Recent advances on numerical simulation in coastal oceanography

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Work in collaboration with F. Marche (IMAG Montpellier)

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Outline

1. Shallow Water Equations - Generalities
2. dG discretization
3. Extension to dispersive equations
4. Handling breaking waves (work with G. Richard)
5. Perspectives

Outline

2D Formulation

$$\partial_t U + \nabla . G(U) = B(U, z) \,.$$

$$U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} , \; G(U) = \begin{pmatrix} hu & hv \\ \frac{1}{2}gh^2 + hu^2 & huv \\ huv & \frac{1}{2}gh^2 + hv^2 \end{pmatrix} , \; B(U, z) = \begin{pmatrix} 0 \\ -gh\partial_x z \\ -gh\partial_y z \end{pmatrix}$$

Application field : some examples

Coastal hydrodynamic

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Shallow Water Equations

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Application field : some examples

Tsunamis

2D Formulation

$$\partial_t U + \nabla . G(U) = B(U, z) \, .$$

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Application field : some examples

Rivers,
dam breaks

Numerical issues

Stability criteria

- Preservation of steady states :
  $\rightarrow$ **(C-property)** [Bermudez & Vázquez, 1994]

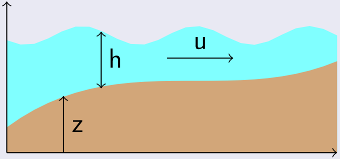
$$h + z = cte \, , \, \mathbf{u} = 0 \, .$$

- Robustness : preservation of the water depth positivity.
- Entropy inequalities.

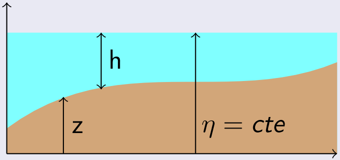
Notable advances :

▷ [Greenberg, Leroux, 1996] , [Gosse, Leroux , 1996] scalar case
▷ [Garcia-Navarro, Vázquez-Cendón , 1997] , [Castro, Gonzales, Pares, 2006] Roe schemes, [LeVeque, 1998] *wave-propagation* algorithm
▷ [Perthame, Simeoni, 2001] , [Perthame, Simeoni, 2003] *kinetic schemes*
▷ [Gallouët, Hérard, Seguin, 2003] , [Berthon, Marche, 2008] *VFRoe schemes*
▷ [Audusse *et al*, 2004] *Hydrostatic Reconstruction*, [Ricchiuto *et al*, 2007] RD schemes , [Lukácová-Medvidová, Noelle, Kraft, 2007] FVEG schemes, [Xing, Zhang, Shu, 2010] , [Berthon, Chalons, 2016] ...

1D Configuration



1D Lake at rest configuration



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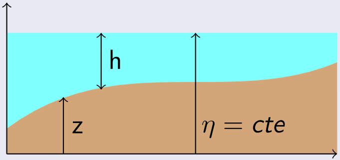

First works :

- [Zhou, Causon, Mingham, 2001] *Surface Gradient Method*
- [Rogers, Fujihara, Borthwick, 2001] , [Russo, 2005] , [Xing, Shu, 2005]

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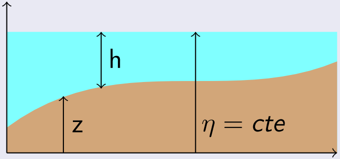

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formulation.

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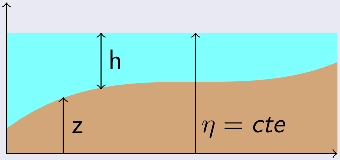


Pre balanced formulation

$$\partial_t V + \nabla . H(V, z) = S(V, z) .$$

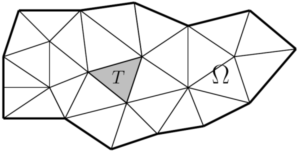
$$V = \begin{pmatrix} \eta \\ hu \\ hv \end{pmatrix} , \; H(V, z) = \begin{pmatrix} hu & hv \\ \frac{1}{2} g(\eta^2 - 2\eta z) + hu^2 & huv \\ huv & \frac{1}{2} g(\eta^2 - 2\eta z) + hv^2 \end{pmatrix} .$$

Topography source term : $\quad S(V, z) = \begin{pmatrix} 0 \\ -g\eta \partial_x z \\ -g\eta \partial_y z \end{pmatrix} .$

Outline

$\mathbb{P}^d(T) := \{$2 variables polynomials on T of degree at most $d\}$.

