Runge-Kutta Multirate Schemes for ODEs and Conservation Laws

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Outline

1. Introduction to the problem
2. Multirate Schemes
3. Multirate with Continuous Extensions (MRKCE)
4. Numerical Tests
5. Multirate strategy for Conservation Laws

Test problem

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & \epsilon \\ \epsilon & -a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where $a > 1$, ϵ is the coupling coefficient. The general solution is

$$y(t) = k_1 \bar{v}_1 \exp(\lambda_1 t) + k_2 \bar{v}_2 \exp(\lambda_2 t).$$

Real eigenvalues and for ϵ small enough

$$\bar{v}_1 = \begin{bmatrix} 1 \\ O(\epsilon) \end{bmatrix}, \qquad \bar{v}_2 = \begin{bmatrix} O(\epsilon) \\ 1 \end{bmatrix}$$

$$\lambda_1 = -1 + O(\epsilon^2), \qquad \lambda_2 = -a + O(\epsilon^2)$$

$\Rightarrow$ separably stiff system $(a \gg 1)$

Active vs Latent

Consider a Cauchy Problem

$$\begin{cases} y' = f(t,y), & f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \\ y(0) = y_0, & y_0 \in \mathbb{R}^m \end{cases}$$

- moderately stiff: $\frac{|\lambda_{\max}|}{|\lambda_{\min}|} \in (4, 100)$
- partitioned in two sets of variables

$$y = \begin{pmatrix} y_A \\ y_L \end{pmatrix}, \; y_A \in \mathbb{R}^{m_A}, \; y_L \in \mathbb{R}^{m_L}, \; m_A + m_L = m$$

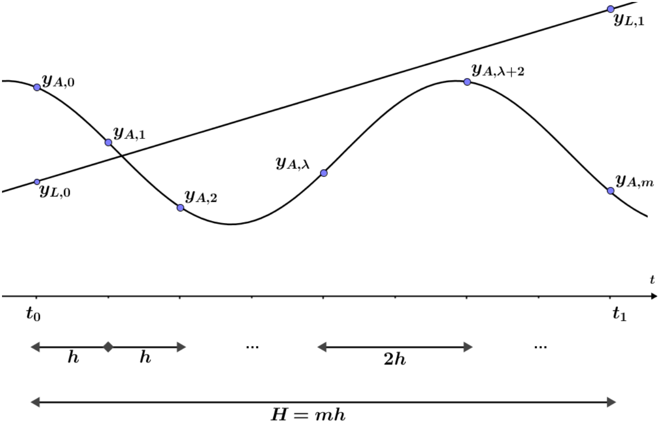
$$y' = f(y) \Leftrightarrow \begin{cases} y_A'(t) = f_A(y_A, y_L), & y_A(t_0) = y_{A,0} \\ y_L'(t) = f_L(y_A, y_L), & y_L(t_0) = y_{L,0} \end{cases}$$

$y_A(t)$ active or fast $\qquad$ $y_L(t)$ latent or slow

- separably stiff: $\min_i |\lambda_i^A| \gg \max_i |\lambda_i^L|$.

Explicit Multirate methods

- $y_L(t)$ are approximated with time-step H
- $y_A(t)$ are approximated with time-step $h = \frac{H}{m}, m \in \mathbb{N}$

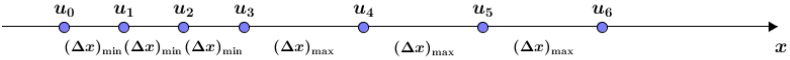


Benefit: we use an explicit scheme, reducing the computational cost and avoiding stability problems.

Applications

1. Electronic circuits
   - coupled digital and analogical circuits
   - inverter chain

2. Discretization of PDEs with the method of lines and non-uniform grids



$$u_i' = \varphi(u_{i-1}, u_i, u_{i+1}; \Delta x_i), \; u_i(t) \approx u(t, x_{i-1} + \Delta x_i)$$

Runge-Kutta Schemes

Multirate methods with explicit Runge-Kutta schemes.

Definition

An explicit Runge-Kutta (RK) scheme with s stages for the approximation of the Cauchy Problem is defined by

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i, \quad n = 0, 1, \dots, N-1$$

$$k_i = f(t_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} k_j), \quad i = 1, 2, \dots, s$$

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \dots & a_{1s} \\ c_2 & a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \vdots & & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\ \hline & b_1 & b_2 & \dots & b_s \end{array}$$

Butcher Tableau

$$\Rightarrow \begin{array}{c|c} c & A \\ \hline & b^t \end{array} \quad \text{or} \quad (A, b, c)$$

Runge-Kutta Scheme

$m = 2$ time-steps of amplitude h of a Runge-Kutta scheme (A, b, c)

$$k_i^1 = f(y_0 + h \sum_{j=1}^{i-1} a_{ij} k_j^1) \qquad k_i^2 = f(y_1 + h \sum_{j=1}^{i-1} a_{ij} k_j^2)$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i k_i^1 \qquad y_2 = y_1 + h \sum_{i=1}^{s} b_i k_i^2$$

Runge-Kutta Scheme

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$$k_i^2 = f(y_0 + h \sum_{i=1}^{s} b_i k_i^1 + h \sum_{j=1}^{i-1} a_{ij} k_j^2)$$

$$y_2 = y_0 + h \sum_{i=1}^{s} b_i k_i^1 + h \sum_{i=1}^{s} b_i k_i^2$$

is equivalent to $m = 1$ time-step of amplitude $H = 2h$ of the method

$$\begin{array}{c|cc} \frac{1}{2} c & \frac{1}{2} A & \\ \frac{1}{2}(\mathbf{1}_s + c) & \frac{1}{2} \mathbf{1}_s b^t & \frac{1}{2} A \\ \hline & \frac{1}{2} b^t & \frac{1}{2} b^t \end{array}$$

Runge-Kutta Scheme

$m = 2$ time-steps of amplitude h of a Runge-Kutta scheme (A, b, c)

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$$\begin{array}{c|cc} \frac{1}{2}c & \frac{1}{2}A & \\ \frac{1}{2}(\mathbf{1}_s + c) & \frac{1}{2}\mathbf{1}_s b^t & \frac{1}{2}A \\ \hline & \frac{1}{2}b^t & \frac{1}{2}b^t \end{array}$$

In the case of a generic m the method is

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{b}^t \end{array} = \begin{array}{c|cccc} \frac{1}{m}c & \frac{1}{m}A & & & \\ \frac{1}{m}(\mathbf{1}_s + c) & \frac{1}{m}\mathbf{1}_s b^t & \frac{1}{m}A & & \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{1}{m}((m-1)\mathbf{1}_s + c) & \frac{1}{m}\mathbf{1}_s b^t & \frac{1}{m}\mathbf{1}_s b^t & \dots & \frac{1}{m}A \\ \hline & \frac{1}{m}b^t & \frac{1}{m}b^t & \dots & \frac{1}{m}b^t \end{array}$$

