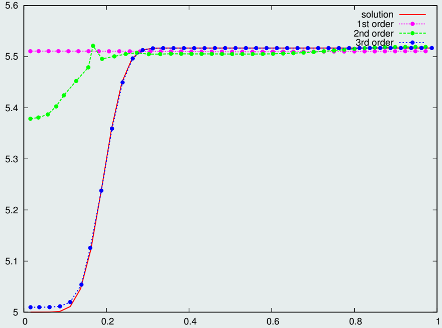
(d) Exact solution

Figure : Final deformed grids at time $t = 1$, on a 20×18 polar mesh

Gresho-like vortex problem



(e) Velocity profiles



(f) Pressure profiles

Figure : Velocity and pressure profiles at time $t = 1$, on a 20×18 polar grid

Gresho-like vortex problem

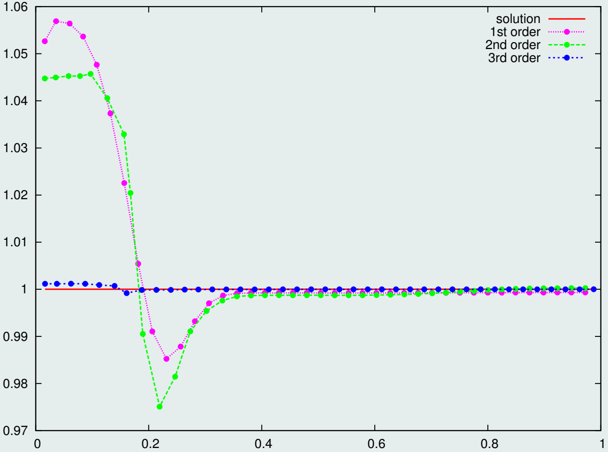
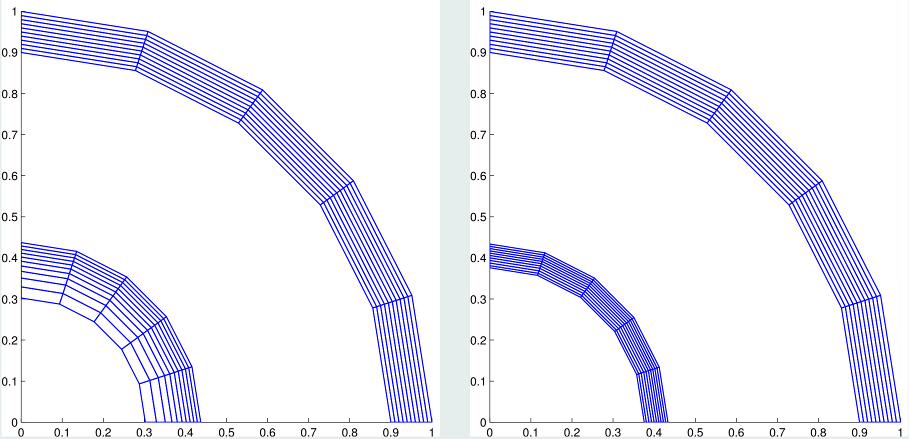