$\mathcal{V}_{\mathfrak{h}} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_{\mathfrak{h}}\,,\ v_{|T} \in \mathbb{P}^d(T)\}$



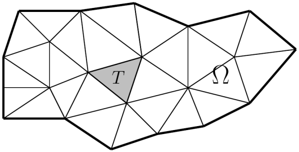
Approximate solution : $V_{\mathfrak{h}}(\mathbf{x}, t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,, \quad \forall \mathbf{x} \in T\,.$

Local weak formulation (1)

$$\partial_t V + \nabla.H(V, z) = S(V, z)$$

$\mathbb{P}^d(T) := \{$2 variables polynomials on T of degree at most $d\}$.

$\mathcal{V}_\mathfrak{h} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_\mathfrak{h} \,,\ v_{|T} \in \mathbb{P}^d(T)\}$



Approximate solution : $V_\mathfrak{h}(\mathbf{x}, t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x}) \,, \quad \forall \mathbf{x} \in T \,.$

Local weak formulation (2)

$$\partial_t V \phi_\mathfrak{h}(\mathbf{x}) + \nabla . H(V, z)\phi_\mathfrak{h}(\mathbf{x}) = S(V, z)\phi_\mathfrak{h}(\mathbf{x})$$

$\mathbb{P}^d(T) := \{$2 variables polynomials on T of degree at most $d\}$.

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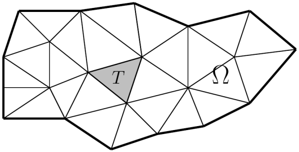

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Local weak formulation (3)

$$\int_T \partial_t V \phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x} + \int_T \nabla . H(V,z)\phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x} = \int_T S(V,z)\phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x}$$

$\mathbb{P}^d(T) := \{$2 variables polynomials on T of degree at most $d\}$.

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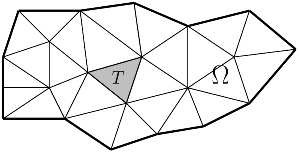


Approximate solution : $V_{\mathfrak{h}}(\mathbf{x}, t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,, \quad \forall \mathbf{x} \in T\,.$

Local weak formulation (4)

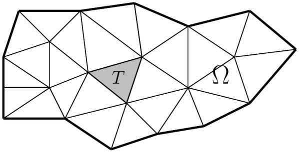
$$\int_T \partial_t V \phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x} + \int_T \nabla.H(V,z)\phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x} = \int_T S(V,z)\phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x}$$

$$\int_T \partial_t V \phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x} - \int_T H(V,z).\nabla\phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x} + \int_{\partial T} H(V,z).\vec{n}\phi_{\mathfrak{h}}(s) ds = \int_T S(V,z)\phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x}$$

dG method : general background

$\mathbb{P}^d(T) := \{$2 variables polynomials on T of degree at most $d\}$.

$\mathcal{V}_\mathfrak{h} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_\mathfrak{h}\,,\ v_{|T} \in \mathbb{P}^d(T)\}$



Approximate solution : $V_\mathfrak{h}(\mathbf{x},t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,,\quad \forall \mathbf{x} \in T\,.$

Through a semi-discrete formulation (1)

- $V \to V_\mathfrak{h}$

$$\int_T \partial_t \left(\sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\right)\phi_\mathfrak{h}(\mathbf{x})d\mathbf{x} - \int_T H(V_\mathfrak{h}, z_\mathfrak{h}).\nabla\phi_\mathfrak{h}(\mathbf{x})d\mathbf{x} +$$

$$\int_{\partial T} H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}\phi_\mathfrak{h}(s)ds = \int_T S(V_\mathfrak{h}, z_\mathfrak{h})\phi_\mathfrak{h}(\mathbf{x})d\mathbf{x}$$