Multirate Runge-Kutta

$$\begin{cases} y_A'(t) = f_A(y_A, y_L), & y_A(t_0) = y_{A,0} \\ y_L'(t) = f_L(y_A, y_L), & y_L(t_0) = y_{L,0} \end{cases}$$

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Definition

Let (A, b, c) be an explicit Runge-Kutta scheme with s stages and let $h = \frac{H}{m}$. We define Multirate method (MRK) a numerical scheme in which the stages of the time-step n are computed as

$$k_{A,i}^{\lambda} = f_A\left(y_{A,n}^{\lambda} + h\sum_{j=1}^{i-1} a_{ij}k_{A,j}^{\lambda},\ Y_{L,i}^{\lambda}\right),\ i = 1, 2, \dots, s,\ \lambda = 0, 1, \dots, m-1$$

$$k_{L,i} = f_L\left(Y_{A,i},\ y_{L,n} + H\sum_{j=1}^{i-1} a_{ij}k_{L,j}\right),\ i = 1, 2, \dots, s$$

where $Y_{L,i}^{\lambda} \approx y_L(t_0 + (\lambda + c_i)h)$ and $Y_{A,i} \approx y_A(t_0 + c_iH)$.

$\Rightarrow$ necessity of computing the stage-values.

Multirate Runge-Kutta

We employ the already computed stages. Then in compact form:

$$\begin{aligned}
\text{Active:}\quad y_{A,n+1} &= y_{A,n} + H\tilde{b}^t \mathbf{k}_A \\
\mathbf{k}_A &= f_A\left(y_{A,n} + H\tilde{A}\mathbf{k}_A,\, y_{L,n} + HB\mathbf{k}_L\right), \\
\text{Latent:}\quad y_{L,n+1} &= y_{L,n} + Hb^t \mathbf{k}_L \\
\mathbf{k}_L &= f_L\left(y_{A,n} + HC\mathbf{k}_A,\, y_{L,n} + HA\mathbf{k}_L\right).
\end{aligned}$$

Multirate tableau:

$\tilde{c}$	$\tilde{A}$	B	
	C	A	c
	$\tilde{b}^t$	b^t	

$\Rightarrow$ we write the Multirate as a Partitioned.

Partitioned Runge-Kutta

The components of the system are integrated with the same time-step H of two Runge-Kutta methods $(\hat{A}, \hat{b}_A, \hat{c}_A)$ with q stages and $(\hat{D}, \hat{b}_D, \hat{c}_D)$ with r stages.

$$
\begin{array}{c|ccccc|ccc|c}
c_{A,1} & \alpha_{11} & \dots & \dots & \dots & \alpha_{1q} & \beta_{11} & \dots & \beta_{1r} & c_{B,1} \\
\vdots & \vdots & & & & \vdots & \vdots & & \vdots & \vdots \\
c_{A,q} & \alpha_{q1} & \dots & \dots & \dots & \alpha_{qq} & \beta_{q1} & \dots & \beta_{qr} & c_{B,q} \\
\hline
c_{C,1} & \xi_{11} & \dots & \dots & \dots & \xi_{1q} & \delta_{11} & \dots & \delta_{1r} & c_{D,1} \\
\vdots & \vdots & & & & \vdots & \vdots & & \vdots & \vdots \\
c_{C,r} & \xi_{r1} & \dots & \dots & \dots & \xi_{rq} & \delta_{r1} & \dots & \delta_{rr} & c_{D,r} \\
\hline
 & b_{A,1} & \dots & \dots & \dots & b_{A,q} & b_{D,1} & \dots & b_{D,r} &
\end{array}
$$

$$k_{A,i} = f_A(y_{A,0} + H \sum_{j=1}^{q} \alpha_{ij} k_{A,j},\ y_{L,0} + H \sum_{j=1}^{r} \beta_{ij} k_{L,j}),\ \ i = 1, 2, \ldots, q$$

$$k_{L,i} = f_L(y_{A,0} + H \sum_{j=1}^{q} \xi_{ij} k_{A,j},\ y_{L,0} + H \sum_{j=1}^{r} \delta_{ij} k_{L,j}),\ \ i = 1, 2, \ldots, r$$

Partitioned Runge-Kutta: order conditions

Under the simplifying hypotheses [Hairer, 1981]:

order	active	latent	additionally, $\forall\theta \in \{\frac{1}{m}, \ldots, 1\}$
1	$\tilde{b}^t \mathbf{1}_{ms} = 1$	$b^t \mathbf{1}_s = 1$	$\tilde{b}^t(\theta) \mathbf{1}_{ms} = \theta$
2	$\tilde{b}^t \tilde{c} = \frac{1}{2}$	$b^t c = \frac{1}{2}$	$\tilde{b}^t(\theta) \tilde{c} = \frac{\theta^2}{2}$
3	$\sum_{i=1}^{ms} \tilde{b}_i \tilde{c}_i^2 = \frac{1}{3}$ $\tilde{b}^t \tilde{A} \tilde{c} = \frac{1}{6}$	$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}$ $b^t A c = \frac{1}{6}$ $b^t C \tilde{c} = \frac{1}{6}$	$\sum_{i=1}^{ms} \tilde{b}_i(\theta) \tilde{c}_i^2 = \frac{\theta^3}{3}$ $\tilde{b}^t(\theta) \tilde{A} \tilde{c} = \frac{\theta^3}{6}$ $\tilde{b}^t(\theta) B c = \frac{\theta^3}{6}$

where $x(l/m) = [\underbrace{x, \ldots, x}_{l}, \underbrace{0, \ldots, 0}_{m-l}]$.

Theorem

If the Partitioned scheme is a Multirate scheme then the order conditions marked in blue are automatically satisfied.

Goal

New method for determining the coefficients of $Y^{\lambda}_{L,i}$ and $Y_{A,i}$.

- Compute all the slow stages during the first active microstep

$\Rightarrow$ C is a lower triangular matrix

$\Rightarrow$ the relations $C\mathbf{1}_{ms} = c,\ \ b^t C c = \frac{1}{6}$ are sufficient to define C

$\Rightarrow$ we can immediately compute the approximation of the latent component $y_{L,n+1} \approx y_L(t_n + H)$

- $Y^{\lambda}_{L,i} \approx y_L(t_n + \frac{c_i+\lambda}{m} H)$ approximations of y_L in the points $t^{\lambda}_i = t_n + \frac{c_i+\lambda}{m} H$, in $[t_n, t_n + H]$

We use the Continuous Extensions of the Runge-Kutta schemes.

