1D STEADY-STATE EULER SYSTEM

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CONCLUSIONS AND PERSPECTIVES

- a strategy to construct high accurate FV schemes has been presented to solve the 1D steady-state Burgers equation and Euler system
- identification of an extra equation to achieve an entropic stationary shock
- two solvers are considered: a NL implicit solver and a so-called explicit TM solver
- numerical experiments show that the optimal order of accuracy is reached for smooth solutions
- for non-smooth solutions, the *a posteriori* MOOD stabilization leads to non-oscillatory solutions

- is the NL solver more efficient than the explicit TM scheme?
- can the MOOD loop be improved?
- where we gain with the cascade $\mathbb{P}_5 \to \mathbb{P}_2 \to \mathbb{P}_1 \to \mathbb{P}_0$ comparing with the parachutte $\mathbb{P}_0$ scheme when there is a shock?
- can we reduce the cone of influence of the errors centered in the shock position?
- what if there are sonic points?

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MOTIVATION

- to solve steady-state hyperbolic equations using finite volume schemes
- prototypes: Burgers' equation (scalar case) and Euler's equation (vectorial case)
- regular solutions: high accuracy
- solutions with a shock: stability (no oscillations) and accuracy (as possible)
- approach: MOOD (Multidimensional Optimal Order Detection)

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inviscid Burgers' equation

- we seek the velocity function $\phi = \phi(x)$, solution of the 1D steady-state inviscid Burgers' equation

$$\frac{\mathrm{d}\mathbb{F}(\phi)}{\mathrm{d}x} = f, \text{ in } \Omega = (0,1)$$

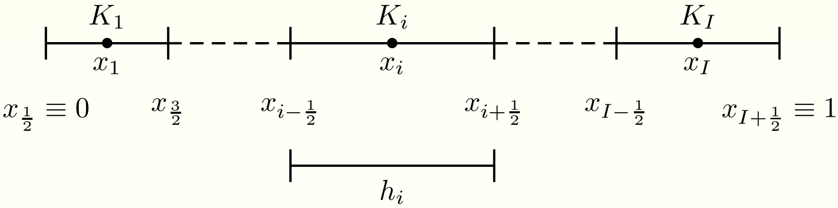
- with Dirichlet boundary conditions

$$\phi = \phi_{\mathsf{lf}}, \text{ on } x = 0$$
$$\phi = \phi_{\mathsf{rg}}, \text{ on } x = 1$$

- with

$$\mathbb{F}(\phi) = \frac{\phi^2}{2}$$
$$f = f(x)$$

NOTATION



- K_i — cell i
- I — number of cells
- $x_{i-\frac{1}{2}}$, $x_{i+\frac{1}{2}}$ — boundary points of cell i
- h_i — length of cell i
- x_i — centroid of cell i

FV scheme (I)

- integrating equation $\frac{\mathrm{d}\mathbb{F}(\phi)}{\mathrm{d}x} = f$ over cell K_i results in

$$\frac{1}{h_i}\left(\mathbb{F}_{i+\frac{1}{2}} - \mathbb{F}_{i-\frac{1}{2}}\right) - \bar{f}_i = 0$$

  with

$$\mathbb{F}_{i+\frac{1}{2}} = \mathbb{F}(\phi(x_{i+\frac{1}{2}}))$$
$$\bar{f}_i = \frac{1}{h_i}\int_{K_i} f(\xi)\,\mathrm{d}\xi$$

- let

$$\mathcal{F}_{i+\frac{1}{2}} \approx F_{i+\frac{1}{2}}$$
$$f_i \approx \bar{f}_i$$

  the residual at cell K_i

$$\mathcal{G}_i = \frac{1}{h_i}\left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}\right) - f_i$$

FV scheme (II)

- goal — to compute an approximation ϕ_i of the mean value of ϕ in each cell of the mesh
- the approximation to the mean value of f over cell K_i, f_i, will be computed by gaussian quadrature
- we will consider the Rusanov numerical flux
- to achieve high-order numerical approximations, we introduce local polynomial reconstructions of the underlying solutions
- stencil: the stencil S_i of cell K_i is composed of the $\mathrm{d_i} + 1$ closest neighbour cells excluding cell K_i

POLYNOMIAL RECONSTRUCTIONS

- reconstruction: the polynomial $\widehat{\phi}_i(x; \mathrm{d_i})$ is based on the data associated to the stencil under a least-square technique

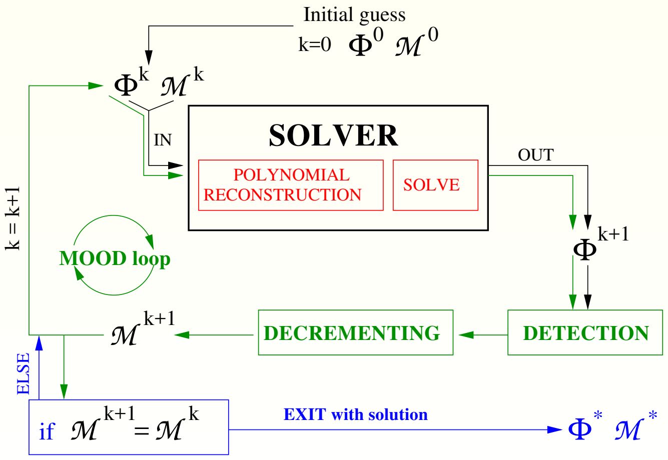
$$\phi_i(x; \mathrm{d}_i) = \sum_{\alpha=0}^{\mathrm{d}_i} \mathcal{R}_{i,\alpha}(x - m_i)^\alpha$$

$$\min_{\mathcal{R}_{i,\mathbf{0}},\ldots,\mathcal{R}_{i,\mathbf{d}_i}} \quad \sum_{j \in \widehat{S}_i} \omega_j \left[\frac{1}{h_j} \int_{c_j} \phi_i(x; \mathrm{d}_i) \mathrm{d}x - \phi_j \right]^2$$

$$\text{s.t.} \quad \frac{1}{h_i} \int_{c_i} \phi_i(x; \mathrm{d}_i) \mathrm{d}x = \phi_i \text{ (mean value conservation)}$$

