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A high order FV/FE projection method for compressible low-Mach number flows

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Outline

Incompressible flows

Incompressible Navier-Stokes equations

$$\operatorname{div} \rho\, \mathbf{u} = 0,$$

$$\frac{\partial \rho\, \mathbf{u}}{\partial t} + \operatorname{div} \mathcal{F}_i^{\mathbf{w_u}}(\mathbf{w_u}, \mathbf{u}(x,t)) + \nabla \pi - \operatorname{div} \tau = \mathbf{f_u},$$

$$\tau = \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right).$$

- Time: t.
- Density: $\rho \in \mathbb{R}$.
- Velocity vector: $\mathbf{u} = \mathbf{u}(x, y, z, t)$.
- Conservative velocity vector: $\mathbf{w_u} = \rho\, \mathbf{u}$
- Pressure: $\pi = \pi(x, y, z, t)$.
- Viscous term of the stress tensor: τ.
- Dynamic viscosity: μ.
- Source term: $\mathbf{f_u}$.
- Flux: $\mathcal{F}_i^{\mathbf{w_u}}(\mathbf{w_u}, \mathbf{u}(x,t)) = u_i \mathbf{w_u}$, $i = 1, 2, 3$.

Turbulence

Reynolds-averaged viscous stress tensor

$k-\varepsilon$ standard

$$\tau = \tau_{\mathbf{u}} + \tau^R, \quad \tau_{\mathbf{u}} = \mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right),$$
$$\tau^R = \mu_t\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) - \frac{2}{3}\rho k I.$$

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon}, \qquad C_\mu = 0.09.$$

- Reynolds stress tensor, τ^R.
- Turbulent viscosity, μ_t.
- Turbulent kinetic energy, k.
- Conservative turbulent kinetic energy, w_k.
- Dissipation rate, ε.
- Conservative dissipation rate, w_ε.

$K - \varepsilon$ standard

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon},$$
$$\frac{\partial \rho k}{\partial t} + \mathrm{div}\,\mathcal{F}^{w_k}\left(w_k, \mathbf{u}(x,t)\right) - \mathrm{div}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\nabla k\right] + \rho\varepsilon = G_k + f_k,$$
$$\frac{\partial \rho \varepsilon}{\partial t} + \mathrm{div}\,\mathcal{F}^{w_\varepsilon}\left(w_\varepsilon, \mathbf{u}(x,t)\right) - \mathrm{div}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\nabla \varepsilon\right] + \rho C_{2\varepsilon}\frac{\varepsilon^2}{k} = C_{1\varepsilon}\frac{\varepsilon}{k}G_k + f_\varepsilon,$$
$$G_k \approx \frac{\mu_t}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2,$$
$$C_{1\varepsilon} = 1.44, \quad C_{2\varepsilon} = 1.92, \quad C_\mu = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\varepsilon = 1.3.$$

- Flux: $\mathcal{F}^{w_k}\left(w_k, \mathbf{u}(x,t)\right) = \mathbf{u} w_k$, $\mathcal{F}^{w_\varepsilon}\left(w_\varepsilon, \mathbf{u}(x,t)\right) = \mathbf{u} w_\varepsilon$.
- Turbulent production: G_k.
- Source MMS terms: f_k, f_ε.
- Prandtl numbers: σ_k, σ_ε.
- Closure constants: $C_{1,\varepsilon}$, $C_{2,\varepsilon}$.

Species transport and energy

Species conservation equations

$$\frac{\partial \rho\, \mathbf{y}}{\partial t} + \operatorname{div} \mathcal{F}^{\mathbf{w}_\mathbf{y}}\left(\mathbf{w}_\mathbf{y}, \mathbf{u}(x,t)\right) - \operatorname{div}\left[\left(\rho \mathcal{D} + \frac{\mu_t}{Sc_t}\right)\nabla\, \mathbf{y}\right] = \mathbf{f}_\mathbf{y}\,.$$

Energy conservation equation

$$\frac{\partial \rho h}{\partial t} + \operatorname{div} \mathcal{F}^{\mathbf{w}_h}\left(\mathbf{w}_h, \mathbf{u}(x,t)\right) - \operatorname{div}\left[\left(\rho \mathcal{D} + \frac{\mu_t}{Sc_t}\right)\nabla\, h\right] = -\operatorname{div} \mathbf{q}_r + f_h.$$

- Species: $\mathbf{y} = (y_1, \ldots, y_{N_e})$.
- Number of species: N_e.
- Conservative species: $\mathbf{w}_\mathbf{y} = \rho\, \mathbf{y}$.
- Enthalpy: h.
- Conservative enthalpy: $w_h = \rho h$.
- Flux: $\mathcal{F}^{\mathbf{w}_\mathbf{y}}\left(\mathbf{w}_\mathbf{y}, \mathbf{u}(x,t)\right) = \mathbf{u}\mathbf{w}_\mathbf{y}$, $\mathcal{F}^{\mathbf{w}_h}\left(\mathbf{w}_h, \mathbf{u}(x,t)\right) = \mathbf{u}\mathbf{w}_h$.
- Mass diffusivity coefficient: $\mathcal{D}$.
- Schmidt number: Sc_t.
- Heat flux: $\mathbf{q}_r$.
- Source terms: $\mathbf{f}_\mathbf{y}$, f_h.

Conservative variables

We introduce the conservative variables

$$\mathbf{w_u} = \begin{pmatrix} \rho u_1 \\ \rho u_2 \\ \rho u_3 \end{pmatrix}, \qquad \hat{\mathbf{w}} = \begin{pmatrix} \rho y_1 \\ \vdots \\ \rho y_{N_e} \\ \rho h \\ \rho k \\ \rho \varepsilon \end{pmatrix} = \begin{pmatrix} w_{y_1} \\ \vdots \\ w_{y_{N_e}} \\ w_h \\ w_k \\ w_\varepsilon \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} \mathbf{w_u} \\ \hat{\mathbf{w}} \end{pmatrix}.$$

Then, the flux can be expressed as

$$\mathcal{F}^{\mathbf{w_u}} = \left(\mathcal{F}_1^{\mathbf{w_u}} | \mathcal{F}_2^{\mathbf{w_u}} | \mathcal{F}_3^{\mathbf{w_u}}\right)_{3\times 3}, \quad \mathcal{F}_i^{\mathbf{w_u}} = u_i\, \mathbf{w_u} = \frac{w_i}{\rho}\, \mathbf{w_u}$$

$$\mathcal{F} = \left(\mathcal{F}_1 | \mathcal{F}_2 | \mathcal{F}_3\right)_{(3+N_e+1+2)\times 3}, \quad \mathcal{F}_i = \begin{pmatrix} \mathcal{F}_i^{\mathbf{w_u}} \\ \\ \frac{w_i}{\rho}\hat{\mathbf{w}} \end{pmatrix}_{3+N_e+1+2} = u_i\, \mathbf{w}, \quad i = 1, 2, 3.$$