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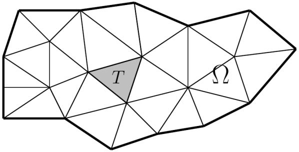
$\mathcal{V}_\mathfrak{h} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_\mathfrak{h} \,,\; v_{|T} \in \mathbb{P}^d(T)\}$



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Through a semi-discrete formulation (2)

- $V \to V_\mathfrak{h}$
- $\phi_\mathfrak{h} \to \theta_j$

$$\int_T \partial_t \left(\sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x}) \right) \theta_j(\mathbf{x}) d\mathbf{x} - \int_T H(V_\mathfrak{h}, z_\mathfrak{h}).\nabla\theta_j(\mathbf{x}) d\mathbf{x} +$$

$$\int_{\partial T} H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}\theta_j(s) ds = \int_T S(V_\mathfrak{h}, z_\mathfrak{h})\theta_j(\mathbf{x}) d\mathbf{x}$$

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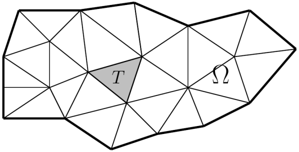

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- $V \rightarrow V_\mathfrak{h}$
- $\phi_\mathfrak{h} \rightarrow \theta_j$

$$\int_T \partial_t \left(\sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\right)\theta_j(\mathbf{x})d\mathbf{x} - \int_T H(V_\mathfrak{h}, z_\mathfrak{h}).\nabla\theta_j(\mathbf{x})d\mathbf{x} +$$

$$\sum_{k=1}^{3} \int_{\Gamma_{ij(k)}} H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)}\theta_j(s)ds = \int_T S(V_\mathfrak{h}, z_\mathfrak{h})\theta_j(\mathbf{x})d\mathbf{x}$$

Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

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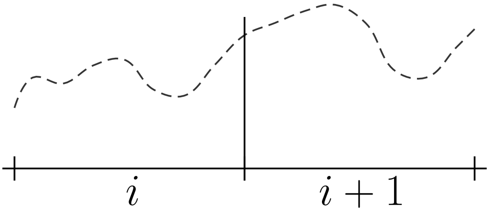

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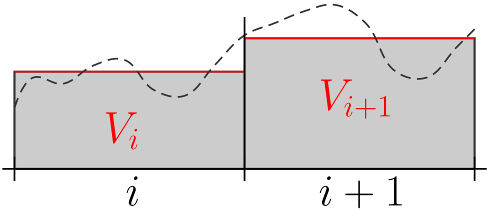

VF

Numerical fluxes

Contributions on the edges

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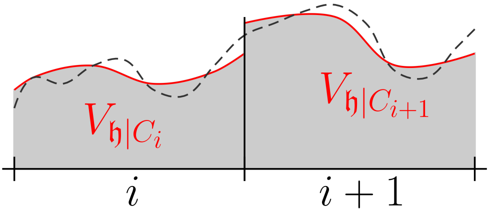

DG

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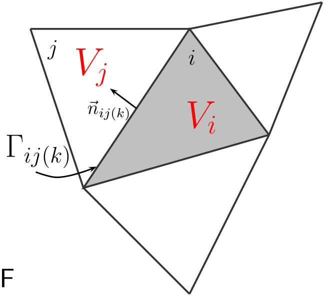

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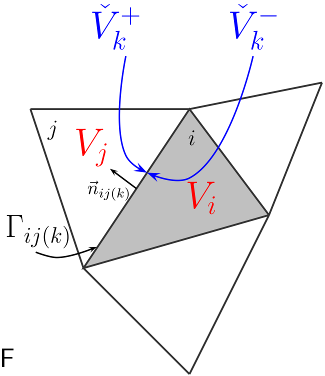

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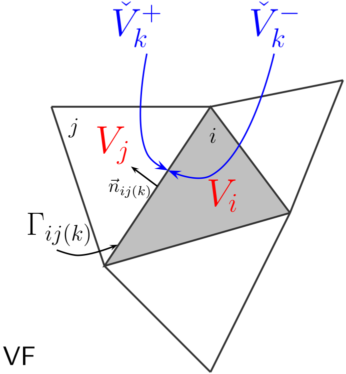

