High-order finite volume scheme for cardiac electrophysiology

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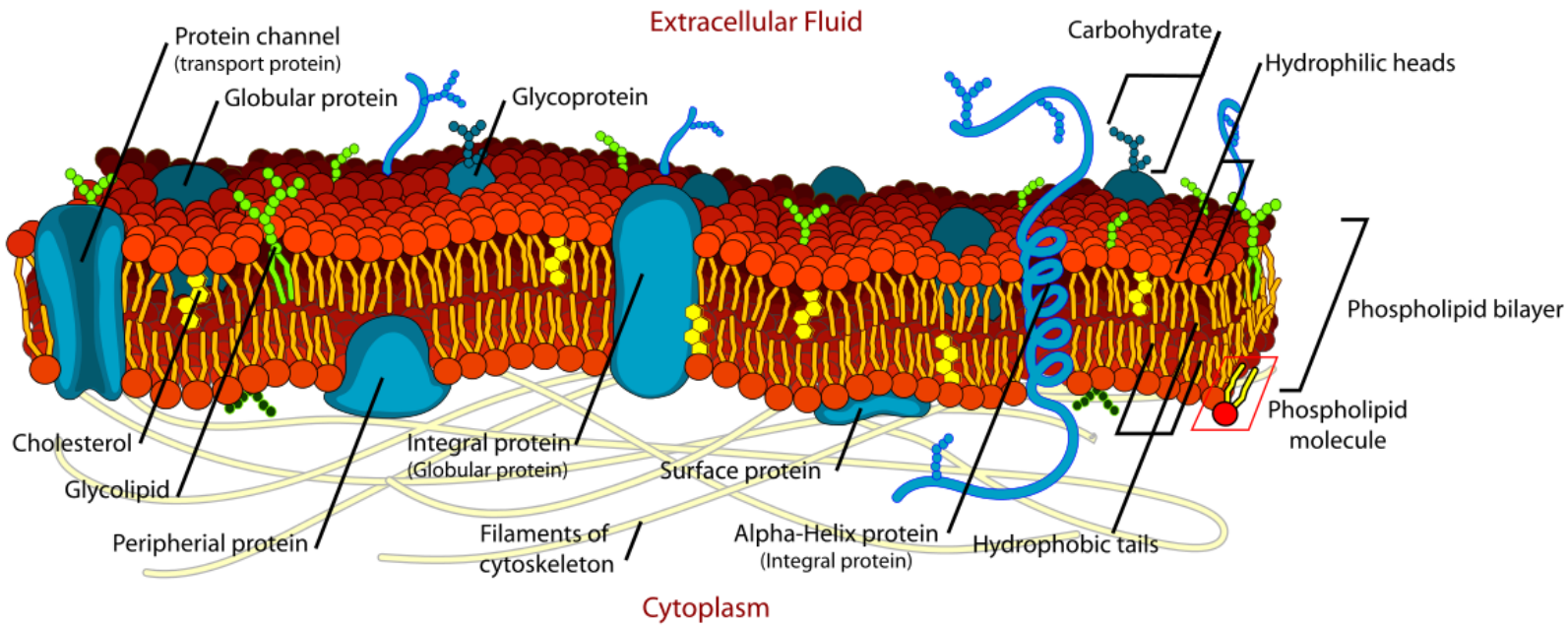
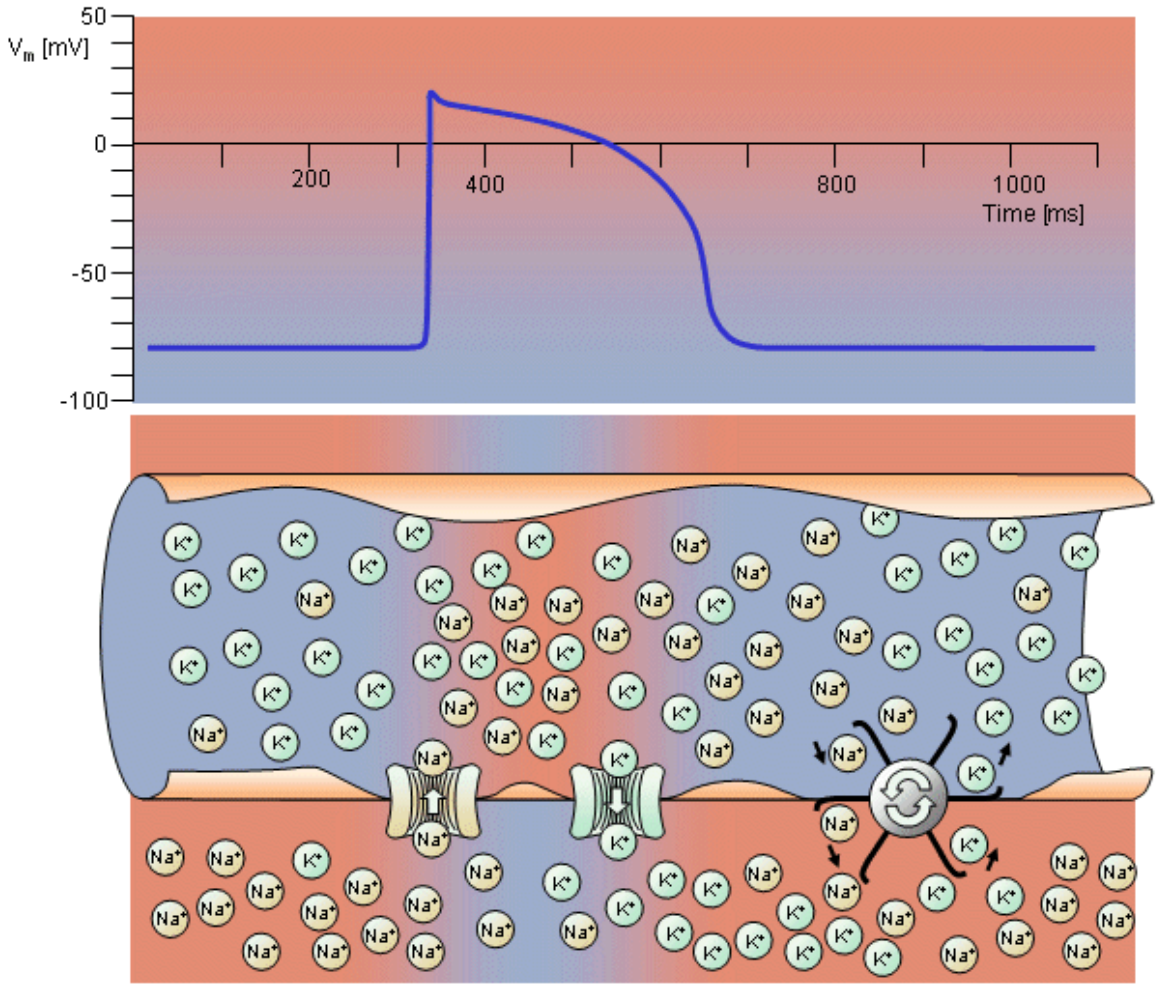
Cell level - membrane