Hence, the system of equations can be rewritten:

$$
\begin{aligned}
&\operatorname{div} \mathbf{w_u} = 0,\\
&\frac{\partial \mathbf{w_u}}{\partial t} + \operatorname{div} \mathcal{F}^{\mathbf{w_u}}(\mathbf{w_u}, \mathbf{u}) + \nabla \pi - \operatorname{div}\left[\frac{1}{\rho}(\mu+\mu_t)\left(\nabla \mathbf{w_u} + \nabla \mathbf{w_u}^T\right) - \frac{2}{3} w_k I\right] = \mathbf{f_u},\\
&\frac{\partial \mathbf{w_y}}{\partial t} + \operatorname{div} \mathcal{F}^{\mathbf{w_y}}(\mathbf{w_y}, \mathbf{u}) - \operatorname{div}\left[\left(\rho\mathcal{D} + \frac{\mu_t}{Sc_t}\right)\nabla\left(\frac{1}{\rho}\mathbf{w_y}\right)\right] = \mathbf{f_y},\\
&\frac{\partial w_h}{\partial t} + \operatorname{div} \mathcal{F}^{\mathbf{w}_h}(\mathbf{w}_h, \mathbf{u}) - \operatorname{div}\left[\left(\rho\mathcal{D} + \frac{\mu_t}{Sc_t}\right)\nabla\frac{w_h}{\rho}\right] = -\operatorname{div} \mathbf{q}_r + f_h,\\
&\frac{\partial w_k}{\partial t} + \operatorname{div} \mathcal{F}^{\mathbf{w}_k}(\mathbf{w}_k, \mathbf{u}) - \operatorname{div}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\nabla\frac{w_k}{\rho}\right] + w_\varepsilon = G_k + f_k,\\
&\frac{\partial w_\varepsilon}{\partial t} + \operatorname{div} \mathcal{F}^{\mathbf{w}_\varepsilon}(\mathbf{w}_\varepsilon, \mathbf{u}) - \operatorname{div}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\nabla\frac{w_\varepsilon}{\rho}\right] + \rho C_{2\varepsilon}\frac{w_\varepsilon^2}{w_k} = C_{1\varepsilon}\frac{w_\varepsilon}{w_k}G_k + f_\varepsilon,\\
&\mu_t = C_\mu \frac{w_k^2}{w_\varepsilon}, \qquad G_k = \frac{\mu_t}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2.
\end{aligned}
$$

Laminar low Mach number flows

Low Mach number equations

We will assume that the Mach number, M, is small enough for splitting the pressure into a spatially constant function, $\bar{\pi}$, and a small perturbation, π:

$$p(x, y, z, t) = \bar{\pi}(t) + \pi(x, y, z, t), \quad \frac{\pi}{\bar{\pi}} = O(M^{-2}) \tag{1}$$

with

- $\bar{\pi}(t)$ provided,
- π neglected in the state equation and retained in the momentum equation.

Compressible Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{u} = 0,$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \pi - \operatorname{div} \tau + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \mathbf{f_u},$$

$$\tau = \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) - \frac{2}{3} \mu \operatorname{div} \mathbf{u} \, I,$$

$$\bar{\pi} = \rho R \theta, \qquad R = \mathcal{R} \sum_{i=1}^{N_e} \frac{Y_i}{\mathcal{M}_i}, \quad \mathcal{R} = 8314 \ \mathrm{J}/(\mathrm{kmolK}).$$

- Density: $\rho = \rho(x, y, z, t)$.
- Gas constant: R.
- Universal constant: $\mathcal{R}$.
- Molecular mass (species Y_i): $\mathcal{M}_i$.
- Temperature: θ.

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\, \mathbf{u} = 0, \tag{2}$$

$$\frac{\partial \rho\, \mathbf{u}}{\partial t} + \underbrace{\frac{\partial F_1(\mathbf{u}, \rho)}{\partial x} + \frac{F_2(\mathbf{u}, \rho)}{\partial y} + \frac{F_3(\mathbf{u}, \rho)}{\partial z}}_{\operatorname{div}(\mathbf{F}(\mathbf{u}, \rho))} + \nabla \pi - \operatorname{div} \tau = \mathbf{f}_{\mathbf{u}},$$

$$\bar{\pi} = \rho R \theta. \tag{3}$$

From (2) and (3) we get the following divergence condition:

$$\operatorname{div}(\rho\, \mathbf{u}) = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\bar{\pi}}{R\theta}\right)$$

so that, the system of equations to be solved reads

$$\frac{\partial \rho\, \mathbf{u}}{\partial t} + \operatorname{div}(F(\mathbf{u}, \rho)) + \nabla \pi - \operatorname{div} \tau = \mathbf{f}_{\mathbf{u}},$$

$$\operatorname{div} \rho\, \mathbf{u} = q,$$

where

$$q := -\frac{\partial}{\partial t}\left(\frac{\bar{\pi}}{R\theta}\right).$$

Species transport and energy

Species conservation equations

$$\frac{\partial \rho \mathbf{y}}{\partial t} + \operatorname{div} F^{\mathbf{y}}((\mathbf{u}, \mathbf{y}), \rho) - \operatorname{div}(\rho \mathcal{D} \nabla \mathbf{y}) = 0. \quad (4)$$

Energy conservation equation

$$\frac{\partial \rho h}{\partial t} + \operatorname{div} F^{h}((\mathbf{u}, h), \rho) - \operatorname{div}(\rho \mathcal{D} \nabla h) = 0. \quad (5)$$

Taking into account the mass conservation equation and the chain rule, equations (4) and (5) are rewritten as

$$\frac{\partial \mathbf{y}}{\partial t} + \operatorname{div}(\mathbf{y}\,\mathbf{v}) - \mathbf{y} \operatorname{div} \mathbf{v} - \frac{1}{\rho} \operatorname{div}(\rho \mathcal{D} \nabla \mathbf{y}) = 0,$$

$$\frac{\partial h}{\partial t} + \operatorname{div}(h\,\mathbf{v}) - h \operatorname{div} \mathbf{v} - \frac{1}{\rho} \operatorname{div}(\rho \mathcal{D}_i \nabla h) = 0.$$

The resolution of the Navier-Stokes equations and the species transport and energy conservation equations are coupled by state equation,

$$\bar{\pi} = \rho R \theta \quad \rightsquigarrow \quad q = -\frac{\partial}{\partial t}\left(\frac{\bar{\pi}}{R\theta}\right).$$

The following equation relates the internal enthalpy for a perfect gas and its temperature:

$$h(\theta) = h_{\theta_0} + \int_{\theta_0}^{\theta} c_\pi(r)dr.$$

- Standard enthalpy formation: h_{θ_0}.
- Temperature of formation: θ_0.
- Specific heat at constant pressure: c_π.

Conservative variables

The vector of unknowns of the new system reads

$$\mathbf{w_u} = \begin{pmatrix} \rho u_1 \\ \rho u_2 \\ \rho u_3 \end{pmatrix}, \qquad \hat{\mathbf{w}} = \begin{pmatrix} y_1 \\ \vdots \\ y_{N_e} \\ h \end{pmatrix} = \begin{pmatrix} w_{y_1} \\ \vdots \\ w_{y_{N_e}} \\ w_h \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} \mathbf{w_u} \\ \hat{\mathbf{w}} \end{pmatrix}.$$

Then, the flux can be expressed as

$$\mathcal{F}^{\mathbf{w_u}} = \left(\mathcal{F}_1^{\mathbf{w_u}} | \mathcal{F}_2^{\mathbf{w_u}} | \mathcal{F}_3^{\mathbf{w_u}}\right)_{3\times 3}, \quad \mathcal{F}_i^{\mathbf{w_u}} = u_i\, \mathbf{w_u} = \frac{w_i}{\rho}\, \mathbf{w_u}$$

$$\mathcal{F} = \left(\mathcal{F}_1 | \mathcal{F}_2 | \mathcal{F}_3\right)_{(3+N_e+1+2)\times 3}, \quad \mathcal{F}_i = \begin{pmatrix} \mathcal{F}_i^{\mathbf{w_u}} \\ \\ \frac{w_i}{\rho} \hat{\mathbf{w}} \end{pmatrix}_{3+N_e+1+2} = u_i\, \mathbf{w}, \quad i = 1, 2, 3.$$

Hence, the system of equations can be rewritten:

$$\operatorname{div} \mathbf{w_u} = q, \qquad q = -\frac{\partial}{\partial t}\left(\frac{\bar{\pi}}{R\theta}\right),$$

$$\frac{\partial \mathbf{w_u}}{\partial t} + \operatorname{div} \mathcal{F}^{\mathbf{w_u}}(\mathbf{w_u}, \rho) + \nabla \pi - \operatorname{div}\left[\mu\left(\nabla\left(\frac{1}{\rho}\mathbf{w_u}\right) + \nabla\left(\frac{1}{\rho}\mathbf{w_u}\right)^T\right)\right.$$

$$\left. -\frac{2}{3}\mu \operatorname{div}\left(\frac{1}{\rho}\mathbf{w_u}\, I\right)\right] = 0,$$