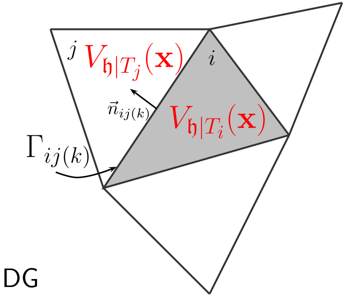

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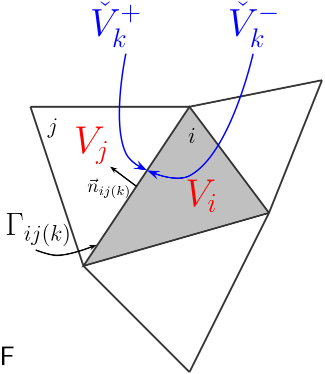

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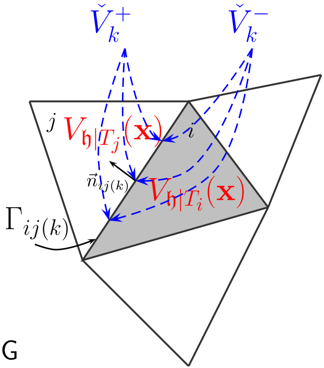

DG

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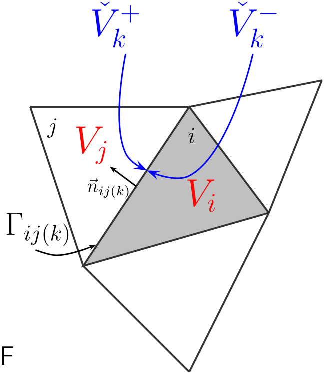

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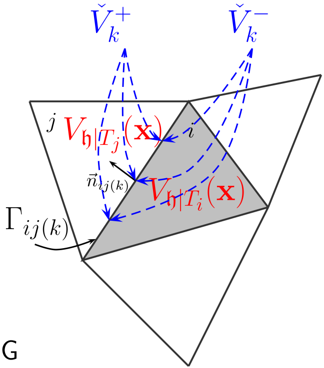

DG

→ Preservation of the motionless steady states.

dG schemes and maximum principle

- [X. Zhang, C.-W. Shu, 2010] Maximum-principle-satisfying high order schemes - 1d and 2d structured meshes
- [Y. Xing, X. Zhang, C.-W. Shu, 2010] Application to 1d SW
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The method

- relies on a special quadrature rule.

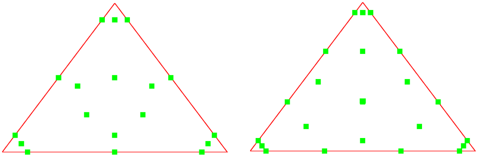


Figure: Nodes locations for the special quadrature - $\mathbb{P}^2$ and $\mathbb{P}^3$

- reduces to the study of a convex combination of **first order Finite Volume schemes**.

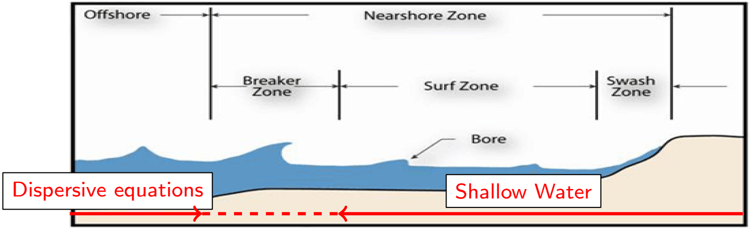
Outline

Objective

Extend the range of applicability of the computations at coast.