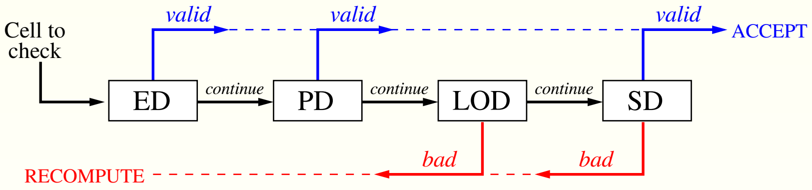
- to construct a generic high-order scheme one has to substitute the left and right states in by states evaluated through high-order polynomial reconstructions

- TM ...
- NL



Initial guess
k=0 Φ^0 $\mathcal{M}^0$
Φ^k $\mathcal{M}^k$
IN
SOLVER
POLYNOMIAL RECONSTRUCTION
SOLVE
OUT
Φ^{k+1}
k = k+1
MOOD loop
$\mathcal{M}^{k+1}$
DECREMENTING
DETECTION
ELSE
if $\mathcal{M}^{k+1} = \mathcal{M}^k$
EXIT with solution
Φ^* $\mathcal{M}^*$

Burgers chain detector



ED: Extrema Detector
PD: Plateau Detector
LOD: Local Oscillation Detector
SD: Smoothness Detector

Burgers: manufactured regular solution (I)

- manufactured regular solution

$$\phi(x) = \sin(3\pi x)\exp(x) + 2$$

then

$$f(x) = (\exp(x)\sin(3\pi x) + 2)(\exp(x)\sin(3\pi x) + 3\pi\exp(x)\cos(3\pi x))$$
$$\phi_{\mathrm{lf}} = 2$$
$$\phi_{\mathrm{rg}} = 2$$

Burgers: manufactured regular solution (ii)

	I	NS E_1	NS $\mathcal{O}_1$	NS E_∞	NS $\mathcal{O}_\infty$	TM E_1	TM $\mathcal{O}_1$	TM E_∞	TM $\mathcal{O}_\infty$
$\mathbb{P}_0$	70	7.8E−02	—	1.9E−01	—	7.8E−02	—	1.9E−01	—
	80	6.8E−02	1.1	1.6E−01	1.0	6.8E−02	1.1	1.6E−01	1.0
	90	6.0E−02	1.0	1.5E−01	1.0	6.0E−02	1.0	1.5E−01	1.0
	100	5.4E−02	1.0	1.3E−01	1.0	5.4E−02	1.0	1.3E−01	1.0
$\mathbb{P}_1$	70	1.8E−03	—	7.4E−03	—	1.8E−03	—	7.4E−03	—
	80	1.4E−03	2.0	5.4E−03	2.4	1.4E−03	2.0	5.4E−03	2.4
	90	1.1E−03	2.0	4.1E−03	2.3	1.1E−03	2.0	4.1E−03	2.3
	100	8.6E−04	2.0	3.2E−03	2.3	8.6E−04	2.0	3.2E−03	2.3
$\mathbb{P}_5$	70	1.1E−07	—	8.8E−07	—	1.8E−07	—	6.2E−07	—
	80	4.9E−08	6.0	3.8E−07	6.2	1.0E−07	4.3	3.6E−07	4.0
	90	2.4E−08	6.1	1.8E−07	6.2	5.7E−08	4.8	2.1E−07	4.7
	100	1.2E−08	6.2	1.1E−07	5.2	3.3E−08	5.1	1.2E−07	5.0

Burgers: solution with a shock

- data

$$f(x) = -\pi \cos(\pi x)\phi(x)$$
$$\phi_{\mathrm{lf}} = 1$$
$$\phi_{\mathrm{rg}} = -0.1$$

- analytical solution

$$\phi(x) = \begin{cases} 1 - \sin(\pi x) & \text{if } 0 \leq x \leq 0.1486 \\ -0.1 - \sin(\pi x) & \text{if } 0.1486 \leq x \leq 1 \end{cases}$$

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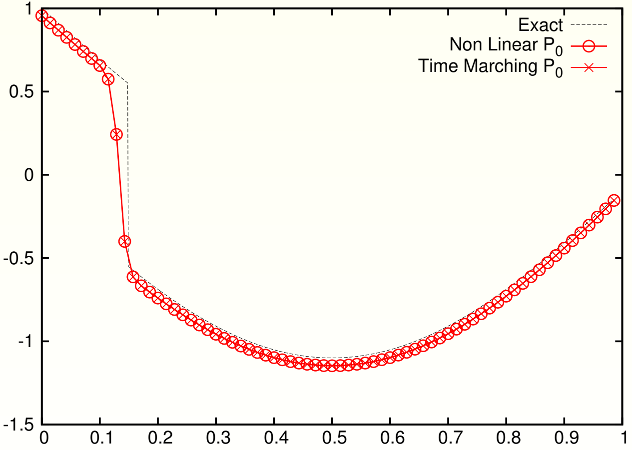
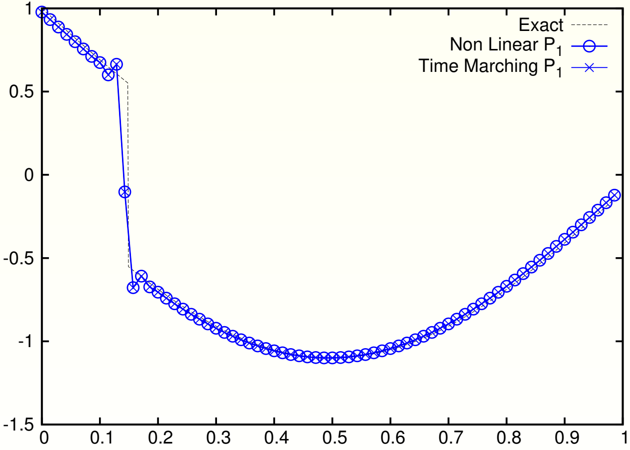
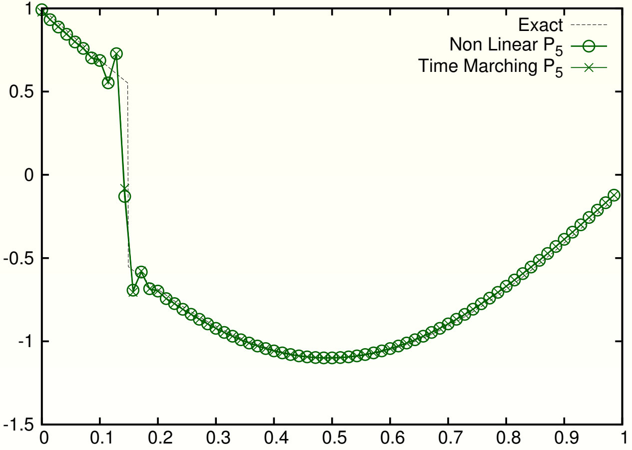
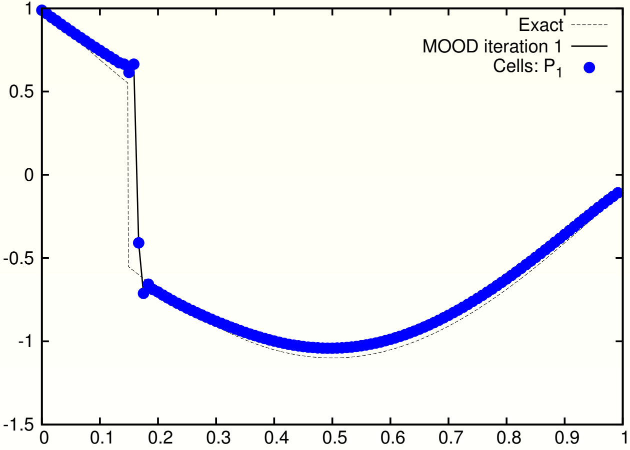