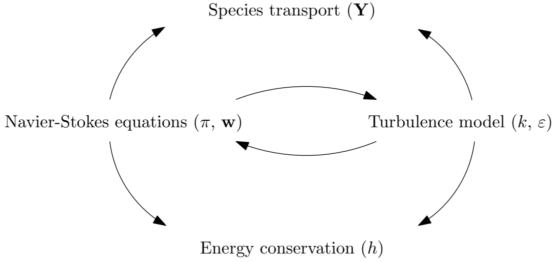
$$\frac{\partial \mathbf{w_y}}{\partial t} + \operatorname{div} \mathcal{F}^{\mathbf{w_y}}(\mathbf{w}, \rho) - \mathbf{w_y} \operatorname{div} \mathbf{u} - \frac{1}{\rho}\operatorname{div}(\rho \mathcal{D} \nabla \mathbf{w_y}) = 0,$$

$$\frac{\partial w_h}{\partial t} + \operatorname{div} \mathcal{F}^{w_h}(\mathbf{w}, \rho) - w_h \operatorname{div} \mathbf{u} - \frac{1}{\rho}\operatorname{div}(\rho \mathcal{D}_i \nabla w_h) = 0,$$

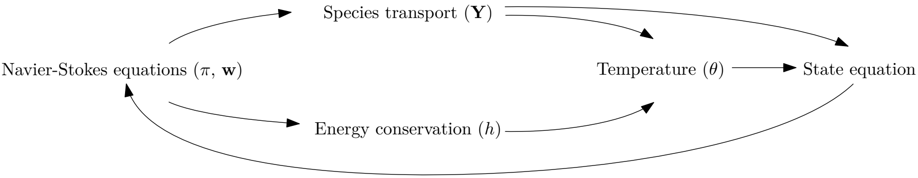
$$h(\theta) = h_{\theta_0} + \int_{\theta_0}^{\theta} c_\pi(r)\,dr.$$

Equations dependency

Incompressible flow with turbulence:



Laminar low Mach number flow:



Numerical Method

Numerical discretization

Let $\mathbf{W}^n$, π^n be an approximation of the conservative variables, $\mathbf{w}(x, y, z, t^n)$, and of the pressure perturbation, $\pi(x, y, z, t^n)$. Then $\mathbf{W}^{n+1}$, π^{n+1} can be defined from the following system of equations:

$$\frac{\widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1} - \mathbf{W}_{\mathbf{u}}^{n}}{\Delta t} + \operatorname{div}\left(\mathcal{F}^{\mathbf{w_u}}\left(\mathbf{W}^n, \rho^n\right)\right) + \nabla \pi^n - \operatorname{div}\left(\tau^n\right) = \mathbf{f}_{\mathbf{u}}^n, \tag{6}$$

$$\frac{\mathbf{W}_{\mathbf{u}}^{n+1} - \widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1}}{\Delta t} + \nabla\left(\pi^{n+1} - \pi^n\right) = 0, \tag{7}$$

$$\operatorname{div} \mathbf{W}_{\mathbf{u}}^{n+1} = q^{n+1}, \tag{8}$$

$$\tau^n = \mu\left[\nabla\left(\frac{1}{\rho}\mathbf{W}_{\mathbf{u}}^n\right) + \nabla\left(\frac{1}{\rho}\mathbf{W}_{\mathbf{u}}^n\right)^T - \frac{2}{3}\operatorname{div}\left(\frac{1}{\rho}\mathbf{W}_{\mathbf{u}}^n I\right)\right],$$

$$\frac{\widetilde{\mathbf{W}}_{\mathbf{y}}^{n+1} - \mathbf{W}_{\mathbf{y}}^{n}}{\Delta t} + \mathrm{div}\, \mathcal{F}^{\mathbf{w_y}}(\mathbf{W}^{n}, \rho) - \mathrm{div}\left[\rho \mathcal{D}\, \nabla \left(\frac{1}{\rho} \mathbf{W}_{\mathbf{y}}^{n}\right)\right] = 0, \tag{9}$$

$$\frac{\mathbf{W}_{\mathbf{y}}^{n+1} - \widetilde{\mathbf{W}}_{\mathbf{y}}^{n+1}}{\Delta t} = \mathbf{f}_{\mathbf{y}}, \tag{10}$$

$$\frac{\widetilde{W}_{h}^{n+1} - W_{h}^{n}}{\Delta t} + \mathrm{div}\, \mathcal{F}^{w_h}(\mathbf{W}^{n}, \rho) - \mathrm{div}\left[\rho \mathcal{D}\, \nabla \frac{W_{h}^{n}}{\rho}\right] = -\,\mathrm{div}\, \mathbf{Q}_{r}^{n}, \tag{11}$$

$$\frac{W_{h}^{n+1} - \widetilde{W}_{h}^{n+1}}{\Delta t} = f_{h}, \tag{12}$$

$$h^{n+1} = h_{\theta_0} + \int_{\theta_0}^{\theta^{n+1}} c_{\pi}(r)dr, \tag{13}$$

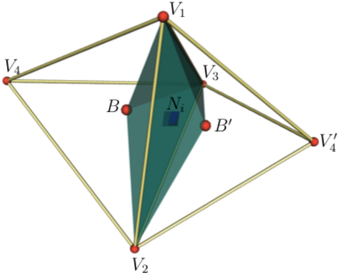
$$\rho^{n+1} = \frac{\overline{\pi}}{\mathcal{R}\theta^{n+1} \sum_{i=1}^{N_e} \frac{Y_i^{n+1}}{\mathcal{M}_i}}. \tag{14}$$

Overall method

- Transport-difusion stage:
  - Equations (6), (9) and (11) are solved by a FVM.
  - Species and energy variables are updated from equations (10) and (12).
- Pre-projection stage: temperature and density are computed from the updated species and enthalpy using (13) and (14).
- Projection stage: a FEM is applied to (7)-(8) in order to obtain the pressure correction.
- Post-projection stage: $\widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1}$ is updated by using δ^{n+1}, (7) (the new approximation, $\mathbf{W}_{\mathbf{u}}^{n+1}$, satisfies the divergence condition).

A dual finite volume mesh

- Barycenter of the faces of the tetrahedra: $\{N_i, \quad i = 1, \ldots, M\}$.
- Set of the neighbours of N_i: $\mathcal{K}_i$.
- Volume of C_i: $vol(C_i)$.
- Boundary of C_i: $\Gamma_i = \partial C_i = \bigcup_{j \in \mathcal{K}_i} \Gamma_{ij}$.



- Outward normal to Γ_{ij}: η_{ij}, $(\eta_{ij} = \tilde{\eta}_{ij} * \|\eta_{ij}\|)$.
- Area of Γ_i: $S_i = \sum_{j \in \mathcal{K}_i} S(\Gamma_{ij}) = \sum_{j \in \mathcal{K}_i} \|\eta_{ij}\|$.
- At the boundary $(N_i \in \Gamma)$: $\begin{cases} \Gamma_{ib} = \Gamma_i \bigcap \Omega, \\ \eta_{ib} \quad \text{outward normal.} \end{cases}$