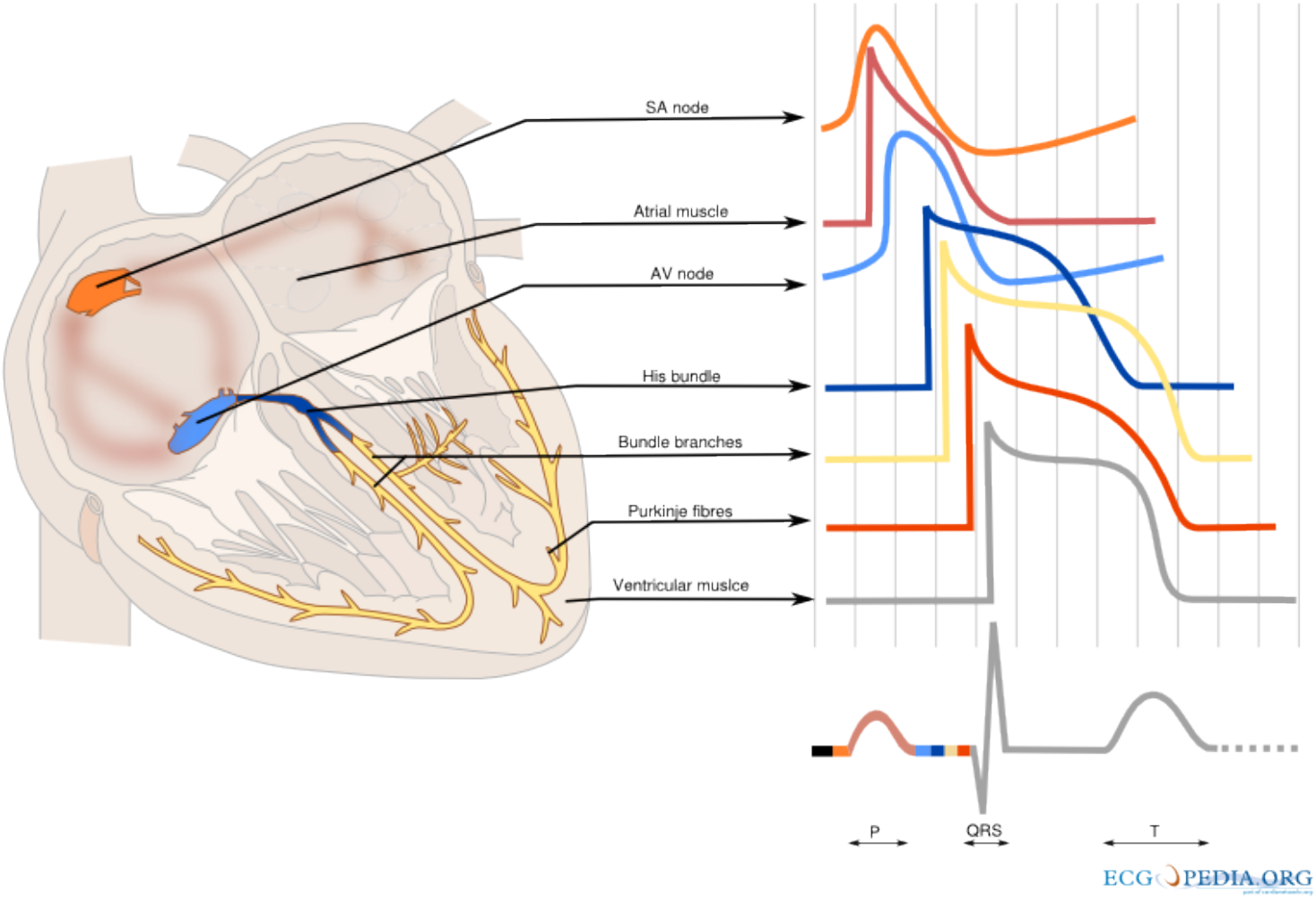


The cell membrane

Cell level - action potential



Macroscopic level



The monodomain model

$$\partial_t V + I_{ion}(V, w) = \text{div}\big(D\nabla V\big),$$
$$\partial_t w = G(V, w),$$

where :

- V $[mV]$ is the transmembrane voltage,
- $I_{ion} = I_{ion}(V, w)$ $[A.F^{-1}.cm^{-2}]$ is the normalized ioinc current par unit surface,
- $D = \dfrac{G}{A_m C_m}$ $[mS.\mu F^{-1}]$ is the normalized diffusion tensor,
- w contains all auxiliary variables.