Exact
Non Linear P_0
Time Marching P_0
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

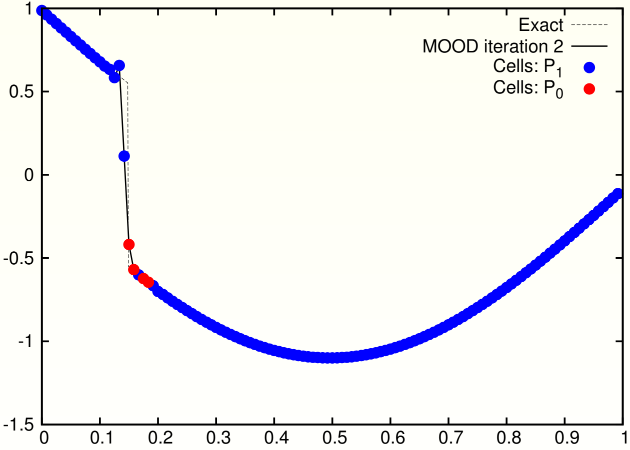
NL + TM | UNLIMITED $\mathbb{P}_1$

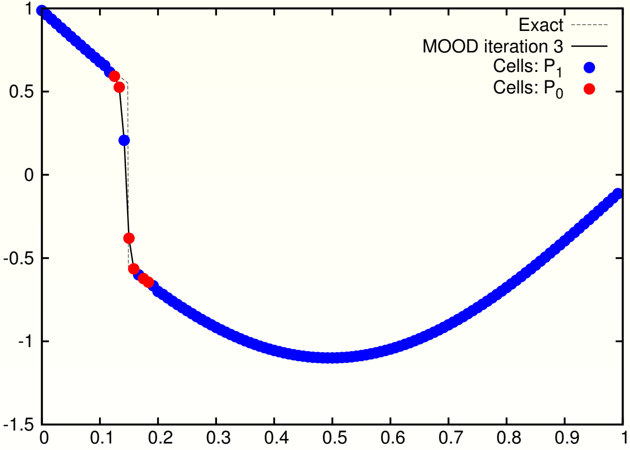


NL + TM | UNLIMITED $\mathbb{P}_5$

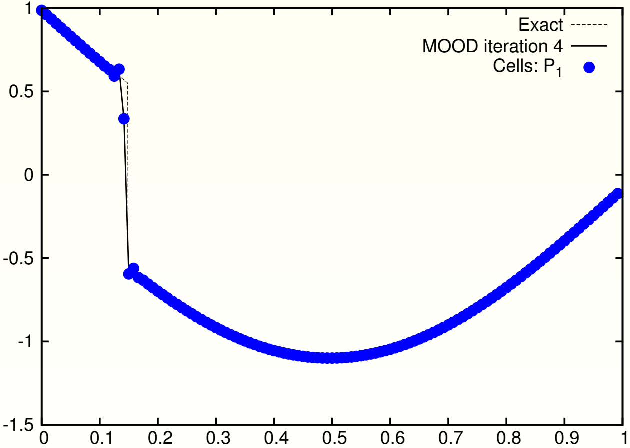


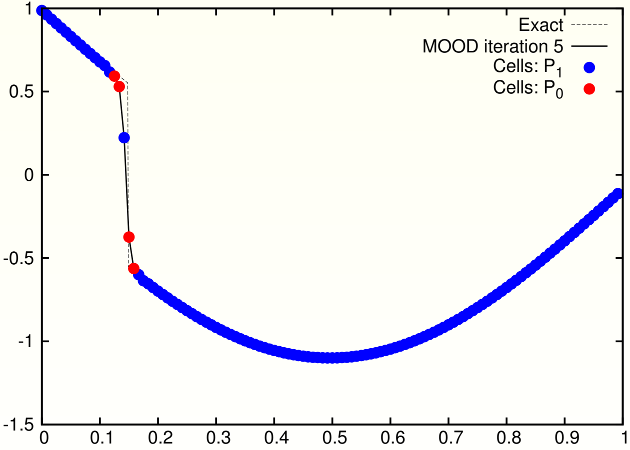

Exact
MOOD iteration 1
Cells: P_1
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

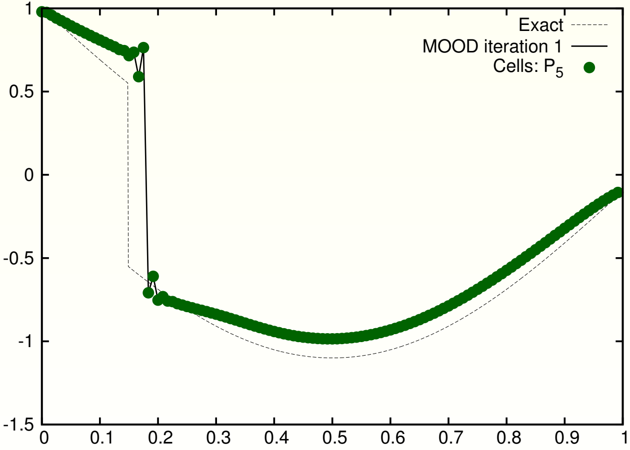

Exact
MOOD iteration 2
Cells: P1
Cells: P0
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