Transport-diffusion stage

Finite volume discretization

Integrating equations (6), (9) and (11), over the finite volume C_i and applying Gauss theorem we obtain

$$\frac{\mathrm{vol}\left(C_i\right)}{\Delta t}\left(\widetilde{\mathbf{W}}_{\mathbf{u}\,i}^{n+1}-\mathbf{W}_{\mathbf{u}\,i}^{n}\right)+\int_{\Gamma_i}\mathcal{F}^{\mathbf{w}_\mathbf{u}}\left(\mathbf{W}_{\mathbf{u}}^{n},\rho^n\right)\tilde{\eta}dS=-\int_{C_i}\nabla\pi^n dV$$
$$+\int_{\Gamma_i}\left(\tau^n\right)\tilde{\eta}dS+\int_{C_i}\mathbf{f}_{\mathbf{u}}^{n}dV,$$

$$\frac{\mathrm{vol}\left(C_i\right)}{\Delta t}\left(\widetilde{\mathbf{W}}_{\mathbf{Y}\,i}^{n+1}-\mathbf{W}_{\mathbf{Y}\,i}^{n}\right)+\int_{\Gamma_i}\mathcal{F}^{\mathbf{w}_\mathbf{Y}}\left(\mathbf{W}^{n},\rho\right)\tilde{\eta}dS-\int_{\Gamma_i}\left(\rho\mathcal{D}\nabla\frac{\mathbf{W}_{\mathbf{Y}}^{n}}{\rho}\right)\tilde{\eta}dS=\int_{C_i}\mathbf{f}_{\mathbf{Y}}dV,$$

$$\frac{\mathrm{vol}\left(C_i\right)}{\Delta t}\left(\widetilde{W}_{hi}^{n+1}-W_{hi}^{n}\right)+\int_{\Gamma_i}\mathcal{F}^{w_h}\left(\mathbf{W}^{n},\rho\right)\tilde{\eta}dS-\int_{\Gamma_i}\left(\rho\mathcal{D}\nabla\frac{W_h^n}{\rho}\right)\tilde{\eta}dS$$
$$=-\int_{\Gamma_i}q_r^n\tilde{\eta}dS+\int_{C_i}f_h dV.$$

Numerical flux

Γ_i is split into the cell interfaces, Γ_{ij}, $N_j \in \mathcal{K}_i$,

$$\int_{C_i} \mathrm{div}\left[\mathcal{F}\left(\mathbf{W}^n, \rho^n\right)\right] d\mathcal{W} = \int_{\Gamma_i} \mathcal{F}\left(\mathbf{W}^n, \rho^n\right) \tilde{\boldsymbol{\eta}} dS = \sum_{N_j \in \mathcal{K}_i} \int_{\Gamma_{ij}} \mathcal{F}\left(\mathbf{W}^n, \rho^n\right) \tilde{\boldsymbol{\eta}}_{ij} dS$$

Numerical flux on a cell boundary, Γ_i

$$\mathcal{Z}\left(\mathbf{W}^n, \rho^n, \tilde{\boldsymbol{\eta}}\right) := \mathcal{F}\left(\mathbf{W}^n, \rho^n\right) \tilde{\boldsymbol{\eta}}.$$

The integral on Γ_{ij} is approximated by an upwind scheme,

$$\int_{\Gamma_{ij}} \mathcal{F}\left(\mathbf{W}^n, \rho^n\right) \tilde{\boldsymbol{\eta}}_{ij} dS \approx \phi\left(\left(\mathbf{W}_i^n, \rho_i^n\right), \left(\mathbf{W}_j^n, \rho_j^n\right), \boldsymbol{\eta}_{ij}\right),$$

with ϕ the numerical flux.

Numerical Flux: Rusanov scheme

$$\phi\left(\left(\mathbf{W}_i^n,\rho_i^n\right),\left(\mathbf{W}_j^n,\rho_j^n\right),\boldsymbol{\eta}_{ij}\right)=\frac{1}{2}\left(\mathcal{Z}^n\left(\mathbf{W}_i^n,\boldsymbol{\eta}_{ij}\right)+\mathcal{Z}^n\left(\mathbf{W}_j^n,\boldsymbol{\eta}_{ij}\right)\right)$$
$$-\frac{1}{2}\alpha_{RS}\left((\mathbf{W}_i^n,\rho_i^n),(\mathbf{W}_j^n,\rho_j^n),\boldsymbol{\eta}_{ij}\right)\left(\mathbf{W}_j^n-\mathbf{W}_i^n\right).$$

- Coupled

$$\alpha_{RS}\left((\mathbf{W}_i^n,\rho_i^n),(\mathbf{W}_j^n,\rho_j^n),\boldsymbol{\eta}_{ij}\right)=\max\left\{2\left|\mathbf{U}_i\cdot\boldsymbol{\eta}_{ij}\right|,2\left|\mathbf{U}_j\cdot\boldsymbol{\eta}_{ij}\right|\right\}.$$

- Decoupled

$$\alpha_{RS}^{\mathbf{w_u}}\left((\mathbf{W}_{u_i}^n,\rho_i^n),(\mathbf{W}_{u_j}^n,\rho_j^n),\boldsymbol{\eta}_{ij}\right)=\max\left\{2\left|\mathbf{U}_i\cdot\boldsymbol{\eta}_{ij}\right|,2\left|\mathbf{U}_j\cdot\boldsymbol{\eta}_{ij}\right|\right\},$$

$$\hat{\alpha}_{RS}\left((\widehat{\mathbf{W}}_i^n,\rho_i^n),(\widehat{\mathbf{W}}_j^n,\rho_j^n),\boldsymbol{\eta}_{ij}\right)=\max\left\{\left|\mathbf{U}_i\cdot\boldsymbol{\eta}_{ij}\right|,\left|\mathbf{U}_j\cdot\boldsymbol{\eta}_{ij}\right|\right\}.$$

We consider two different schemes to obtain second order:

- CVC Kolgan-type scheme

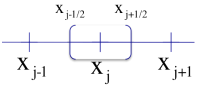
  A. Bermúdez, S. Busto, M. Cobas, J.L. Ferrín, L.Saavedra and M.E. Vázquez-Cendón. "Paths from mathematical problem to technology transfer related with finite volume methods". Proceedings of the XXIV CEDYA / XIV CAM (2015).

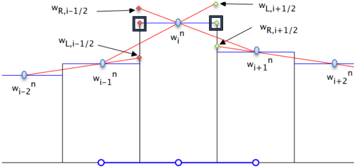
- Local ADER scheme

  S. Busto, E.F. Toro and M.E. Vázquez-Cendón. "Design and analysis of ADER-type schemes for model advection-diffusion-reaction equations". J. Comp. Phys. Volume 327, 15 December, Pages 553–575(2016).

A new Kolgan-type scheme: CVC

$$\frac{\partial W}{\partial t}(x,t) + \frac{\partial f(W)}{\partial x}(x,t) = 0$$





- New upwind second order scheme

$$W_j^{n+1} = W_j^n - \frac{\Delta t}{\Delta x}\left(\phi\left(W_j^n, W_{j+1}^n, W_{L,j+1/2}^n, W_{R,j+1/2}^n\right) - \phi\left(W_{j-1}^n, W_j^n, W_{L,j-1/2}^n, W_{R,j-1/2}^n\right)\right)$$

- Numerical flux

$$\phi(U, V, U_L, V_R) = \frac{f(U)+f(V)}{2} - \frac{1}{2}\left|\mathcal{Q}(U_L, V_R)\right|(V_R - U_L)$$

$$W_{L,j-1/2} = W_{j-1} + \Delta_{j-1}^{L*}, \quad W_{R,j-1/2} = W_j + \Delta_j^{R*},$$
$$W_{L,j+1/2} = W_j + \Delta_j^{L*}, \quad W_{R,j+1/2} = W_{j+1} + \Delta_{j+1}^{R*}.$$

Δ_j^{L*}, Δ_j^{R*} the *left and right limited slopes* at the node x_j:

$$\Delta_j^{L*} = \begin{cases} \max\left[0, \min\left(\frac{1}{2}\left(W_j - W_{j-1}\right), W_{j+1} - W_j\right)\right] & W_{j+1} - W_j > 0 \\ \min\left[0, \max\left(\frac{1}{2}\left(W_j - W_{j-1}\right), W_{j+1} - W_j\right)\right] & W_{j+1} - W_j < 0 \end{cases}$$

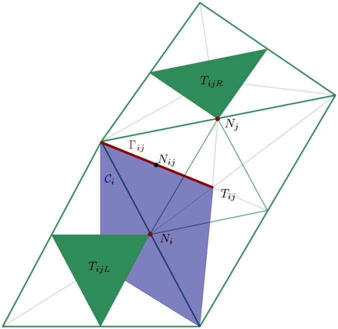
$$\Delta_j^{R*} = \begin{cases} \max\left[0, \min\left(-\frac{1}{2}\left(W_{j+1} - W_j\right), W_{j-1} - W_j\right)\right] & W_{j-1} - W_j > 0 \\ \min\left[0, \max\left(-\frac{1}{2}\left(W_{j+1} - W_j\right), W_{j-1} - W_j\right)\right] & W_{j-1} - W_j < 0 \end{cases}$$