Figure : Density profiles at time $t = 1$, on a 20×18 polar grid

Kidder isentropic compression



(g) 1st order

(h) 2nd order

Figure : Intial and final grids for a Kidder problem on a 10×5 polar mesh

Kidder isentropic compression

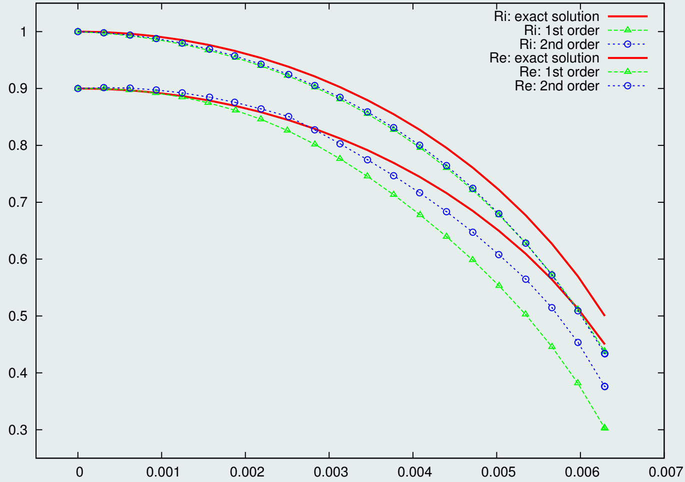
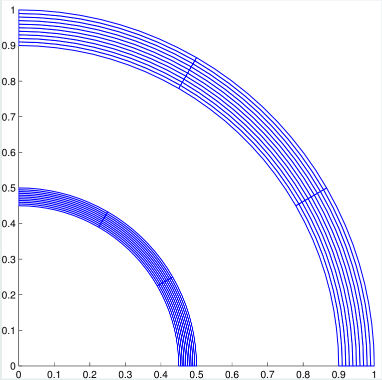
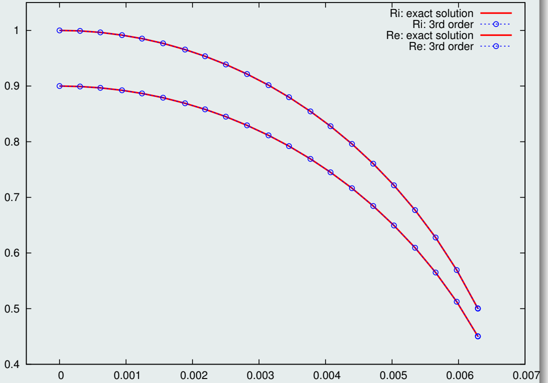


Figure : Interior and exterior shell radii evolution for a Kidder problem on a 10×5 polar mesh

Kidder isentropic compression



(i) Initial and final grids



(j) Shell radii evolution

Figure : 3rd order solution for a Kidder compression problem on a 10×3 polar grid

Accuracy and computational time for a Taylor-Green vortex

D.O.F	N	$E^h_{L_1}$	$E^h_{L_2}$	$E^h_{L_\infty}$	time (sec)
600	24×25	2.67E-2	3.31E-2	8.55E-2	2.01
2400	48×50	1.36E-2	1.69E-2	4.37E-2	11.0

Table: 1st order scheme

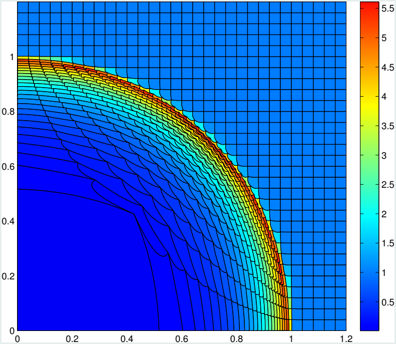
D.O.F	N	$E^h_{L_1}$	$E^h_{L_2}$	$E^h_{L_\infty}$	time (sec)
630	14×15	2.76E-3	3.33E-3	1.07E-2	2.77
2436	28×29	7.52E-4	9.02E-4	2.73E-3	11.3

Table: 2nd order scheme

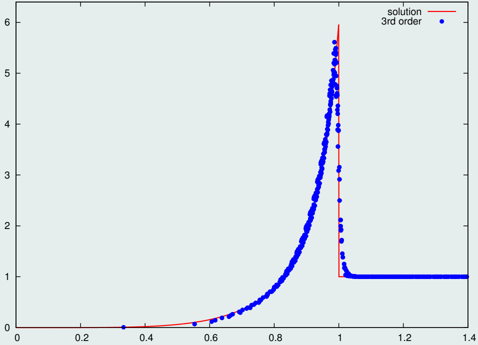
D.O.F	N	$E^h_{L_1}$	$E^h_{L_2}$	$E^h_{L_\infty}$	time (sec)
600	10×10	2.67E-4	3.36E-4	1.21E-3	4.00
2400	20×20	3.43E-5	4.36E-5	1.66E-4	30.6