Ionic models

The ionic current I_{ion} and G are given by a so-called *ionic model* which approximates all ionic processes in the cardiac cells :

- Phenomenological models (Fitzugh-Nagumo, Aliev-Panfilov, Mitchell-Schaeffer,...) are simple,
- Hodgkin-Huxley type models (Beeler-Reuter, Ten Tusscher et al, Luo-Rudy,..) are more complex. Markov chains variants (Iyer et al...) are nowadays widely used by biologists.

Aliev-Panfilov

The Aliev-Panfilov model is a simple 2-equation model :

$$I_{ion} = -\kappa V(V-a)(V-1) - wV,$$
$$g(V,w) = \left(\varepsilon + \mu_1 \frac{w}{\mu_2 + V}\right)\left(-w - \kappa V(V-b-1)\right).$$

Homogeneous monodomain + Aliev-Panfilov $\rightarrow$ bistable system which preserves $V \in (V_{min}, V_{max})$.

Its simplicity allows to obtain theoretical results (*e.g.* estimation of propagation speed).

Hodgkin-Huxley formalism

Mimics the behavior of proteins in the cell membrane :
$I_{ion}(V, w) = \sum_i I_i$ where the I_is are expressed as functions of *gating variables*.
Example : $I_{Na} = g_{Na} m^3 hj(V - E_{Na})$ where :

- $E_{Na} = \dfrac{RT}{F} \ln\left(\dfrac{[Na^+]_e}{[Na^+]_i}\right)$ is Nernst's potential,
- m, h, j are gating variables $\in [0, 1]$ given by :

$$\frac{d}{dt} m = \frac{m_\infty - m}{\tau_m},$$

$w \in \mathbb{R}^N$ also contains other variables (concentrations,...) and N ranges from 8 to 100+.

Main numerical difficulties

- Stiffness
  - in time due to physiological processes (*e.g.* fast Na^+ channels),
  - in time due to ODEs,
  - in space due to depolarization fronts,
- *propagation failure*
  - No propagation when the mesh is too coarse,
  - Wrong propagation speed,
- Anisotropy

Most codes use P_1 or equivalent methods with a mesh length $\simeq 100\mu m$ (which is too coarse !) and adapt A_m.

Semi-discrete scheme

Finite volumes scheme : unknowns are mean values of V and w in each cell K :

$$\frac{d}{dt} V_K + I_K = \frac{1}{|K|} \sum_{e_i \in \mathcal{E}_K} F_i \cdot n_i,$$

$$\frac{d}{dt} w_K = G_K,$$

The scheme is determined by the choices of I_K, G_K and $F_i \cdot n_i$.
Our choice : scheme based on the ideas of Clain, Machado, Nobrega and Pereira (CMAME '13).

Approximation on cell edges

The diffusion part is approximated using a polynomial reconstruction on cell interfaces :

$$V(x,y) = \sum_{i+j \leq m} \gamma_{i,j} (x - x_e)^i (y - y_e)^j,$$

Coeffs $\Gamma = (\gamma_{i,j})_{i,j}$ are obtained by computing the minimum of the function :

$$J(\Gamma) = \frac{1}{2} \sum_{C \in S_e} \omega_C^e (V(x_C, y_C) - V_C)^2,$$

where S_e is a neighborhood of e.

Approximation on cell edges

Therefore, Γ is the solution of the system :

$$(X^\top \Omega^e X)\Gamma = X^\top \Omega^e \mathbb{V},$$

where

- $X = \left[(x_C - x_e)^i (y_C - y_e)^j\right]_{\substack{C \in S_e \\ i+j \leq m}}$,
- $\Omega^e = \text{diag}(\omega_C^e)$,
- $\mathbb{V} = (V_C)_{C \in S_e}$.

In practice, $X^\top \Omega^e X$ is invertible as soon as $\#S_e$ is large enough.

Approximation on cell edges

The matrix $M_e := (X^\top \Omega^e X)^{-1} X^\top \Omega^e$ is computed once and for all. It maps V to the coeffs $\gamma_{i,j}$:

$$\Gamma = M_e \mathbb{V}.$$

Then $F_i \cdot n_i$ is obtained through a quadrature formula :

$$F_i \cdot n_i := \sum_l w_l (G(x_l, y_l) n_i) \cdot \nabla V(x_l, y_l),$$

using the reconstructed V.