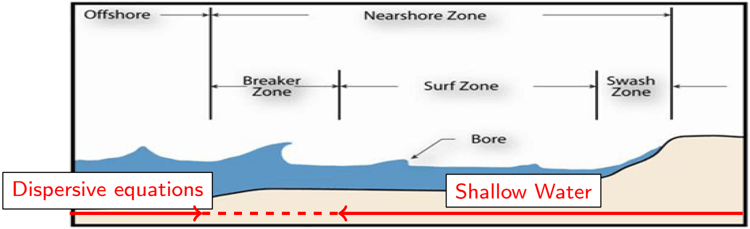
- ▷ Describe the non-linearities **before** the breaking point.
- ▷ Dispersive equations : $O(\mu^2)$-accurate
  Shallow Water equations : $O(\mu)$-accurate .

$$\text{Shallowness parameter : } \mu = \frac{h_0^2}{\lambda_0^2} .$$



Numerical issues

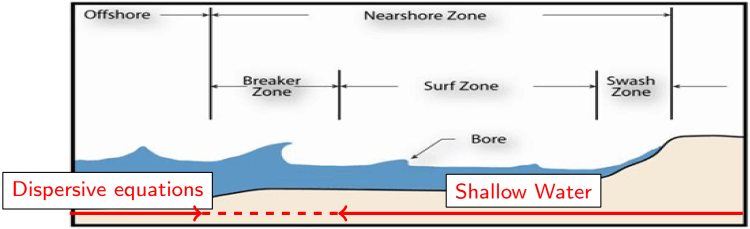
- ▷ Non conservative terms, high order derivatives, wave-breaking, non linearities.
- ▷ Maintain the stability of the method (positivity, well balancing), even on unstructured environments.



Interest of dispersive equations

Numerical issues

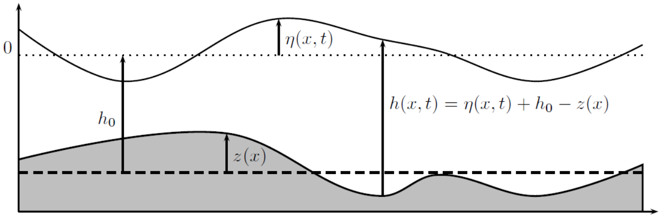
- ▷ Non conservative terms, high order derivatives, wave-breaking, non linearities.
- ▷ Maintain the stability of the method (positivity, well balancing), even on unstructured environments.



State of the art

- ▷ 1d works : [Antunes Do Carmo *et al*] (FD, 1993), [Cienfuegos *et al*] (FV, 2006), [Dutykh *et al*] (FV, 2013), [Panda *et al*] (dG, 2014), [AD, Marche] (dG, 2015)
- ▷ 2d works : [Marche, Lannes] (Hybrid FV/FD, cartésien, 2015), [Popinet] (Hybrid FV/FD, cartésien, 2015)
- ▷ Unstructured meshes : Weakly non linear models (Boussinesq - type). [Kazolea, Delis, Synolakis] (FV, 2014), [Filippini, Kazolea, Ricchiuto] (Hybrid FV/FE, 2016)

- [P. Bonneton *et al*, 2011] 1d derivation and optimized model. Hybrid method.
- [F. Chazel, D. Lannes, F. Marche, 2011] 3 parameters model
- [M. Tissier *et al*, 2012] Wave breaking issues
- [D. Lannes, F. Marche, 2015] *A new class of fully nonlinear and weakly dispersive Green-Naghdi models for efficient 2D simulations*



Revoke the time dependency

$$\begin{cases} \partial_t \eta + \partial_x(hu) = 0\,, \\ \left[1+\alpha\mathfrak{T}[h_b]\right]\left(\partial_t hu + \partial_x(hu^2) + \frac{\alpha-1}{\alpha} gh\partial_x\eta\right) + \frac{1}{\alpha} gh\partial_x\eta \\ \qquad + h\left(\mathcal{Q}_1(u) + g\mathcal{Q}_2(\eta)\right) + g\mathcal{Q}_3\left(\left[1+\alpha\mathfrak{T}[h_b]\right]^{-1}(gh\partial_x\eta)\right) = 0\,. \end{cases}$$