Table: 3rd order scheme

Sedov point blast problem - spurious deformations



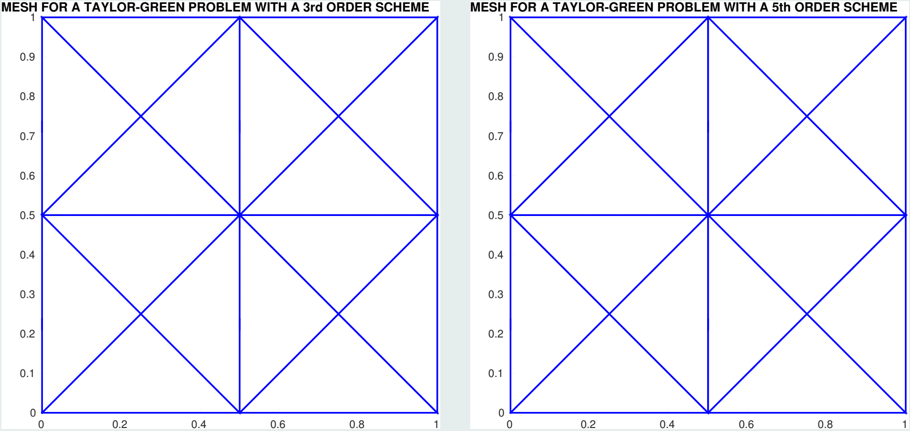
(k) Density field



(l) Density profiles

Figure : Third-order solution at time $t = 1$ for a Sedov problem on a 30×30 Cartesian mesh

Taylor-Green vortex

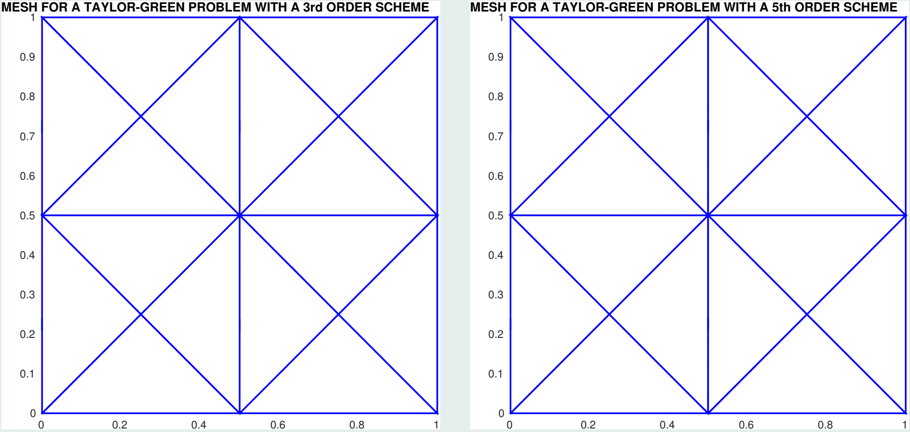


(m) 3rd order

(n) 5th order

Figure : Final deformed grids at time $t = 0.6$, for 16 triangular cells meshes

Taylor-Green vortex

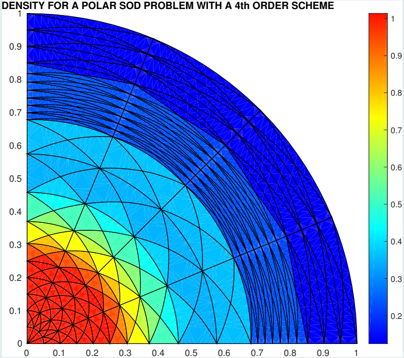


(m) 3rd order

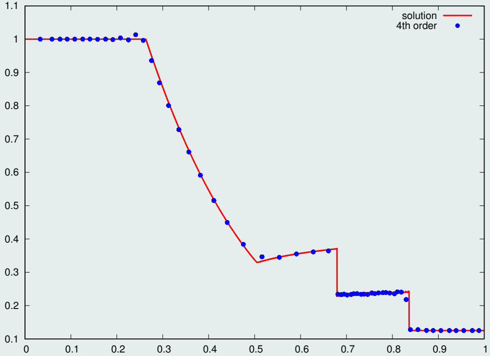
(n) 5th order

Figure : Final deformed grids at time $t = 0.6$, for 16 triangular cells meshes

Sod shock tube problem - symmetry preservation



(o) Density field



(p) Density profiles

Figure : 4th order solution for a Sod shock tube problem on a polar grid made of 308 triangular cells

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- F. VILAR, C.-W. SHU AND P.-H. MAIRE, *Positivity-preserving cell-centered Lagrangian schemes for multi-material compressible flows: Form first-order to high-orders. Part I: The 1D case.* JCP, 2016.
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