M. Semplice, G. Visconti (2016), *Multirate schemes with continuous extensions for separably stiff problems*. In preparation

Continuous Extensions and DDEs

$$\begin{cases} y'(t) = f(t, y(t), y(t - \tau(t, y(t)))), & t_0 \leq t \leq t_f \\ y(t) = \phi(t), & t \leq t_0 \end{cases} \tag{DDE}$$

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right) \qquad \text{(Logistic equation with delay)}$$

$\Rightarrow$ is useful having a continuous approximation of the solution.

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Definition (Zennaro 1986)

The Natural Continuous Extension (NCE) of an explicit Runge-Kutta scheme (A, b, c) of order p is the continuous approximation

$$u(t_0 + \rho H) = y_0 + H \sum_{i=1}^{s} b_i(\rho) k_i$$

where $\rho \in [0, 1]$, $b_i(\rho)$, $i = 1, \ldots, s$ polynomials of degree $\le d$ and such that

$$\int_{t_0}^{t_0+H} G(t)[y'(t) - u'(t)]dt = O(h^{p+1})$$

Coefficients of the NCEs

NCEs coefficients of the explicit Runge-Kutta methods of order $p = s = 1, 2, 3$

- Runge-Kutta with one stage and of order 1:

$$d = \left[\tfrac{p+1}{2}\right] = s = p = 1 \qquad b_1(\rho) = \rho$$

- Runge-Kutta with two stages and of order 2:

$$d = \left[\tfrac{p+1}{2}\right] = 1 \qquad b_i(\rho) = b_i\rho, \;\; i = 1, 2$$

$$d = s = p = 2 \qquad \begin{cases} b_1(\rho) = (b_1 - 1)\rho^2 + \rho \\ b_2(\rho) = b_2\rho^2 \end{cases}$$

- Runge-Kutta with three stages and of order 3:

$$d = \left[\tfrac{p+1}{2}\right] = 2 \qquad b_i(\rho) = w_i\rho^2 + (b_i - w_i)\rho, \;\; i = 1, 2, 3$$

where, for each $k \in \mathbb{R}$,

$$w_1 = -\frac{k(c_3 - c_2) + c_2}{2c_2c_3}, \qquad w_2 = \frac{k}{2c_2}, \qquad w_3 = \frac{1-k}{2c_3}$$

Multirate with NCEs

We use the continuous extensions for the latent stage-values $Y_{L,i}^{\lambda} \approx y_L(t_n + (\frac{c_i+\lambda}{m})H)$:

$$Y_{L,i}^{\lambda} = y_{L,n} + H \sum_{j=1}^{s} b_j(\rho_i^{\lambda}) k_{L,j}$$

$$\rho_i^{\lambda} = \tfrac{c_i+\lambda}{m},\ i = 1, \dots, s,\ \lambda = 0, \dots, m-1$$

The simplifying hypothesis $B\mathbf{1}_s = \tilde{c}$ is verified:

$$\sum_{j=1}^{s} b_j(\rho_i^{\lambda}) = \rho_i^{\lambda} = \frac{c_i + \lambda}{m},\ \ i = 1, 2,\ \ \lambda = 1, \dots, m-1$$

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$$\sum_{j=1}^{s} b_j(\rho_i^{\lambda}) = \rho_i^{\lambda} = \frac{c_i + \lambda}{m},\ \ i = 1, 2,\ \ \lambda = 1, \ldots, m-1$$

By means of two theorems we obtain the structure of B and C for multirate methods of order 2 and 3, with a generic number of microsteps, built with the coefficients of the continuous extensions.

Theorem

If the explicit Runge-Kutta scheme (A, b, c) *has order* $p = s = 2$*, choosing*

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ c_2 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

the resulting Multirate method is of order 2 *for the latent part.*
If (A, b, c) *is of order* $p = s = 3$*, choosing*

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ c_2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ c_3 - \frac{m}{6b_3c_2} & \frac{m}{6b_3c_2} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

with m *number of microsteps, the resulting Multirate method is of order* 3 *for the latent part.*

Multirate with NCEs

Theorem

Let (A, b, c) be an explicit Runge-Kutta scheme of order $p = s = 2, 3$ and let

$$B = \left[\begin{array}{c} B_0 \\ \hline \\ b_j(\rho_i^\lambda) \\ \\ \end{array} \right]$$

where B_0 is a lower triangular matrix univocally determined by (A, b, c) and by the number of microsteps m, $b_j(\rho)$'s are the NCEs coefficients evaluated in $\rho_i^\lambda = \frac{c_i+\lambda}{m}$, $i = 1, \ldots, s$, $\lambda = 1, \ldots, m-1$.
Then the resulting Multirate method is of order $2, 3$ for the active part.

Multirate with NCEs - Examples

0	0	0			0	0	
$\frac{1}{2}$	$\frac{1}{2}$	0			$\frac{1}{2}$	0	
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{3}{8}$	$\frac{1}{8}$	
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	
	0	0			0	0	0
	1	0			1	0	1
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	

0	0	0					0	0	
$\frac{1}{3}$	$\frac{1}{3}$	0					$\frac{1}{3}$	0	
$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0			$\frac{5}{18}$	$\frac{1}{18}$	
$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	0			$\frac{4}{9}$	$\frac{2}{9}$	
$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{4}{9}$	$\frac{2}{9}$	
1	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{1}{2}$	$\frac{1}{2}$	
	0	0					0	0	0
	1	0					1	0	1
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	

Multirate with NCEs - Examples

$$
\begin{array}{c|cccccc|ccc|c}
0 & 0 & 0 & 0 & & & & 0 & 0 & 0 & \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & & & & \frac{1}{4} & 0 & 0 & \\
\frac{1}{2} & -\frac{1}{2} & 1 & 0 & & & & 0 & \frac{1}{2} & 0 & \\
\frac{1}{2} & \frac{1}{12} & \frac{1}{3} & \frac{1}{12} & 0 & 0 & 0 & \frac{3k+5}{24} & \frac{4-3k}{12} & \frac{3k-1}{24} & \\
\frac{3}{4} & \frac{1}{12} & \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & 0 & 0 & \frac{9k+21}{96} & \frac{8-3k}{16} & \frac{9k+3}{96} & \\
1 & \frac{1}{12} & \frac{1}{3} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \\
\hline
 & 0 & 0 & 0 & & & & 0 & 0 & 0 & 0 \\
 & \frac{1}{2} & 0 & 0 & & & & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
 & -3 & 4 & 0 & & & & -1 & 2 & 0 & 1 \\
\hline
 & \frac{1}{12} & \frac{1}{3} & \frac{1}{12} & \frac{1}{12} & \frac{1}{3} & \frac{1}{12} & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} &
\end{array}
$$

Continuous Extensions of MRKCE

MRKCE are also endowed with a Continuous Extension making them suitable to solve DDEs.