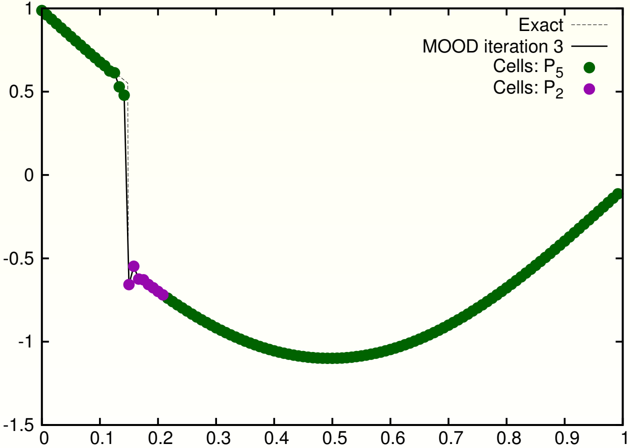

Exact
MOOD iteration 3
Cells: P1
Cells: P0
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

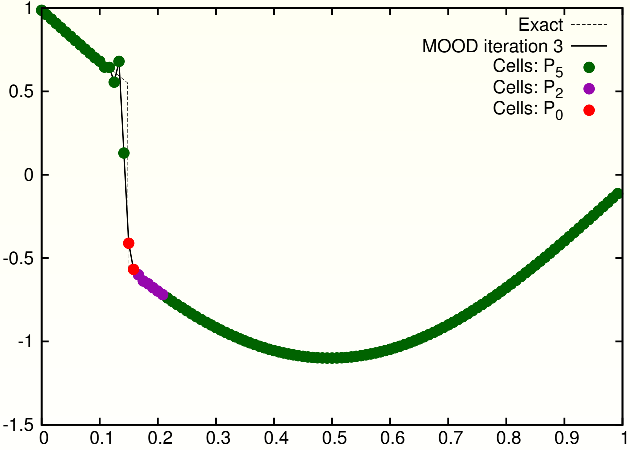
NL | $\mathbb{P}_1 \to \mathbb{P}_0$ | IT4 — REINITIALIZATION



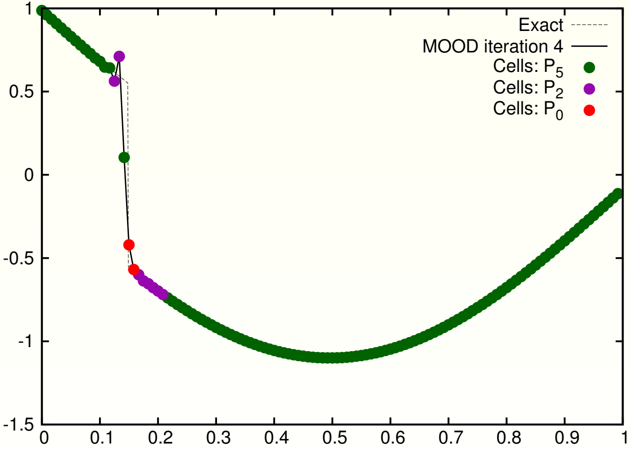

Exact
MOOD iteration 5
Cells: P_1
Cells: P_0
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Exact
MOOD iteration 1
Cells: P5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

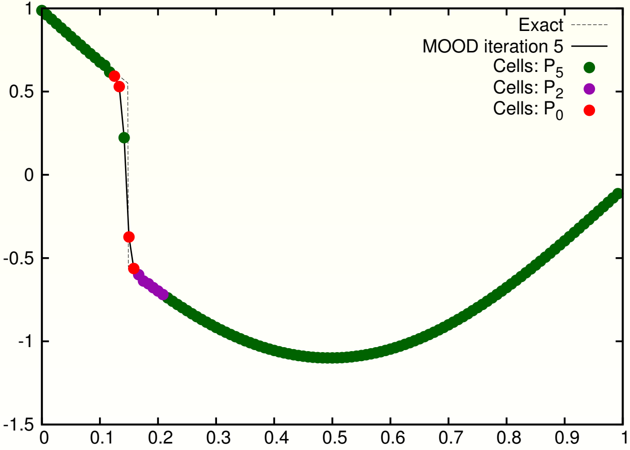

Exact
MOOD iteration 3
Cells: P5
Cells: P2
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

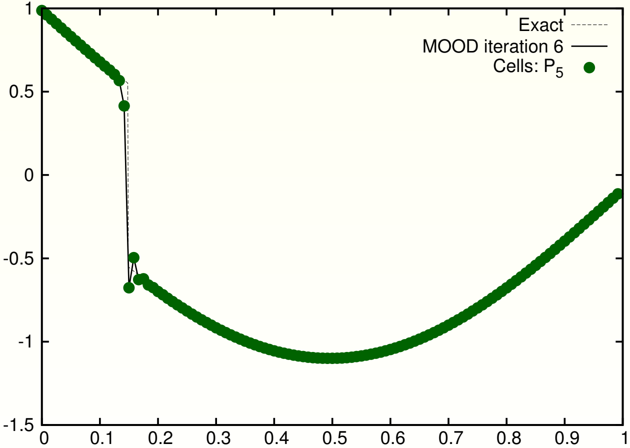

Exact
MOOD iteration 3
Cells: P5
Cells: P2
Cells: P0
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

NL | $\mathbb{P}_5 \to \mathbb{P}_2 \to \mathbb{P}_0$ | IT4



NL | $\mathbb{P}_5 \to \mathbb{P}_2 \to \mathbb{P}_0$ | IT5 (4+6)