L. Cea and M.E. Vázquez-Cendón. "Analysis of a new Kolgan-type scheme motivated by the shallow water equations". Appl. Num. Math. **62** 489–506 (2012).

$$\mathbf{W}_{ijL}^n = \mathbf{W}_i^n + \Delta^{ijL}, \qquad \mathbf{W}_{ijR}^n = \mathbf{W}_j^n - \Delta^{ijR},$$

Δ^{ijL} and Δ^{ijR} are the left and right limited slopes at the face defined with the Galerkin gradients computed at the upwind tetrahedra T_{ijL} and T_{ijL}, and taking into account some limiter:

$$\Delta^{ijL} = Lim\left(\frac{1}{2}\left(\nabla \mathbf{W}^n{}_{|T_{ijL}}\right) \cdot \overline{N_i N_j}, \mathbf{W}_j^n - \mathbf{W}_i^n\right),$$
$$\Delta^{ijR} = Lim\left(\frac{1}{2}\left(\nabla \mathbf{W}^n{}_{|T_{ijR}}\right) \cdot \overline{N_i N_j}, \mathbf{W}_j^n - \mathbf{W}_i^n\right),$$



the numerical flux reads

$$\phi\left((\mathbf{W}_i^n, \rho_i^n), (\mathbf{W}_j^n, \rho_j^n), (\mathbf{W}_{ijL}^n, \rho_{ijL}^n), (\mathbf{W}_{ijR}^n, \rho_{ijR}^n), \boldsymbol{\eta}_{ij}\right.$$
$$= \frac{1}{2}\left(\mathcal{Z}(\mathbf{W}_i^n, \rho_i, \boldsymbol{\eta}_{ij}) + \mathcal{Z}(\mathbf{W}_j^n, \rho_j^n, \boldsymbol{\eta}_{ij})\right)$$
$$-\frac{1}{2}\alpha_{RS}\left((\mathbf{W}_{ijL}^n, \rho_{ijL}^n), (\mathbf{W}_{ijR}^n, \rho_{ijR}^n), \boldsymbol{\eta}_{ij}\right)\left(\mathbf{W}_{ijR}^n - \mathbf{W}_{ijL}^n\right).$$

Local ADER

Step 1. ENO-based data reconstruction[1]. First-degree polynomials of the conservative variable are defined in each cell at the neighbouring of the faces:

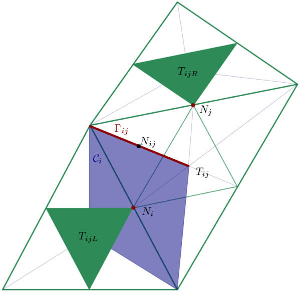
$$p_{i\,N_{ij}}(N) = W_i + (N - N_i)\,(\nabla W)^i_{ij}\,,$$

$$p_{j\,N_{ij}}(N) = W_j + (N - N_j)\,(\nabla W)^j_{ij}\,.$$

$$(\nabla W)^i_{ij} = \begin{cases} (\nabla W)_{T_{ijL}} & \left|(\nabla W)_{T_{ijL}}\cdot\Delta N\right| \le \left|(\nabla W)_{T_{ij}}\cdot\Delta N\right|, \\ (\nabla W)_{T_{ij}} & \left|(\nabla W)_{T_{ijL}}\cdot\Delta N\right| > \left|(\nabla W)_{T_{ij}}\cdot\Delta N\right|; \end{cases}$$

$$(\nabla W)^j_{ij} = \begin{cases} (\nabla W)_{T_{ijR}} & \left|(\nabla W)_{T_{ijR}}\cdot\Delta N\right| \le \left|(\nabla W)_{T_{ij}}\cdot\Delta N\right|, \\ (\nabla W)_{T_{ij}} & \left|(\nabla W)_{T_{ijR}}\cdot\Delta N\right| > \left|(\nabla W)_{T_{ij}}\cdot\Delta N\right|, \end{cases}$$

$$\Delta N := N_{ij} - N_j.$$



[1] E.F. Toro "Riemann solvers and numerical methods for fluid dynamics: a practical introduction. Third edition". Springer, 2009.

Local ADER

Step 1. ENO-based data reconstruction[1]. First-degree polynomials of the conservative variable are defined in each cell at the neighbouring of the faces:

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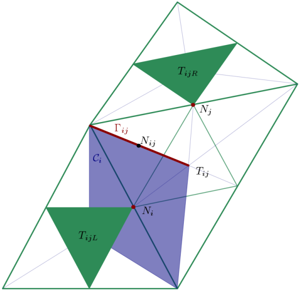
$$p_{j\,N_{ij}}(N) = W_j + (N - N_j)\,(\nabla W)^j_{ij}\,.$$

Step 2. Computation of boundary extrapolated values at the barycentre of the face Γ_{ij}, N_{ij}.

$$W_{i\,N_{ij}} = p_{i\,N_{ij}}(N_{ij}) = W_i + (N_{ij} - N_i)\,(\nabla W)^i_{ij}\,,$$

$$W_{j\,N_{ij}} = p_{j\,N_{ij}}(N_{ij}) = W_j + (N_{ij} - N_j)\,(\nabla W)^j_{ij}\,.$$



[1] E.F. Toro "Riemann solvers and numerical methods for fluid dynamics: a practical introduction. Third edition". Springer, 2009.

Step 3. Taylor series expansion in time and Cauchy - Kovalevskaya procedure are applied to locally approximate the conservative variables at time $\tau = \frac{\Delta t}{2}$:

$$\overline{\mathbf{W}_{i\,N_{ij}}} = \mathbf{W}_{i\,N_{ij}} - \frac{\Delta t}{2\mathcal{L}_{ij}} \left(\mathcal{Z}(\mathbf{W}_i, \boldsymbol{\eta}_{ij}) + \mathcal{Z}(\mathbf{W}_j, \boldsymbol{\eta}_{ij}) \right) + \frac{\Delta t}{2\mathcal{L}_{ij}^2} \left(\alpha_{iN_{ij}} \nabla \mathbf{W}_{|T_{ijL}} \boldsymbol{\eta}_{ij} + \alpha_{jN_{ij}} \nabla \mathbf{W}_{|T_{ijR}} \boldsymbol{\eta}_{ij} \right),$$

$$\overline{\mathbf{W}_{j\,N_{ij}}} = \mathbf{W}_{j\,N_{ij}} - \frac{\Delta t}{2\mathcal{L}_{ij}} \left(\mathcal{Z}(\mathbf{W}_i, \boldsymbol{\eta}_{ij}) + \mathcal{Z}(\mathbf{W}_j, \boldsymbol{\eta}_{ij}) \right) + \frac{\Delta t}{2\mathcal{L}_{ij}^2} \left(\alpha_{iN_{ij}} \nabla \mathbf{W}_{|T_{ijL}} \boldsymbol{\eta}_{ij} + \alpha_{jN_{ij}} \nabla \mathbf{W}_{|T_{ijR}} \boldsymbol{\eta}_{ij} \right),$$

$\mathcal{L}_{ij} = \min \left\{ \frac{\mathrm{vol}(C_i)}{\mathrm{S}(C_i)}, \frac{\mathrm{vol}(C_j)}{\mathrm{S}(C_j)} \right\}$, $\alpha_{i\,N_{ij}}$ diffusion coefficient.