- Latent part: for $t \in [t_n, t_n + H]$

$$y_L(t_n + \theta H) = y_{L,n} + H \sum_{i=1}^{s} b_i(\theta) k_{L,i} \quad \theta \in [0,1]$$

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$$y_L(t_n + \theta H) = y_{L,n} + H \sum_{i=1}^{s} b_i(\theta) k_{L,i} \quad \theta \in [0,1]$$

- Active part: for $t \in [t_n + \lambda h, t_n + (\lambda + 1)h]$

$$y_A(t_n + \big(\lambda + \theta)h\big) = y^{\lambda}_{A,n} + h \sum_{i=1}^{s} b_i(\theta) k^{\lambda}_{A,i}, \quad \theta \in [0,1]$$

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$$y_L(t_n + \theta H) = y_{L,n} + H \sum_{i=1}^{s} b_i(\theta) k_{L,i} \quad \theta \in [0,1]$$

- Active part: for $t \in [t_n, t_n + H]$

$$y_A(t_n + \theta H) = y_{A,n} + H \sum_{i=1}^{s} \sum_{\lambda=1}^{m} b^{\lambda}_{A,i}(\theta) k^{\lambda}_{A_i}, \quad \theta \in [0,1]$$

$$b^{A}_{(\lambda,i)}(\theta) = \begin{cases} 0 & \text{if } \theta \in [0, \frac{\lambda}{m}] \\ \frac{b_i(m\theta - \lambda)}{m} & \text{if } \theta \in [\frac{\lambda}{m}, \frac{\lambda+1}{m}] \\ \frac{b_i}{m} & \text{if } \theta \in [\frac{\lambda+1}{m}, 1] \end{cases} \qquad \text{for } \lambda = 0, \ldots, m-1$$

Test problem

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & \epsilon \\ \epsilon & -a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where $a > 1$, ϵ is the coupling coefficient. The general solution is

$$y(t) = k_1 \bar{v}_1 \exp(\lambda_1 t) + k_2 \bar{v}_2 \exp(\lambda_2 t).$$

Real eigenvalues and for ϵ small enough

$$\bar{v}_1 = \begin{bmatrix} 1 \\ O(\epsilon) \end{bmatrix}, \qquad \bar{v}_2 = \begin{bmatrix} O(\epsilon) \\ 1 \end{bmatrix}$$

$$\lambda_1 = -1 + O(\epsilon^2), \qquad \lambda_2 = -a + O(\epsilon^2)$$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & \epsilon \\ \epsilon & -a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Consider $\epsilon \approx 0$ and moderate velues of a.

Linear stability

$$y_1 = R(H, a, \epsilon) y_0, \quad y_0 = \begin{bmatrix} y_{A,0} \\ y_{L,0} \end{bmatrix}$$

Multirate schemes are absolutely stable $\iff |\rho(R)| < 1$.

$$\begin{cases} \mathbf{k}_A = \epsilon y_{L,0} - a y_{A,0} + H\epsilon B \mathbf{k}_L - H a \tilde{A} \mathbf{k}_A \\ y_{A,1} = y_{A,0} + H \tilde{b}^t \mathbf{k}_A \\ \mathbf{k}_L = -y_{L,0} + \epsilon y_{A,0} - H A \mathbf{k}_L + H \epsilon C \mathbf{k}_A \\ y_{L,1} = y_{L,0} + H b^t \mathbf{k}_L \end{cases}$$

$$\mathcal{B} = \begin{bmatrix} \tilde{b}^t & \mathbf{0} \\ \mathbf{0} & b^t \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} -a\tilde{A} & \epsilon B \\ \epsilon C & -aA \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} -a\mathbf{1}_{ms} & \epsilon \mathbf{1}_{ms} \\ \epsilon \mathbf{1}_s & -\mathbf{1}_s \end{bmatrix}$$

Compact formulation of a Multirate scheme:

$$\begin{cases} y_1 = y_0 + H\mathcal{B}\mathbf{k} \\ \mathbf{k} = \mathcal{A} y_0 + H\mathcal{M}\mathbf{k} \end{cases}$$

$$\Rightarrow y_1 = (I_1 + H\mathcal{B}(I_2 - H\mathcal{M})^{-1}\mathcal{A}) y_0 = R(H, a, \epsilon) y_0.$$

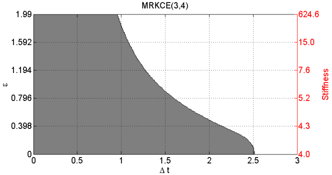
Theorem

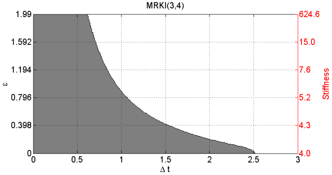
Let $m = \lceil a \rceil$ *be the smallest integer such that* $m > a$ *and let* $\epsilon = 0$*. The time integrator of the active component is stable for any choice of the time-step* H *for which the time integrator of the latent component is stable.*

Theorem

Let $m = \lceil a \rceil$ be the smallest integer such that $m > a$ and let $\epsilon = 0$. The time integrator of the active component is stable for any choice of the time-step H for which the time integrator of the latent component is stable.

Stability regions $(|R(H, \cdot, \epsilon)| < 1)$

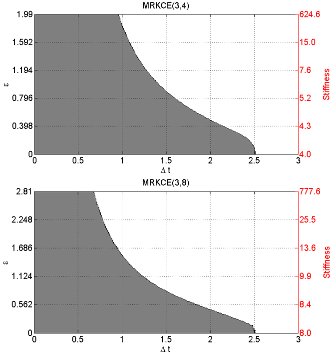




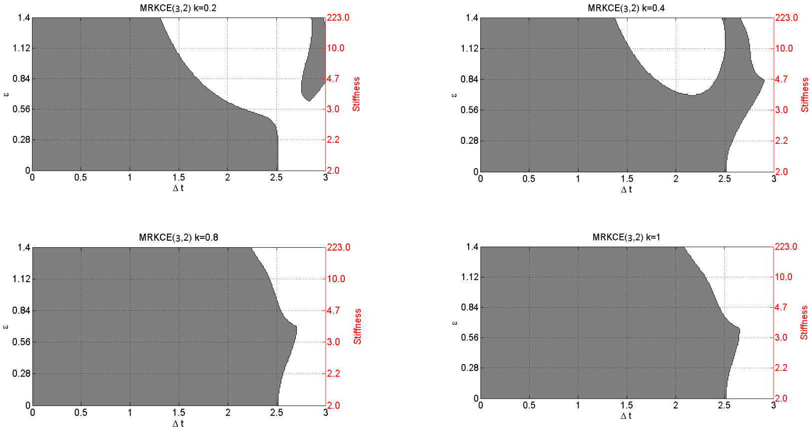
Theorem

Let $m = \lceil a \rceil$ be the smallest integer such that $m > a$ and let $\epsilon = 0$. The time integrator of the active component is stable for any choice of the time-step H for which the time integrator of the latent component is stable.