Exact
MOOD iteration 6
Cells: P_5
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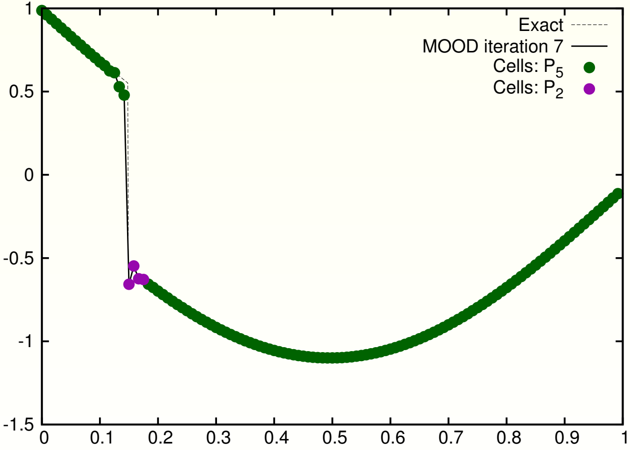
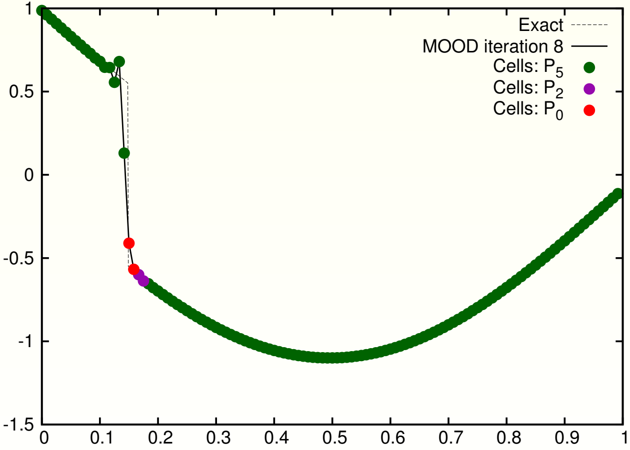
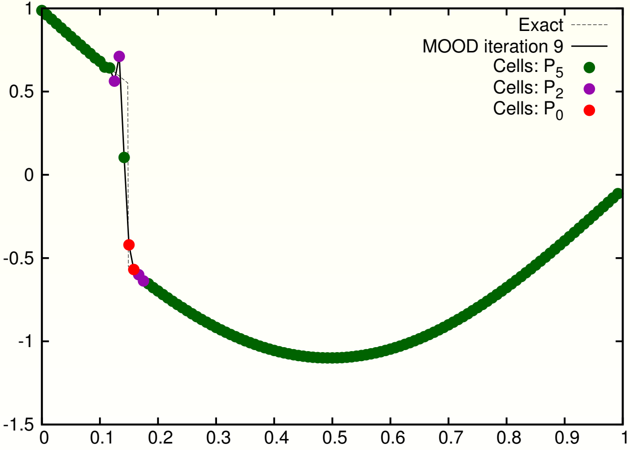
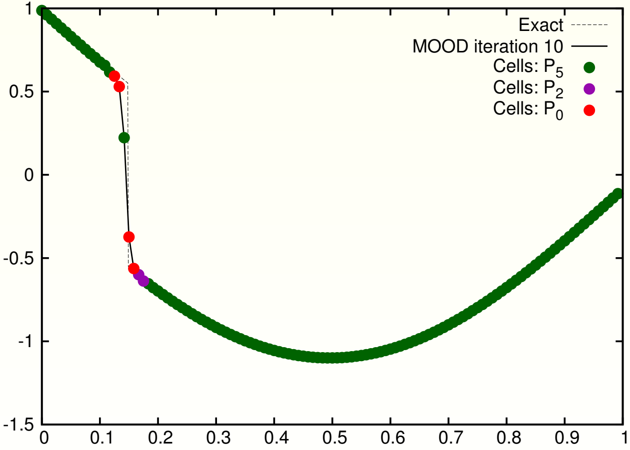
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Exact
MOOD iteration 7
Cells: P5
Cells: P2
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

NL | $\mathbb{P}_5 \to \mathbb{P}_2 \to \mathbb{P}_0$ | IT8



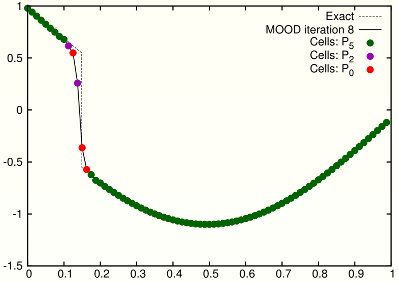

Exact
MOOD iteration 9
Cells: P5
Cells: P2
Cells: P0
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



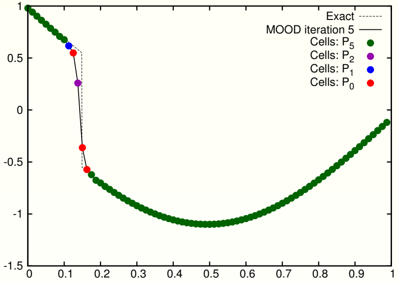
Exact
MOOD iteration 10
Cells: P5
Cells: P2
Cells: P0
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

NL | $I = 80$

$\mathbb{P}_5 \to \mathbb{P}_2 \to \mathbb{P}_0$

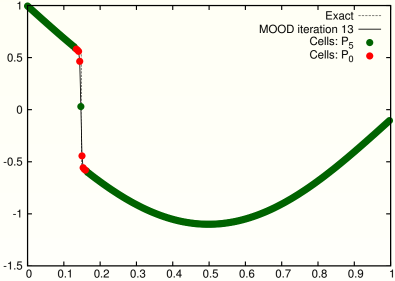


$\mathbb{P}_5 \to \mathbb{P}_2 \to \mathbb{P}_1 \to \mathbb{P}_0$

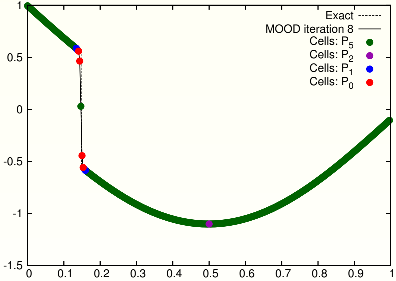


NL | $I = 320$

$\mathbb{P}_5 \rightarrow \mathbb{P}_2 \rightarrow \mathbb{P}_0$



$\mathbb{P}_5 \rightarrow \mathbb{P}_2 \rightarrow \mathbb{P}_1 \rightarrow \mathbb{P}_0$



Euler system formulation

- Euler equations

$$\frac{\mathrm{d}\mathbb{F}(W)}{\mathrm{d}x} = \frac{\mathrm{d}S}{\mathrm{d}x}, \text{ in } \Omega = (0,1)$$