Step 4. Computation of the numerical flux considering Rusanov scheme:

$$\phi\left(\left(\overline{\mathbf{W}^n_{i\,N_{ij}}}, \rho^n_i \right), \left(\overline{\mathbf{W}^n_{j\,N_{ij}}}, \rho^n_j \right), \eta_{ij} \right) = \frac{1}{2} \left(\mathcal{Z}\left(\overline{\mathbf{W}^n_{i\,N_{ij}}}, \eta_{ij} \right) + \mathcal{Z}\left(\overline{\mathbf{W}^n_{j\,N_{ij}}} \eta_{ij} \right) \right) - \frac{1}{2} \alpha_{RS} \left((\overline{\mathbf{W}^n_{i\,N_{ij}}}, \rho^n_i), (\overline{\mathbf{W}^n_{j\,N_{ij}}}, \rho^n_j), \boldsymbol{\eta}_{ij} \right) \left(\overline{\mathbf{W}^n_{j\,N_{ij}}} - \overline{\mathbf{W}^n_{i\,N_{ij}}} \right).$$

Pressure term

Pressure term

$$\int_{C_i} \nabla \pi^n dV = \sum_{N_j \in \mathcal{K}_i} \int_{\Gamma_{ij}} \pi^n \tilde{\eta}_{ij} dS.$$

The pressure is approximated by the average of the values at the vertices of the face.

$$\int_{\Gamma_{ij}} \pi^n \tilde{\eta}_{ij} dS \approx \frac{1}{3} \left(\pi^n (\mathbf{V}_1) + \pi^n (\mathbf{V}_2) + \pi^n (B)\right) \mathrm{area} (\Gamma_{ij}) \tilde{\eta}_{ij} =$$

$$= \left[\frac{5}{12} \left(\pi^n (\mathbf{V}_1) + \pi^n (\mathbf{V}_2)\right) + \frac{1}{12} \left(\pi^n (\mathbf{V}_3) + \pi^n (\mathbf{V}_4)\right) \right] \eta_{ij}.$$

Viscous terms

Viscous term of the momentum equation

$$\int_{C_i} \operatorname{div} \tau^n dV = \sum_{N_j \in \mathcal{K}_i} \int_{\Gamma_{ij}} \tau^n \tilde{\eta}_{ij} dS = \sum_{N_j \in \mathcal{K}_i} \int_{\Gamma_{ij}} \mu \nabla \mathbf{u} \, \tilde{\eta}_{ij} dS.$$

Gradient approximation:

1. orthogonal and non-orthogonal flux in the face [2];
2. `Galerkin approximation at the tetrahedron of the face;`
3. average of the Galerkin approximation at the upwind tetrahedra;
4. average of the Galerkin approximation at the three tetrahedra.

[2] A. Bermúdez, J.L.Ferrín, L. Saavedra and M.E. Vázquez-Cendón "A projection hybrid finite volume/ element method for low-Mach number flows". J. Comp. Phys. **271** 360–378, 2014.

Galerkin approximation at the tetrahedron of the face:

We consider a numerical diffusion function such that

$$\int_{\Gamma_{ij}} \mu \nabla \mathbf{U}^n \tilde{\eta}_{ij} \mathrm{dS} \approx \psi_{\mathbf{u}} \left(\mathbf{U}_i^n, \mathbf{U}_j^n, \boldsymbol{\eta}_{ij} \right).$$

Two different discretizations are considering depending on the convection term treatment:

CVC Kolgan type scheme

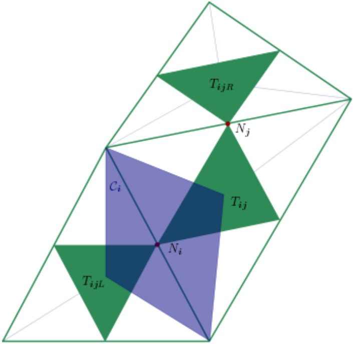
$$\psi_{\mathbf{u}} \left(\mathbf{U}_i^n, \mathbf{U}_j^n, \boldsymbol{\eta}_{ij} \right) = \mu \left(\nabla \mathbf{U}^n \right)_{|T_{ij}} \boldsymbol{\eta}_{ij}.$$

Local ADER scheme

$$\psi_{\mathbf{u}} \left(\overline{\mathbf{U}_i^n}, \overline{\mathbf{U}_j^n}, \boldsymbol{\eta}_{ij} \right) = \mu \left(\nabla \overline{\mathbf{U}^n} \right)_{|T_{ij}} \boldsymbol{\eta}_{ij},$$

$$\overline{\mathbf{U}_i^n} = \mathbf{U}_i^n + \frac{\Delta t}{4} \mathrm{tr} \left(\nabla \mathbf{V}^n_{|T_{ijL}} + \nabla \mathbf{V}^n_{|T_{ij}} \right),$$

$$\mathbf{V}_i^n = \mu \frac{1}{2} \left(\nabla U^n_{|T_{ijL}} + \nabla U^n_{|T_{ij}} \right).$$



Viscous terms for $\widehat{\mathbf{W}}$

The viscous terms for the species and energy equations read

$$\int_{C_i} \frac{1}{\rho} \operatorname{div}\left(\boldsymbol{\mathcal{D}}^n \nabla \widehat{\mathbf{W}}^n\right) dV = \frac{1}{\rho} \sum_{N_j \in \mathcal{K}_i} \int_{\Gamma_{ij}} \boldsymbol{\mathcal{D}}^n \nabla \widehat{\mathbf{W}}^n \, \tilde{\boldsymbol{\eta}}_{ij} \, dS. \quad (15)$$

Like for the momentum equation we introduce a new numerical diffusion function, $\psi_{\hat{\mathbf{w}}}$, such that

$$\int_{\Gamma_{ij}} \boldsymbol{\mathcal{D}}^n \nabla \widehat{\mathbf{W}}^n \tilde{\boldsymbol{\eta}} dS \approx \psi_{\widehat{\mathbf{w}}}\left(\widehat{\mathbf{W}}_i^n, \widehat{\mathbf{W}}_j^n, \boldsymbol{\eta}_{ij}\right),$$

$$\psi_{\widehat{\mathbf{w}}}\left(\widehat{\mathbf{W}}_i^n, \widehat{\mathbf{W}}_j^n, \boldsymbol{\eta}_{ij}\right) = \boldsymbol{\mathcal{D}}_{ij}^n \left(\nabla \widehat{\mathbf{W}}^n\right)_{|T_{ij}} \boldsymbol{\eta}_{ij}.$$

Pre-projection stage: Low Mach number flows

Pre-projection stage. Low Mach number flows

- The temperature is calculated by solving

$$h^{n+1} = h_{\theta_0} + \int_{\theta_0}^{\theta^{n+1}} c_\pi(r)dr$$

using Newton's method.

- The density is obtained from the state equation.

- The computed density is used to approximate the source term of the projection stage equations:

$$q^{n+1} = \frac{\rho^{n+1} - \rho^n}{\Delta t}.$$

Projection stage

Projection stage

We use a finite element method in order to solve

$$\frac{\mathbf{W}_{\mathbf{u}}^{n+1} - \widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1}}{\Delta t} + \nabla\left(\pi^{n+1} - \pi^{n}\right) = 0,$$

$$\operatorname{div} \mathbf{W}_{\mathbf{u}}^{n+1} = q^{n+1}.$$

Let $z \in V_0$ be a test function, $V_0 := \left\{ z \in H^1(\Omega) : \int_\Omega z = 0 \right\}$, then

$$\int_\Omega \nabla\left(\pi^{n+1} - \pi^{n}\right) \cdot \nabla z dV = \frac{1}{\Delta t} \int_\Omega \widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1} \cdot \nabla z dV - \frac{1}{\Delta t} \int_\Omega \mathbf{W}_{\mathbf{u}}^{n+1} \cdot \nabla z dV.$$

Using the divergence condition and the Green formula, we obtain

$$\int_{\Omega} \mathbf{W}_{\mathbf{u}}^{n+1} \cdot \nabla z dV = \int_{\Gamma} \mathbf{W}_{\mathbf{u}}^{n+1} \cdot \eta \nabla z dS - \int_{\Omega} \operatorname{div}\left(\mathbf{W}_{\mathbf{u}}^{n+1}\right) z dV = \int_{\Gamma} g^{n+1} z - \int_{\Omega} q^{n+1} z dS.$$