Stability regions $(|R(H, \cdot, \epsilon)| < 1)$



Linear stability



Convergence test

$$p = \frac{\log(e_{n+1}) - \log(e_n)}{\log(H_{n+1}) - \log(H_n)} = \frac{\log(\frac{e_{n+1}}{e_n})}{\log(\frac{H_{n+1}}{H_n})}$$

MRKCE Method	p_A	p_L
$(p = 2, a = m = 2)$	2.0288	1.9858
$(p = 2, a = m = 3)$	2.0126	1.9981
$(p = 2, a = m = 4)$	1.9986	1.9814
$(p = 2, a = m = 8)$	2.3617	3.5888
$(p = 3, a = m = 2, k = 0)$	3.0154	3.1847
$(p = 3, a = m = 3, k = 0)$	2.9962	3.0480
$(p = 3, a = m = 4, k = 0)$	3.0596	3.0474
$(p = 3, a = m = 8, k = 0)$	3.0640	3.0642
$(p = 3, a = m = 2, k = 0.2)$	3.0153	3.1855
$(p = 3, a = m = 2, k = 0.4)$	3.0153	3.1862
$(p = 3, a = m = 2, k = 0.6)$	3.0153	3.1869
$(p = 3, a = m = 2, k = 0.8)$	3.0153	3.1877
$(p = 3, a = m = 2, k = 1.0)$	3.0152	3.1884

Table: EOC for some multirate schemes based on NCEs

Work-precision tests

Nonlinear test problem (Robertson, 1966):

$$\begin{cases} y_1' = 0.4y_2 - 20y_1y_3 - 3y_1^2 \\ y_2' = -.04y_2 + 2y_1y_3 \\ y_3' = 0.15y_1^2 \end{cases}$$

with initial conditions $y(0) = [0, 1, 0]$, for $t \in [0, 100]$.

$$\lambda_1 \simeq 2.5, \quad \lambda_2 \simeq 10^{-1}, \quad \lambda_3 \simeq 0$$

$$v_1 \simeq [1, 0, 0], \quad \text{span}(v_2, v_3) \simeq \text{span}(e_2, e_3)$$

Heun

type	m	$H_{\max}$
	1	0.3
MRKI	2	0.59*
MRKCE	2	0.60
MRKI	4	1.15*
MRKCE	4	1.18

RK3

type	m	$H_{\max}$
	1	0.38
MRKI	2	0.77*
MRKCE	2	$0.77 \div 1.0^\dagger$
MRKI	4	1.56*
MRKCE	4	$1.56 \div 1.68^\dagger$

Bogacki-Shampine

type	m	$H_{\max}$
	1	0.38
MRKI	2	0.75
MRKCE	2	$1.00 \div 1.05^\dagger$
MRKI	3	1.14
MRKCE	3	$1.14 \div 1.58^\dagger$
MRKI	4	1.61
MRKCE	4	$1.61 \div 1.71^\dagger$

Table: Boundary of the stability regions for different multirate strategies.

Work-precision tests

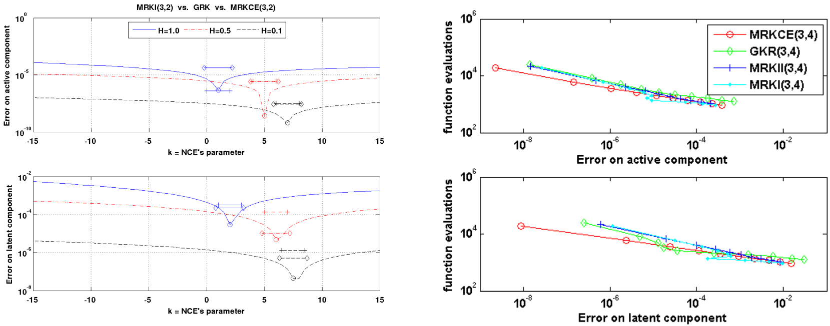


Figure: Scheme comparison on the Robertson problem.

CUSP problem (Hairer-Nørsett-Wanner, 1993)

$$\begin{cases} y_i'(t) = -A\big((y_i(t)^3 + a_i(t)y_i(t) + b_i(t-\tau)\big) + D\big(y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)\big) \\ a_i'(t) = b_i(t) + 0.07v_i(t) + D\big(a_{i-1}(t) - 2a_i(t) + a_{i+1}(t)\big) \\ b_i'(t) = \big(1 - a_i(t)^2\big)b_i(t) - a_i(t) - 0.4y_i(t-\tau) + 0.035v_i(t) + D\big(b_{i-1}(t) - 2b_i(t) + b_{i+1}(t)\big) \end{cases}$$

CUSP problem (Hairer-Nørsett-Wanner, 1993)

$$\begin{cases} y_i'(t) = -A\big((y_i(t)^3 + a_i(t)y_i(t) + b_i(t-\tau)\big) + D\big(y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)\big) \\ a_i'(t) = b_i(t) + 0.07v_i(t) + D\big(a_{i-1}(t) - 2a_i(t) + a_{i+1}(t)\big) \\ b_i'(t) = \big(1 - a_i(t)^2\big)b_i(t) - a_i(t) - 0.4y_i(t-\tau) + 0.035v_i(t) + D\big(b_{i-1}(t) - 2b_i(t) + b_{i+1}(t)\big) \end{cases}$$

H	$m = 1, \tau = 1/7$		$m = 2, \tau = 1/7$		$m = 2, \tau = 1/3$		$m = 4, \tau = 1/7$	
1.00E-2	1.94E-4		7.59E-4		4.35E-4		1.70E-3	
6.81E-3	8.91E-5	2.02	3.47E-4	2.04	1.99E-4	2.04	7.83E-4	2.03
4.64E-3	4.12E-5	2.01	1.60E-4	2.02	9.13E-5	2.02	3.60E-4	2.02
3.16E-3	1.91E-5	2.01	7.40E-5	2.00	4.22E-5	2.01	1.66E-4	2.02
2.15E-3	8.82E-6	2.01	3.41E-5	2.01	1.95E-5	2.01	7.63E-5	2.02
1.47E-3	4.09E-6	2.00	1.58E-5	2.00	9.03E-6	2.00	3.54E-5	2.00
1.00E-3	1.90E-6	2.00	7.35E-6	2.00	4.19E-6	2.00	1.64E-5	2.00

Table: Convergence test for the DDE: rates of convergence for the latent part of the MRKCE(2,m) schemes based on the Heun method.