- conservative variable $W = (\rho, \rho u, E)^T$
- total energy $E = \frac{1}{2}\rho u^2 + \rho e$ (e is the specific internal energy)
- for an ideal gas, this system is closed by the equation of state $e = \frac{p}{\rho(\gamma-1)}$ (γ the ratio of specific heats with $\gamma = \frac{7}{5}$ in our studies)
- physical flux $\mathbb{F}(W) = (\rho u, \rho u^2 + p, u(E+p))^T$
- the source term $\frac{\mathrm{d}S}{\mathrm{d}x}$, $S = (D, F, H)^T = (D(x), F(x), H(x))^T$
- Dirichlet boundary conditions

$$W(0) = W_{\mathrm{lf}}$$
$$W(1) = W_{\mathrm{rg}}$$

EULER'S EQUATIONS (I)

- existence of solution requires that the following compatibility condition are satisfied

$$F > 0,\ \ HD \geq 0,\ \ \frac{\gamma^2}{\gamma^2 - 1} \geq 2\frac{DH}{F^2}$$

- if D is positive, the supersonic solution writes

$$\hat{\rho}_{\mathsf{sup}}(D, F, H) = \frac{2\gamma F - \sqrt{4\gamma^2 F^2 - 4(\gamma+1)(\gamma+1)DH^2}}{4(\gamma-1)\frac{H}{D}}$$

- if D is positive, the subsonic solution writes

$$\hat{\rho}_{\mathsf{sub}}(D, F, H) = \frac{2\gamma F + \sqrt{4\gamma^2 F^2 - 4(\gamma+1)(\gamma+1)DH^2}}{4(\gamma-1)\frac{H}{D}}$$

- $W_{\mathsf{lf}} = W(D(0), F(0), H(0))$ and $W_{\mathsf{rg}} = W(D(1), F(1), H(1))$
- there are two choices for each condition: subsonic or supersonic

Euler's equations (II)

- Sup-Sup: smooth solution (in most of the cases)
- Sub-Sub: smooth solution (in most of the cases)

Theorem

A steady-state solution admits an entropic stationary shock at point x_c if Assume that a steady-state solution admits an entropic stationary genuinely nonlinear shock at point x_c. Then there only exist two admissible situations:

(a) *the solution is supercritical on the left and subcritical on the right with $D > 0$ and $H > 0$;*

(b) *the solution is subcritical on the left and supercritical on the right with $D < 0$ and $H < 0$.*

- Sub-Sup: non entropic shock
- Sup-Sub: solution with an entropic shock (need 1 more equation to fix the shock position)

Euler's equations (iii)

Theorem

Assume that density $\widetilde{\rho}(x,t)$, the mass flow $\widetilde{\rho u}(x,t)$, and the total energy $\widetilde{E}$ are continuous at the boundaries for $t > 0$, that is, there are no jumps between the boundary conditions and the solution at x_{L} and x_{R}. Then for any time $t > 0$, we have

$$\int_0^1 \widetilde{\phi}(x,t)\,\mathrm{d}x = \int_0^1 \phi^0(x)\,\mathrm{d}x, \qquad \widetilde{\phi} = \widetilde{\rho}, \widetilde{\rho u}, \widetilde{E}.$$

Moreover, if the solution converges to a steady-state solution in time, denoted by ρ, u, and E, then we have

$$\begin{aligned}
\int_0^1 \rho^0(x)\,\mathrm{d}x &= \int_0^1 \rho(x)\,\mathrm{d}x, \\
\int_0^1 \rho^0(x)u^0(x)\,\mathrm{d}x &= \int_0^1 D(x)\,\mathrm{d}x, \\
\int_0^1 E^0(x)\,\mathrm{d}x &= \int_0^1 E(x)\,\mathrm{d}x.
\end{aligned}$$

Euler's equations (IV)

Corrolary

- *Assume that the initial conditions satisfy the theorem condition with $D > 0$, $H > 0$.*
- *Then the steady-state solution in constituted of the supersonic solution on $[0, x_c[$ and the subsonic solution on $]x_c, 1]$ with an entropic shock located at the unique point x_c such that*

$$\int_0^{x_c} \rho_{\mathrm{sup}}(x)\,\mathrm{d}x + \int_{x_c}^1 \rho_{\mathrm{sub}}(x)\,\mathrm{d}x = \int_0^1 \rho^0(x)\,\mathrm{d}x.$$

this is the equation which fixes the shock position

KEYWORDS

- high-order FV scheme ...
- local polynomial reconstructions ...
- HLL flux ...
- MOOD ...
- nonlinear solver ...
- time marching solver ...

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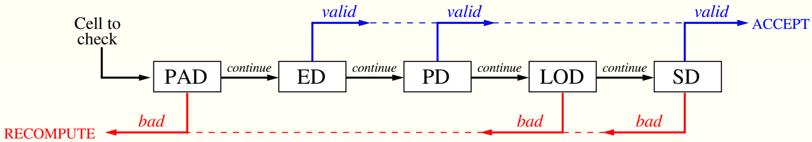
Euler chain detector



PAD: Physical Admissible Detector (density + pressure)
ED: Extrema Detector
PD: Plateau Detector
LOD: Local Oscillation Detector
SD: Smoothness Detector

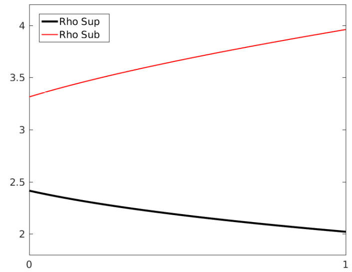
NUMERICAL TESTS

- $D(x) = 1,\ F(x) = 0.027x + 0.6137,\ H(x) = 0.375$
- supersonic branch:

$$\rho_{\text{sup}}(x) = 0.126x - 3.3333\sqrt{1.96(0.027x + 0.6137)^2 - 0.72} + 2.8639$$

- subsonic branch:

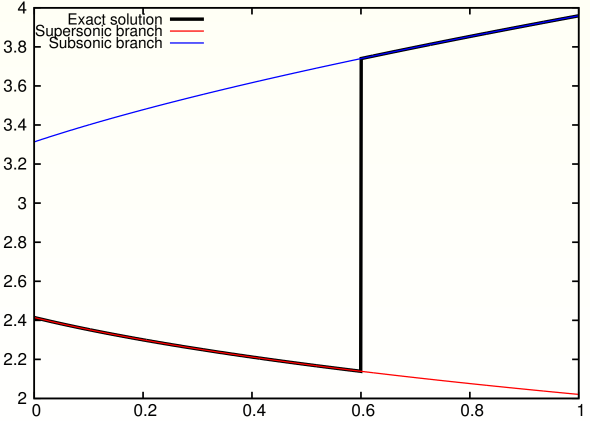
$$\rho_{\text{sub}}(x) = 0.126x + 3.3333\sqrt{1.96(0.027x + 0.6137)^2 - 0.72} + 2.8639$$

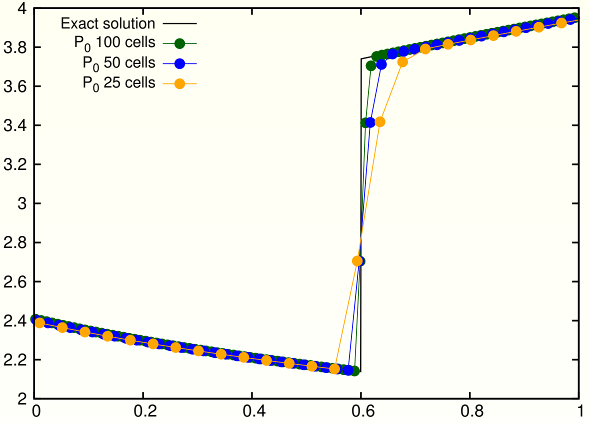


	I	NL E_1	NL $\mathcal{O}_1$	NL E_∞	NL $\mathcal{O}_\infty$	TM E_1	TM $\mathcal{O}_1$	TM E_∞	TM $\mathcal{O}_\infty$
$\mathbb{P}_0$	25	7.8E−03	—	1.3E−02	—	7.8E−02	—	1.9E−01	—
	50	3.9E−03	1.0	6.6E−03	1.0	6.8E−02	1.1	1.6E−01	1.0
	75	2.6E−03	1.0	4.4E−03	1.0	6.0E−02	1.0	1.5E−01	1.0
	100	2.0E−03	1.0	3.3E−03	1.0	5.4E−02	1.0	1.3E−01	1.0
$\mathbb{P}_1$	25	9.0E−05	—	9.3E−04	—	1.8E−03	—	7.4E−03	—
	50	1.9E−05	2.3	2.6E−04	1.8	1.4E−03	2.0	5.4E−03	2.4
	75	7.7E−06	2.2	1.2E−04	1.9	1.1E−03	2.0	4.1E−03	2.3
	100	4.1E−06	2.2	6.9E−05	1.9	8.6E−04	2.0	3.2E−03	2.3
$\mathbb{P}_5$	25	1.5E−07	—	3.4E−06	—	1.8E−07	—	6.2E−07	—
	50	2.9E−09	5.7	1.3E−07	4.7	1.0E−07	4.3	3.6E−07	4.0
	75	2.4E−10	6.1	1.6E−08	5.2	5.7E−08	4.8	2.1E−07	4.7
	100	4.0E−11	6.3	3.4E−09	5.4	3.3E−08	5.1	1.2E−07	5.0

DISCONTINUOUS CASE

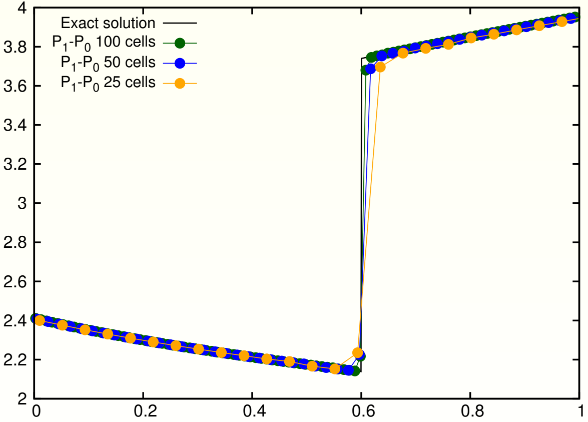
- $x_c = 0.6$
- supersonic-subsonic
- density variable
- NL scheme



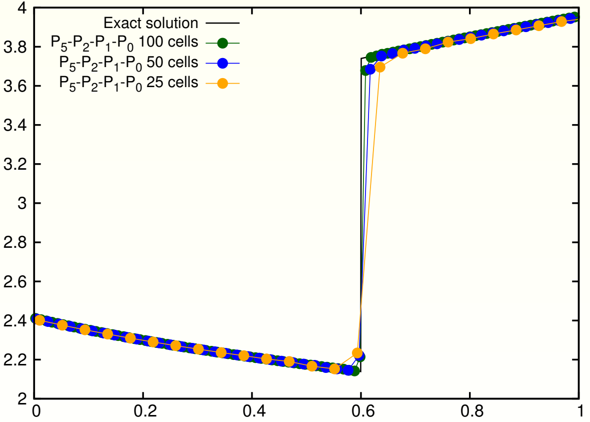


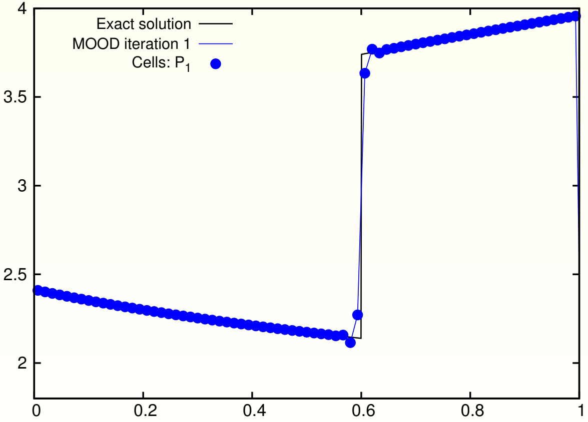
- we observe a convergence towards the exact solution when the mesh size increases without any spurious oscillation
- this validates the robustness of the $\mathbb{P}_0$ scheme that is used as the parachute scheme of the MOOD cascade