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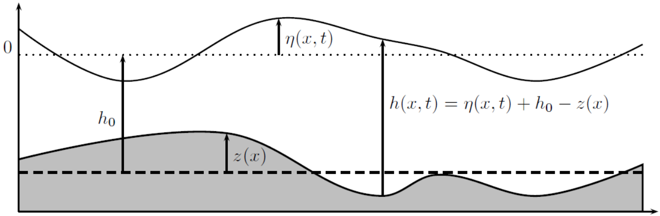

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▷ $\mathfrak{T}[h]w = -\frac{h^3}{3}\partial_x{}^2\left(\frac{w}{h}\right) - h^2\partial_x h\partial_x\left(\frac{w}{h}\right)$, $h_b = h_0 - z$,

▷ $\mathcal{Q}_{i=1,2,3}$: non linear, non conservative terms with second order derivatives.

▷ 2d version : "**diagonal**" sytem : no coupling between u and v !

Isolation of the hyperbolic part

A convenient formulation

$$\partial_t U + \partial_x G(U) = B(U,z) + \mathfrak{D}(U,z)$$

A convenient formulation

$$\underbrace{\partial_t U + \partial_x G(U) = B(U,z)}_{\text{Shallow Water}} + \mathfrak{D}(U,z)$$

▷ **Shallow Water** equations :

$$U = \begin{pmatrix} h \\ hu \end{pmatrix} \quad , \quad G(U) = \begin{pmatrix} hu \\ \frac{1}{2}gh^2 + hu^2 \end{pmatrix} \quad , \quad B(U) = \begin{pmatrix} 0 \\ -gh\partial_x z \end{pmatrix} .$$

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▷ **Dispersive** terms :

$$\mathfrak{D}(V,z) = \begin{pmatrix} 0 \\ \mathfrak{D}_{hu}(V,z) \end{pmatrix} , \text{ with}$$

$$\mathfrak{D}_{hu}(V,z) = \left[1 + \alpha \mathfrak{T}[h_b]\right]^{-1} \Big(\frac{1}{\alpha} gh\partial_x \eta + h\big(\mathcal{Q}_1(u)\big) + g\,\mathcal{Q}_2(\eta)\big)$$

$$+ g\,\mathcal{Q}_3\Big(\left[1 + \alpha \mathfrak{T}[h_b]\right]^{-1}(gh\partial_x \eta)\Big)\Big) - \frac{1}{\alpha} gh\partial_x \eta \,.$$

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- Hyperbolic part : ok
- Dispersive part :

$$D_{\mathfrak{h}}(x,t) = \sum_{l=1}^{N_d} D_l(t)\theta_l(x)\,, \quad x \in C_i\,.$$

- Well balancing and robustness ,
- Treatment of the second order derivatives .

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Simplified case : $T = {\partial_x}^2$

Consider the second order ODE :

$$f - \partial_x^2 u = 0 \,. \tag{1}$$

(1) reduces to a coupled system of first order equations.

$$f + \partial_x v = 0 \quad , \quad v + \partial_x u = 0 \,.$$

Weak formulation

$$\int_{x_i^l}^{x_i^r} f \phi_{\mathfrak{h}} - \int_{x_i^l}^{x_i^r} v \phi_{\mathfrak{h}}' + \widehat{v}_r \phi_{\mathfrak{h}}(x_i^r) - \widehat{v}_l \phi_{\mathfrak{h}}(x_i^l) = 0 \,,$$

$$\int_{x_i^l}^{x_i^r} v \phi_{\mathfrak{h}} - \int_{x_i^l}^{x_i^r} u \phi_{\mathfrak{h}}' + \widehat{u}_r \phi_{\mathfrak{h}}(x_i^r) - \widehat{u}_l \phi_{\mathfrak{h}}(x_i^l) = 0 \,.$$

LDG formalism

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LDG schemes :

[B. Cockburn, C.-W. Shu, 1998] *The Local Discontinuous Galerkin method for time-dependent convection-diffusion systems* .

Outline

Switching method : press ENTER and hope ...