CUSP problem (Hairer-Nørsett-Wanner, 1993)

$$\begin{cases} y_i'(t) = -A\big((y_i(t)^3 + a_i(t)y_i(t) + b_i(t-\tau)\big) + D\big(y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)\big) \\ a_i'(t) = b_i(t) + 0.07v_i(t) + D\big(a_{i-1}(t) - 2a_i(t) + a_{i+1}(t)\big) \\ b_i'(t) = \big(1 - a_i(t)^2\big)b_i(t) - a_i(t) - 0.4y_i(t-\tau) + 0.035v_i(t) + D\big(b_{i-1}(t) - 2b_i(t) + b_{i+1}(t)\big) \end{cases}$$

H	$m = 1, \tau = 1/7$		$m = 2, \tau = 1/7$		$m = 2, \tau = 1/3$		$m = 4, \tau = 1/7$	
1.00E-2	5.54E-4		7.80E-4		8.29E-4		9.59E-4	
6.81E-3	2.60E-4	1.97	3.64E-4	1.98	3.84E-4	2.00	4.16E-4	2.18
4.64E-3	1.17E-4	2.09	1.68E-4	2.01	1.78E-4	2.01	1.92E-4	2.01
3.16E-3	5.40E-5	2.00	7.58E-5	2.08	8.17E-5	2.02	8.87E-5	2.01
2.15E-3	2.50E-5	2.00	3.56E-5	1.97	3.80E-5	2.00	4.08E-5	2.02
1.47E-3	1.16E-5	2.01	1.63E-5	2.04	1.76E-5	2.01	1.91E-5	1.97
1.00E-3	5.34E-6	2.01	7.52E-6	2.01	8.14E-6	2.00	8.90E-6	2.00

Table: Convergence test for the DDE: rates of convergence for the active part of the MRKCE(2,m) schemes based on the Heun method.

CUSP problem (Hairer-Nørsett-Wanner, 1993)

$$\begin{cases} y_i'(t) = -A\big((y_i(t)^3 + a_i(t)y_i(t) + b_i(t-\tau)\big) + D\big(y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)\big) \\ a_i'(t) = b_i(t) + 0.07v_i(t) + D\big(a_{i-1}(t) - 2a_i(t) + a_{i+1}(t)\big) \\ b_i'(t) = \big(1 - a_i(t)^2\big)b_i(t) - a_i(t) - 0.4y_i(t-\tau) + 0.035v_i(t) + D\big(b_{i-1}(t) - 2b_i(t) + b_{i+1}(t)\big) \end{cases}$$

H	$m=1, \tau=1/7$		$m=2, \tau=1/7$		$m=2, \tau=1/3$		$m=3, \tau=1/7$	
1.00E-2	2.37E-6		7.61E-6		6.03E-5		5.36E-5	
6.81E-3	7.13E-7	3.12	2.33E-6	3.09	1.72E-5	3.27	1.53E-5	3.26
4.64E-3	2.21E-7	3.06	7.15E-7	3.07	5.05E-6	3.19	4.44E-6	3.23
3.16E-3	6.84E-8	3.05	2.21E-7	3.06	1.51E-6	3.15	1.37E-6	3.06
2.15E-3	2.11E-8	3.06	6.83E-8	3.06	4.48E-7	3.16	4.24E-7	3.06
1.47E-3	6.46E-9	3.09	2.12E-8	3.04	1.41E-7	3.01	1.31E-7	3.05
1.00E-3	1.88E-9	3.22	6.51E-9	3.08	4.35E-8	3.06	4.07E-8	3.05

Table: Convergence test for the DDE: rates of convergence for the latent part of the MRKCE(3,m) schemes based on the Bogacki-Shampine method.

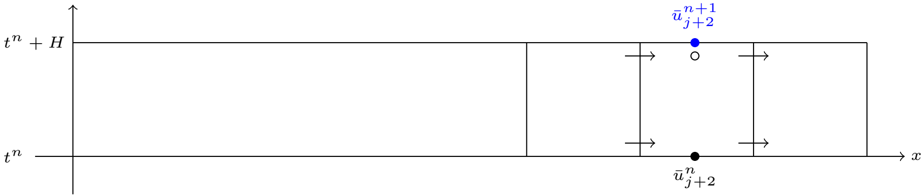
CUSP problem (Hairer-Nørsett-Wanner, 1993)

$$\begin{cases} y_i'(t) = -A\big((y_i(t)^3 + a_i(t)y_i(t) + b_i(t-\tau)\big) + D\big(y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)\big) \\ a_i'(t) = b_i(t) + 0.07v_i(t) + D\big(a_{i-1}(t) - 2a_i(t) + a_{i+1}(t)\big) \\ b_i'(t) = \big(1 - a_i(t)^2\big)b_i(t) - a_i(t) - 0.4y_i(t-\tau) + 0.035v_i(t) + D\big(b_{i-1}(t) - 2b_i(t) + b_{i+1}(t)\big) \end{cases}$$

H	$m=1, \tau=1/7$		$m=2, \tau=1/7$		$m=2, \tau=1/3$		$m=3, \tau=1/7$	
1.00E-2	1.86E-5		8.97E-6		6.07E-5		2.61E-5	
6.81E-3	6.00E-6	2.95	2.54E-6	3.29	1.84E-5	3.12	7.09E-6	3.39
4.64E-3	1.82E-6	3.10	7.59E-7	3.15	5.56E-6	3.11	1.63E-6	3.82
3.16E-3	5.63E-7	3.06	2.24E-7	3.18	1.69E-6	3.10	4.72E-7	3.24
2.15E-3	1.75E-7	3.04	6.70E-8	3.14	5.23E-7	3.06	1.25E-7	3.45
1.47E-3	5.21E-8	3.16	1.78E-8	3.45	1.60E-7	3.08	3.34E-8	3.44
1.00E-3	1.32E-8	3.58	5.04E-9	3.28	4.78E-8	3.15	1.05E-8	3.00

Table: Convergence test for the DDE: rates of convergence for the active part of the MRKCE(3,m) schemes based on the Bogacki-Shampine method.