DISCONTINUOUS CASE | $\mathbb{P}_1 \to \mathbb{P}_0$

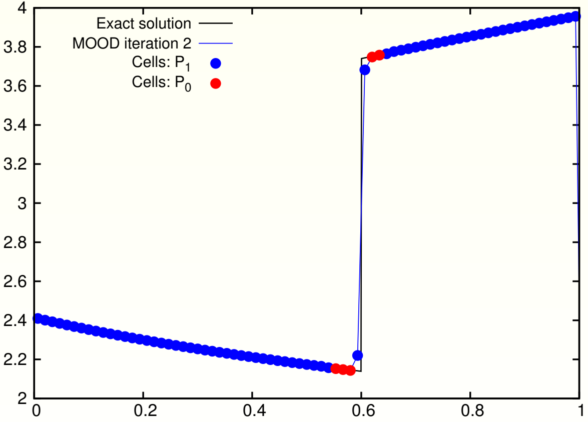


DISCONTINUOUS CASE | $\mathbb{P}_5 \rightarrow \mathbb{P}_2 \rightarrow \mathbb{P}_1 \rightarrow \mathbb{P}_0$

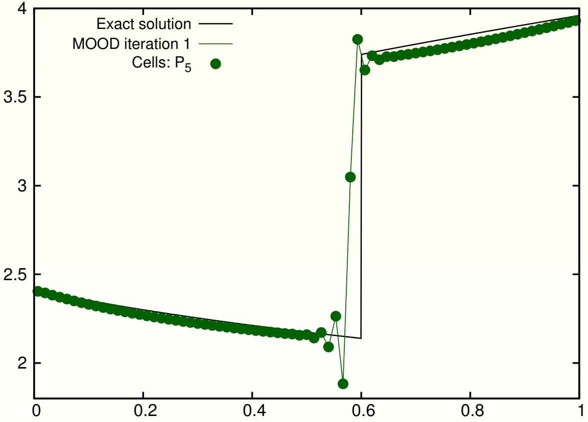


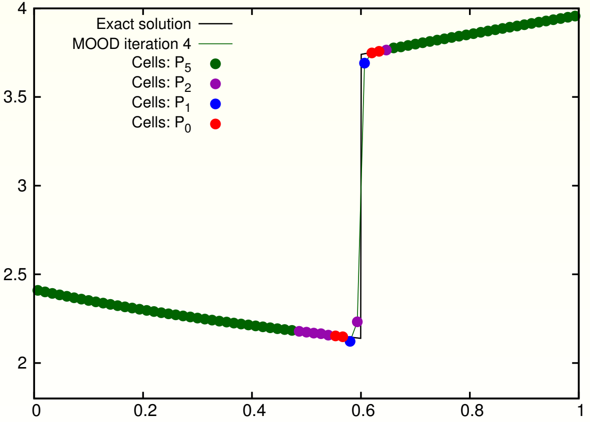

Exact solution
MOOD iteration 1
Cells: P_1
4
3.5
3
2.5
2
0
0.2
0.4
0.6
0.8
1

DISCONTINUOUS CASE | $\mathbb{P}_1 \rightarrow \mathbb{P}_0$ | IT2

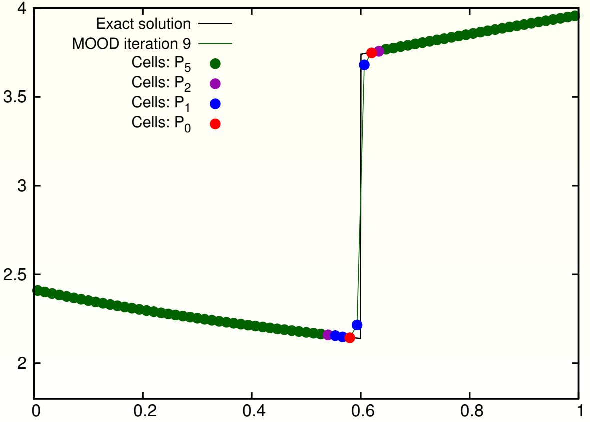


DISCONTINUOUS CASE | $\mathbb{P}_5 \to \mathbb{P}_2 \to \mathbb{P}_1 \to \mathbb{P}_0$ | IT1

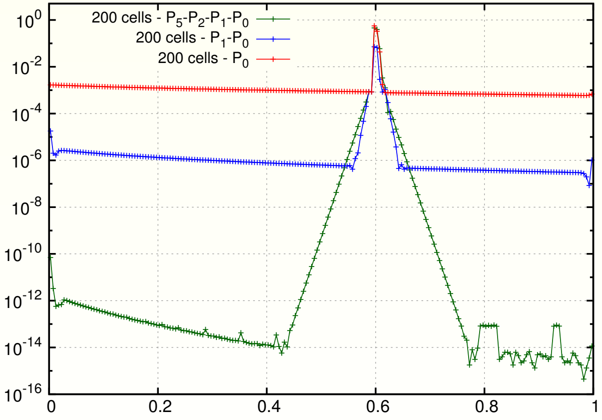



Exact solution
MOOD iteration 4
Cells: P5
Cells: P2
Cells: P1
Cells: P0
4
3.5
3
2.5
2
0
0.2
0.4
0.6
0.8
1

DISCONTINUOUS CASE | $\mathbb{P}_5 \rightarrow \mathbb{P}_2 \rightarrow \mathbb{P}_1 \rightarrow \mathbb{P}_0$ | IT9



DISCONTINUOUS CASE | ERRORS



CONCLUSIONS AND PERSPECTIVES

- a strategy to construct high accurate FV schemes has been presented to solve the 1D steady-state Burgers equation and Euler system
- identification of an extra equation to achieve an entropic stationary shock
- two solvers are considered: a NL implicit solver and a so-called explicit TM solver
- numerical experiments show that the optimal order of accuracy is reached for smooth solutions
- for non-smooth solutions, the *a posteriori* MOOD stabilization leads to non-oscillatory solutions

- is the NL solver more efficient than the explicit TM scheme?
- can the MOOD loop be improved?
- where we gain with the cascade $\mathbb{P}_5 \to \mathbb{P}_2 \to \mathbb{P}_1 \to \mathbb{P}_0$ comparing with the parachutte $\mathbb{P}_0$ scheme when there is a shock?
- can we reduce the cone of influence of the errors centered in the shock position?
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