Protocol : At each time step

- Detection : evaluation of $\mathbb{I}^k$ on each cell k.
  [L. Krivodonova *et al*, 2004] *Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws*
- Determination of the breaking area.
- Switching strategy
  - Suppress of the dispersive terms on the targeted area.
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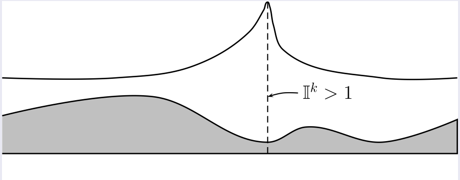

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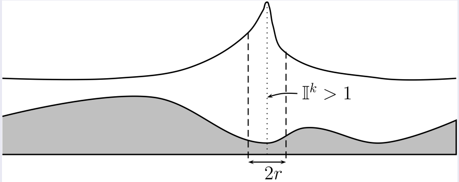

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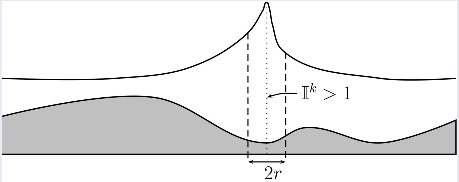

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▷ Account for the mechanical energy dissipation through a third variable φ.

A new model (G. Richard, 2016)

$$\partial_t \tilde{U} + \partial_x \tilde{G}(\tilde{U}) = B(\tilde{U}, z) + \mathfrak{D}(\tilde{U}, z)$$

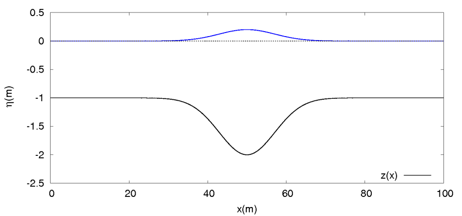
with :

$$\tilde{U} = \begin{pmatrix} h \\ hu \\ h\varphi \end{pmatrix}, \; \tilde{G}(\tilde{U}) = \begin{pmatrix} hu \\ \dfrac{1}{2}gh^2 + h^3\varphi + hu^2 \\ hu\varphi \end{pmatrix}, \; B(\tilde{U}) = \begin{pmatrix} 0 \\ -gh\partial_x z \\ 0 \end{pmatrix}$$

▷ A simple additional transport equation in the hyperbolic part !

Convergence analysis and model comparison

Profiles



Convergence rates

N	N_e 20	40	80	160	320	order
1	2.5e-1	4.2e-2	1.0e-3	2.8e-3	9.6e-4	1.9
2	7.5e-2	7.5e-3	6.2e-4	6.5e-5	7.7e-6	3.2
3	4.5e-3	3.0e-4	1.7e-5	9.4e-7	5.7e-8	4.0
4	7.0e-4	1.6e-5	4.6e-7	1.4e-8	4.4e-10	5.1
5	6.1e-5	7.6e-7	1.0e-8	1.6e-10	3.1e-12	6.1

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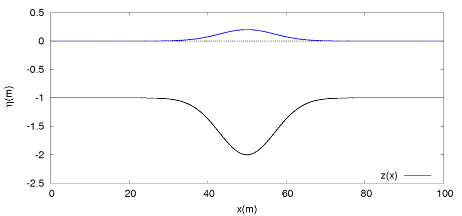

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4	7.0e-4	1.6e-5	4.6e-7	1.4e-8	4.4e-10	5.1
5	6.1e-5	7.6e-7	1.0e-8	1.6e-10	3.1e-12	6.1

Evolution of the ratio $\tau = \rho_o/\rho_c$.
ρ : mean iteration time (based on 1000 iterations).

N	N_e 1000	2000	3000	4000	5000	6000
1	3.23	3.21	3.04	2.96	2.94	2.90
2	4.09	4.27	4.14	4.01	3.90	3.84
3	5.32	5.11	5.03	4.97	4.91	4.87
4	6.01	5.77	5.67	5.63	5.51	5.55
5	6.66	6.38	6.32	6.30	6.26	6.16
6	7.15	6.99	7.05	6.97	6.86	6.54

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→ Modelling, Numerical methods.
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Numerical treatment of the non-hydrostatic terms

→ Weak formulations.
→ Numerical exploration (linear solvers, matrix storage, re-numbering).
→ Alternative approaches (quadrature rules, Finite Volume methods, FEM).

Thanks !