Replacing the previous expression in the variational formulation and introducing the variable $\delta^{n+1} := \pi^{n+1} - \pi^{n}$, to obtain π^{n+1} we need to solve the weak problem

$$\int_{\Omega} \nabla \delta^{n+1} \cdot \nabla z dV = \frac{1}{\Delta t} \int_{\Omega} \widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1} \cdot \nabla z dV + \frac{1}{\Delta t} \int_{\Omega} q^{n+1} z dV - \frac{1}{\Delta t} \int_{\Gamma} g^{n+1} z dS.$$

This weak problem can be seen as corresponding to the following Laplace problem with Neumann conditions

$$\Delta \delta^{n+1} = \frac{1}{\Delta t}\left(\operatorname{div} \widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1} - q^{n+1}\right) \quad \text{in } \Omega,$$

$$\frac{\partial \delta^{n+1}}{\partial \eta} = \frac{1}{\Delta t}\left(\widetilde{\mathbf{W}}_{\mathbf{u}}^{n+1} \cdot \eta - g^{n+1}\right) \quad \text{in } \Gamma.$$

Post-projection stage

Linear momentum density

$$\mathbf{W}_{\mathbf{u}\,i}^{n+1} = \widetilde{\mathbf{W}}_{\mathbf{u}\,i}^{n+1} + \Delta t \nabla \left(\pi_i^{n+1} - \pi_i^{n}\right)$$

Numerical results

Incompressible test

Computational domain:

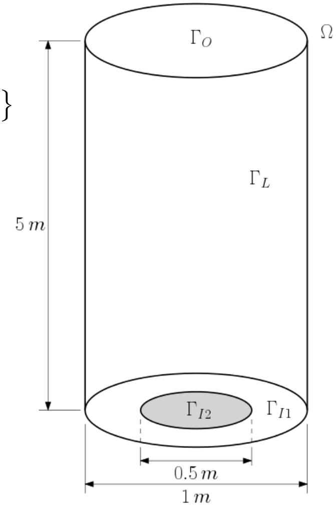
$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2) \leq 0.5^2, z \in [0, 5]\right\}$$

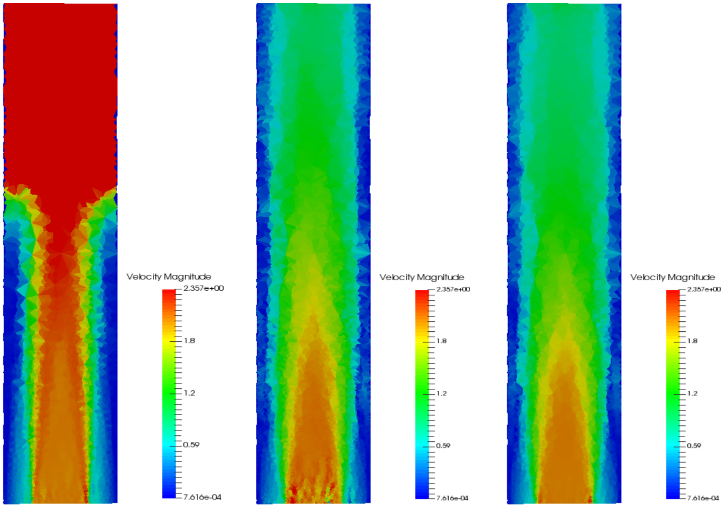
Fluid properties:

- $\rho = 0.38474$
- $\mu = 1.7894e^{-5}$

Boundary conditions:

- Γ_{I1}: $\mathbf{v} = \left(0, 0, \frac{-x^2-y^2}{0.25} + 1\right)$,
- Γ_{I2}: $\mathbf{v} = (0, 0, 2)$,
- Γ_O: outflow,
- Γ_L: $\mathbf{v} = (0, 0, 0)$.




Velocity Magnitude
2.357e+00
1.8
1.2
0.59
7.616e-04
Velocity Magnitude
2.357e+00
1.8
1.2
0.59
7.616e-04
Velocity Magnitude
2.357e+00
1.8
1.2
0.59
7.616e-04

The importance of density

Computational domain:

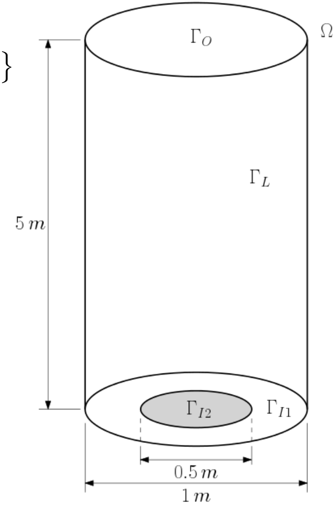
$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2) \leq 0.5^2, z \in [0, 5]\}$$

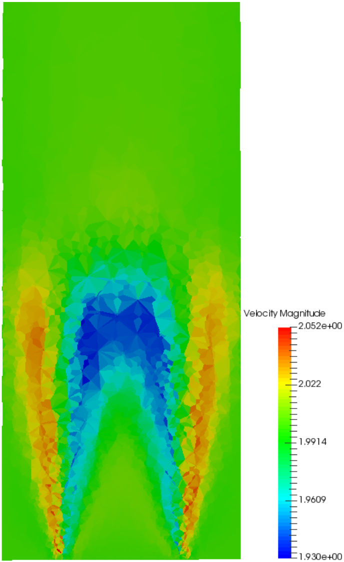
Species:

- O_2
- CO_2

Boundary conditions:

- Γ_{I1}: $\mathbf{v} = (0, 0, 2)$, $\mathbf{Y}_{O_2} = 0.75$, $\mathbf{Y}_{CO_2} = 0.25$, $\theta = 1000$,
- Γ_{I2}: $\mathbf{v} = (0, 0, 2)$, $\mathbf{Y}_{O_2} = 0.25$, $\mathbf{Y}_{CO_2} = 0.75$, $\theta = 1000$,
- Γ_O: outflow,
- Γ_L: $\mathbf{v} = (0, 0, 2)$, $\mathbf{Y}_{O_2} = 0.75$, $\mathbf{Y}_{CO_2} = 0.25$, $\theta = 1000$.




Velocity Magnitude
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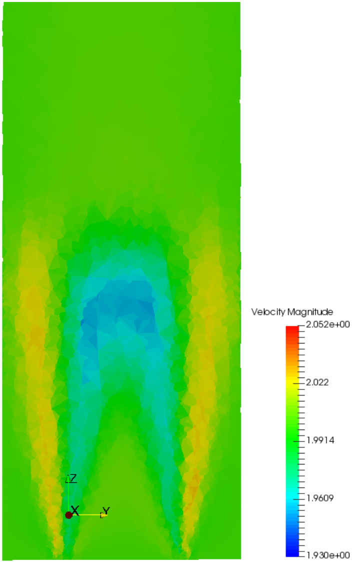
2.022
1.9914
1.9609
1.930e+00


Velocity Magnitude
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2.022
1.9914
1.9609
1.930e+00
Z
X
Y

Second compressible test

Computational domain:

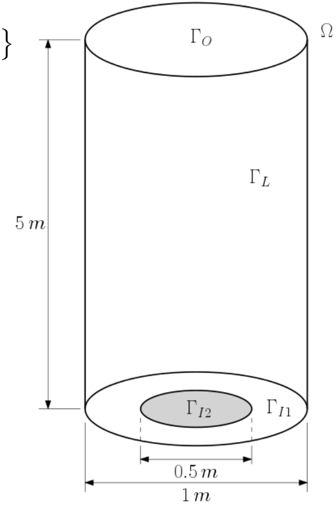
$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2) \leq 0.5^2, z \in [0, 5]\right\}$$

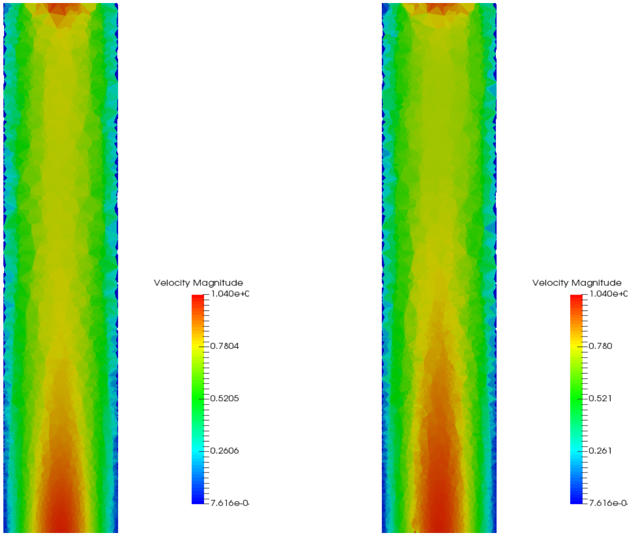
Species:

- O_2
- CO_2

Boundary conditions:

- Γ_{I1}: $\mathbf{v} = \left(0, 0, \frac{-x^2-y^2}{0.25} + 1\right)$, $\mathbf{Y}_{O_2} = 0.75$, $\mathbf{Y}_{CO_2} = 0.25$, $\theta = 1000$,
- Γ_{I2}: $\mathbf{v} = \left(0, 0, \frac{-x^2-y^2}{0.25} + 1\right)$, $\mathbf{Y}_{O_2} = 0.25$, $\mathbf{Y}_{CO_2} = 0.75$, $\theta = 1200$,
- Γ_O: outflow,
- Γ_L: $\mathbf{v} = \left(0, 0, \frac{-x^2-y^2}{0.25} + 1\right)$, $\mathbf{Y}_{O_2} = 0.75$, $\mathbf{Y}_{CO_2} = 0.25$, $\theta = 1000$.




Velocity Magnitude
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7.616e-0
Velocity Magnitude
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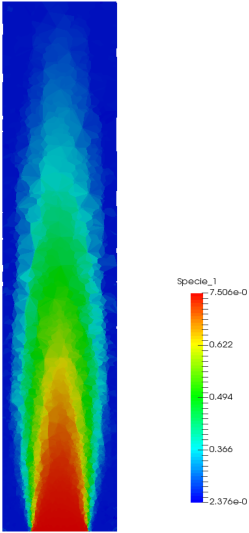
0.261
7.616e-0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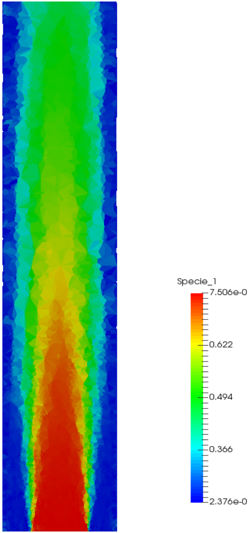
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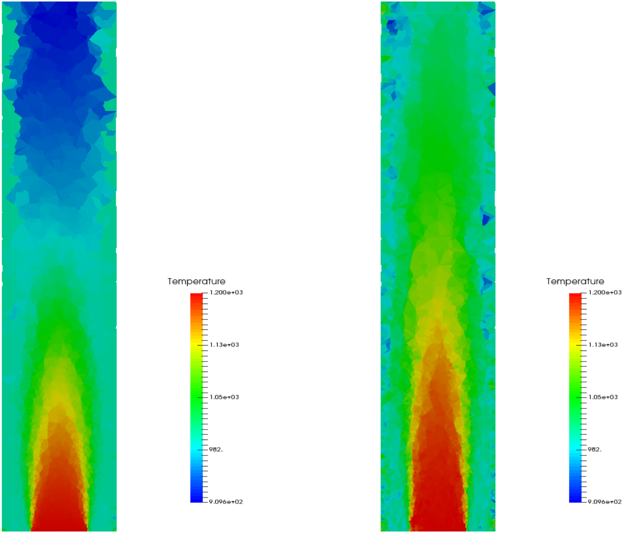
Temperature
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9.096e+02
Temperature
1.200e+03
1.13e+03
1.05e+03
982.
9.096e+02

Ongoing research and conclusions

Divergence terms for low Mach number flows

The equations of species transport and energy conservation in low Mach number flows include two new terms with respect to the incompressible equations:

$$-\mathbf{w_Y}\,\mathrm{div}\,\mathbf{v}$$
$$\Downarrow$$
$$-\int_{C_i}\mathbf{w_Y}\,\mathrm{div}\,\mathbf{v}\,dV$$
$$\approx -\mathbf{w_Y}_{|N_i}\sum_{N_j\in\mathcal{K}_i}\int_{\Gamma_{ij}}\mathbf{v}\cdot\widetilde{\eta}dS$$

$$-\mathbf{w}_h\,\mathrm{div}\,\mathbf{v}$$
$$\Downarrow$$
$$-\int_{C_i}\mathbf{w}_h\,\mathrm{div}\,\mathbf{v}\,dV$$
$$\approx -\mathbf{w}_{h|N_i}\sum_{N_j\in\mathcal{K}_i}\int_{\Gamma_{ij}}\mathbf{v}\cdot\widetilde{\eta}dS$$

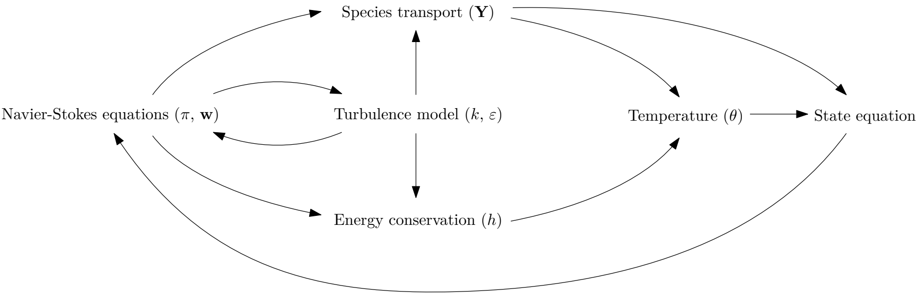
Source term treatment[3,4]?

[3] A. Bermúdez, A. Dervieux, J.A. Desideri and M.E. Vázquez-Cendón "Upwind schemes for the two-dimensional shallow water equations with variable depth using unstructured meshes". Comput.Methods in Appl.Mech.Eng. **155** 49–72, 1998.

[4] M.E. Vázquez-Cendón "Estudio de esquemas descentrados para su aplicación a las leyes de conservación hiperbólicas con términos fuente". PhD thesis, 1994.

Conclusions

Equations dependency



Models

- Incompressible Newtonian flow
- Low Mach number flow

Methodology

- Hybrid projection method FV/FE

FV schemes

- 1^{st} order
- CVC Kolgan-type
- Local ADER (ENO based)

Acknowledgements

This research was partially supported by Spanish MICINN project MTM2013-43745-R; by the Spanish MECD under grant FPU13/00279; by the Xunta de Galicia Consellería de Cultura Educación e Ordenación Universitaria under grant *Axudas de apoio á etapa predoutoral do Plan I2C*; by Xunta de Galicia and FEDER under research project GRC2013-014 and by Fundación Barrié under grant *Becas de posgrado en el extranjero*.

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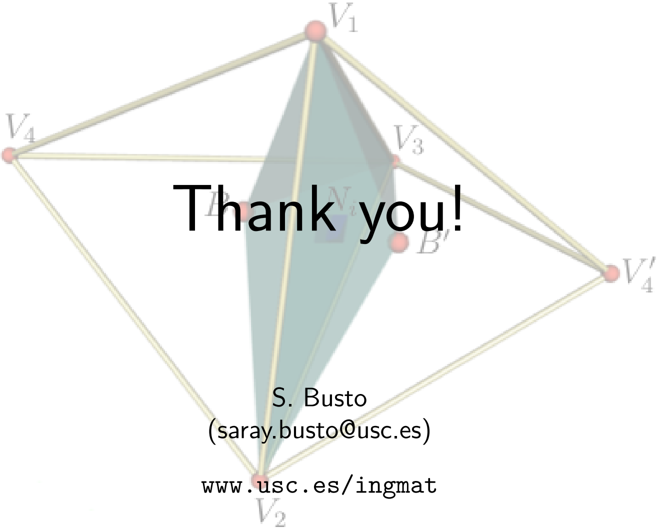

Thank you!

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