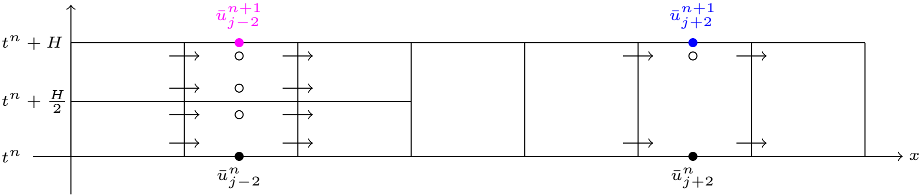
Multirate for $u_t + F(u)_x = 0$



$$\bar{u}_{j+2}^{n+1} = \bar{u}_{j+2}^{n} - \frac{H}{\Delta x} \sum_{i=1}^{2} b_i \left(\mathcal{F}^{(i)}_{j+\frac{5}{2}} - \mathcal{F}^{(i)}_{j+\frac{3}{2}} \right)$$

$$\begin{array}{c|cc} 0 & 0 & \\ c_2 & a_{21} & 0 \\ \hline & b_1 & b_2 \end{array}$$

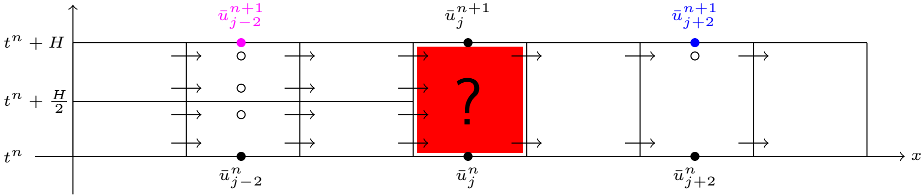
Multirate for $u_t + F(u)_x = 0$



$$\bar{u}^{n+1}_{j-2} = \bar{u}^{n}_{j-2} - \frac{H}{\Delta x} \sum_{i=1}^{4} \tilde{b}_i \left(\mathcal{F}^{(i)}_{j-\frac{3}{2}} - \mathcal{F}^{(i)}_{j-\frac{5}{2}} \right) \qquad \bar{u}^{n+1}_{j+2} = \bar{u}^{n}_{j+2} - \frac{H}{\Delta x} \sum_{i=1}^{2} b_i \left(\mathcal{F}^{(i)}_{j+\frac{5}{2}} - \mathcal{F}^{(i)}_{j+\frac{3}{2}} \right)$$

$$\begin{array}{c|cccc} 0 & 0 & & & \\ c_2/2 & a_{21}/2 & 0 & & \\ 1/2 & b_1/2 & b_2/2 & 0 & \\ (c_2+1)/2 & b_1/2 & b_2/2 & a_{21}/2 & \\ \hline & b_1/2 & b_2/2 & b_1/2 & b_2/2 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & \\ c_2 & a_{21} & 0 \\ \hline & b_1 & b_2 \end{array}$$

Multirate for $u_t + F(u)_x = 0$



$$\bar{u}_{j-2}^{n+1} = \bar{u}_{j-2}^n - \frac{H}{\Delta x} \sum_{i=1}^{4} \tilde{b}_i \left(\mathcal{F}^{(i)}_{j-\frac{3}{2}} - \mathcal{F}^{(i)}_{j-\frac{5}{2}} \right) \quad \bar{u}_{j+2}^{n+1} = \bar{u}_{j+2}^n - \frac{H}{\Delta x} \sum_{i=1}^{2} b_i \left(\mathcal{F}^{(i)}_{j+\frac{5}{2}} - \mathcal{F}^{(i)}_{j+\frac{3}{2}} \right)$$

$$\begin{array}{c|cccc} 0 & 0 & & & \\ c_2/2 & a_{21}/2 & 0 & & \\ 1/2 & b_1/2 & b_2/2 & 0 & \\ (c_2+1)/2 & b_1/2 & b_2/2 & a_{21}/2 & \\ \hline & b_1/2 & b_2/2 & b_1/2 & b_2/2 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & \\ c_2 & a_{21} & 0 \\ \hline & b_1 & b_2 \end{array}$$

Cell-boundary partitioning

Start from $u_t + F(u)_x = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}_j = -\frac{1}{\Delta x}\left[f_{j+1/2} - f_{j-1/2} + g_{j+1/2} - g_{j-1/2}\right]$$

where

$$f_{j+1/2} = \begin{cases} F\left(u(x_{j+1/2,t})\right), & \text{if there are } 4 \text{ fluxes in } x_{j+1/2}, \\ 0, & \text{otherwise} \end{cases}$$

$$g_{j+1/2} = \begin{cases} F\left(u(x_{j+1/2,t})\right), & \text{if there are } 2 \text{ fluxes in } x_{j+1/2}, \\ 0, & \text{otherwise} \end{cases}$$

Cell-boundary partitioning

Start from $u_t + F(u)_x = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}_j = -\frac{1}{\Delta x}\Big[f_{j+1/2} - f_{j-1/2} + g_{j+1/2} - g_{j-1/2}\Big]$$

where

$$f_{j+1/2} = \begin{cases} F\left(u(x_{j+1/2,t})\right), & \text{if there are } 4 \text{ fluxes in } x_{j+1/2}, \\ 0, & \text{otherwise} \end{cases}$$

$$g_{j+1/2} = \begin{cases} F\left(u(x_{j+1/2,t})\right), & \text{if there are } 2 \text{ fluxes in } x_{j+1/2}, \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{u}} = -\frac{1}{\Delta x}\Big[\Delta f + \Delta g\Big]$$

- use the Runge-Kutta (A, b, c) for Δg
- use the Runge-Kutta $(\tilde{A}, \tilde{b}, \tilde{c})$ for Δf

Coupling in the center cell

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}_j = -\frac{1}{\Delta x}\Big[\Delta f + \Delta g\Big]$$

$$\Rightarrow \bar{u}_j^{n+1} = \bar{u}_j^n - \frac{H}{\Delta x}\left[\sum_{i=1}^{ms}\tilde{b}_i K_i + \sum_{i=1}^{s} b_i J_i\right]$$

where

$$K_i = \Delta f\left(\bar{u}_j^n - \frac{H}{\Delta x}\sum_{j=1}^{i-1}\tilde{a}_{ij}K_j\right)$$

$$J_i = \Delta g\left(\bar{u}_j^n - \frac{H}{\Delta x}\sum_{j=1}^{i-1}a_{ij}J_j\right)$$

Coupling in the center cell

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}_j = -\frac{1}{\Delta x}\left[\Delta f + \Delta g\right]$$

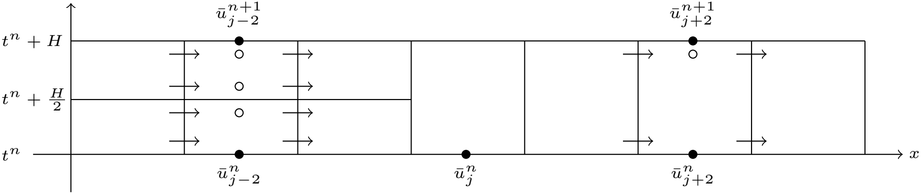
$$\Rightarrow \bar{u}_j^{n+1} = \bar{u}_j^n - \frac{H}{\Delta x}\left[\sum_{i=1}^{ms}\tilde{b}_i K_i + \sum_{i=1}^{s} b_i J_i\right]$$

where

$$K_i = \Delta f\left(\bar{u}_j^n - \frac{H}{\Delta x}\sum_{j=1}^{i-1}\tilde{a}_{ij}K_j - \frac{H}{\Delta x}\sum_{j=1}^{s}\beta_{ij}J_j\right)$$

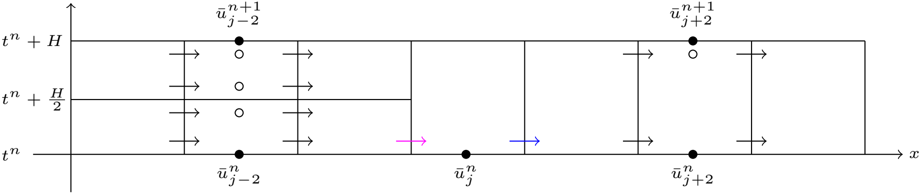
$$J_i = \Delta g\left(\bar{u}_j^n - \frac{H}{\Delta x}\sum_{j=1}^{i-1}a_{ij}J_j - \frac{H}{\Delta x}\sum_{j=1}^{ms}\gamma_{ij}K_j\right)$$

Second order method



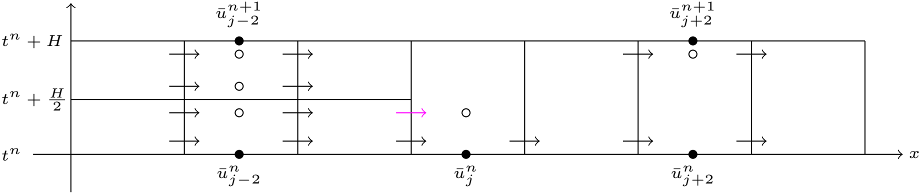
0	0	0					
$\frac{1}{2}$	$\frac{1}{2}$	0					
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0			
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0			
					0	0	0
					1	0	1
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	

Second order method



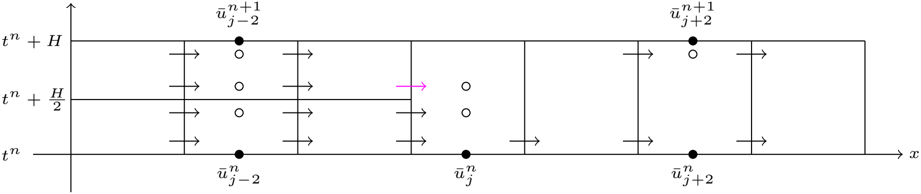
0	0	0			0	0	
$\frac{1}{2}$	$\frac{1}{2}$	0					
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0			
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0			
	0	0			0	0	0
					1	0	1
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	

Second order method



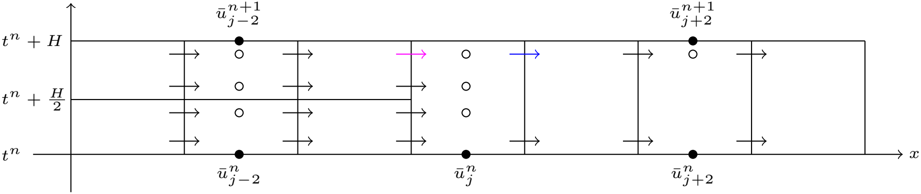
0	0	0			0	0	
$\frac{1}{2}$	$\frac{1}{2}$	0			$\frac{1}{2}$	0	
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0			
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0			
	0	0			0	0	0
					1	0	1
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	

Second order method



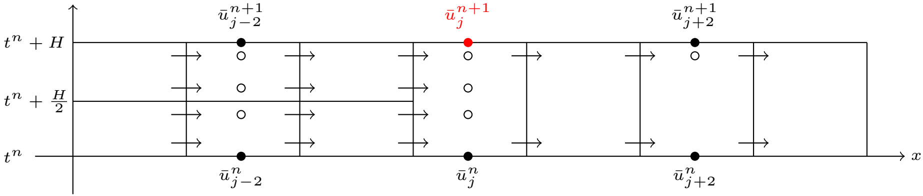
0	0	0			0	0	
$\frac{1}{2}$	$\frac{1}{2}$	0			$\frac{1}{2}$	0	
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	0	
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0			
	0	0			0	0	0
					1	0	1
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	

Second order method



0	0	0			0	0	
$\frac{1}{2}$	$\frac{1}{2}$	0			$\frac{1}{2}$	0	
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	0	
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	1	0	
	0	0			0	0	0
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	1	0	1
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	

Second order method



0	0	0			0	0	
$\frac{1}{2}$	$\frac{1}{2}$	0			$\frac{1}{2}$	0	
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	0	
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	1	0	
	0	0			0	0	0
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	1	0	1
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{H}{\Delta x}\left[\sum_{i=1}^{4} \tilde{b}_i K_i + \sum_{i=1}^{2} b_i J_i\right]$$

Second-order conditions

$$
\begin{array}{c|cccc|cc|c}
0 & 0 & 0 & & & 0 & 0 & \\
\frac{1}{2} & \frac{1}{2} & 0 & & & \frac{1}{2} & 0 & \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & \\
1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 1 & 0 & \\
\hline
 & 0 & 0 & & & 0 & 0 & 0 \\
 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & & 1 & 0 & 1 \\
\hline
 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} &
\end{array}
$$

G. Puppo, M. Semplice (2011), *Numerical entropy and adaptivity for finite volume schemes*, Comm. Comput. Phys., 10(5), pp.1132-1160

Theorem

The Multirate method is of second order if each method is of second order and the extended tableaux satisfy the coupling conditions

$$\sum_{i,j} \tilde{b}_i B^i_j = \frac{1}{2}, \quad \sum_{i,k} b_i C^i_k = \frac{1}{2}.$$

Perspectives

The construction of the previous method can be extend to a generic number of microsteps.

Theorem

Let (A, b, c) *a Runge-Kutta scheme of order* $p = 3$ *with* 3 *stages and let* m *be the number of microsteps. Then we cannot have approximations of order* 3 *for the active component if the first block of* B *is*

$$\begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$$

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$\Rightarrow$ we are trying to use the Multirate Runge-Kutta method based on the Continuous Extensions.

Thank you for your attention!