Volumic approximation

The reaction terms are approximated using a polynomial reconstruction on cells :

$$\tilde{U}(x,y) = U_K + \sum_{1\leq i+j\leq p} \lambda_{i,j}\left[(x-x_K)^i(y-y_K)^j - \frac{1}{|K|}\int_K (x-x_K)^i(y-y_K)^j d\mathbf{x}\right],$$

where $U = (V, w)$. This guarantees that :

$$\frac{1}{|K|}\int_K \tilde{U}(x,y)d\mathbf{x} = U_K.$$

Volumic approximation

Once again, the coefficients are chosen to be the solution of :

$$J(\Lambda) = \frac{1}{2} \sum_{C \in S_K} \omega_C^K \Big(\frac{1}{|C|} \int_C U(x_C, y_C) d\mathbf{x} - U_C \Big)^2,$$

which leads to :

$$(X^\top \Omega^K X)\Lambda = X^\top \Omega^K \mathbb{U},$$

and I_K and G_K are obtained with a quadrature formula :

$$I_K = \sum_l w_l I_{ion}\big(V(x_l, y_l), w(x_l, y_l)\big),$$

$$G_K = \sum_l w_l G\big(V(x_l, y_l), w(x_l, y_l)\big).$$

A few remarks

- Choice of S_e and S_K,
- Weights : $(\omega_C^e)_C$: $\omega_C^e = (X_C - X_e)^{-s}$, with $s > 0$,
- No preconditionning,
- Preservation of admissibility : *a posteriori* limitation on each quadrature point,
- 1 unknown/quadrature point vs cell reconstruction :
  - 1 unkwown/quad. pt. : no need for interpolation by cell but many more unknowns,
  - cell reconst. : quadrature and evaluation of the polynomial at the quadrature points (expensive) but less memory requirement.

Time integration

- Explicit schemes : ionic model $\rightarrow$ small Δt, "reasonable" mesh, preservation of admissibility,...
- If $\Delta t = \mathcal{O}(\Delta x^2)$ then time order = (space order)/2
- Preservation of admissibility : SSP-RK.

Order of convergence

Ad hoc choice of I_{ion} in order to have an analytical solution.

h	p=2	ord.	p=3	ord.	p=4	ord.	p=5	ord.
1.2E-2	2.4E-5	n/a	6.8E-6	n/a	2.9E-6	n/a	9.5E-7	n/a
6.1E-3	5.6E-6	2.09	5.1E-7	3.75	1.3E-7	4.51	2.1E-8	5.52
3.1E-3	1.3E-6	2.09	3.2E-8	3.97	5.2E-9	4.60	3.7E-10	5.80
1.5E-3	3.2E-7	2.03	2.0E-9	4.04	2.4E-10	4.46	6.3E-11	2.57

TABLE : L^2 errors for the analytical test case

Planar waves - AP I

h	p=1	p=3	p=5
9.87E-002	3.25E-001	3.97E-001	4.40E-001
4.80E-002	3.79E-001	4.22E-001	4.43E-001
2.24E-002	4.09E-001	4.37E-001	
1.05E-002	4.21E-001	4.43E-001	
5.34E-003	4.22E-001		

TABLE : Aliev-Panfilov : D =1.E-1, c =4.43E-1 $cm.ms^{-1}$.

Planar wave - AP II

h	p=1	p=3	p=5
9.87E-002	PF	1.63E-001	1.71E-001
4.80E-002	PF	1.30E-001	1.40E-001
2.24E-002	1.05E-001	1.40E-001	1.40E-001
1.05E-002	1.37E-001		
5.34E-003	1.40E-001		

TABLE : Aliev-Panfilov : D =1.E-2, c =1.40E-1 $cm.ms^{-1}$.

Planar wave - AP III

h	p=1	p=3	p=5
9.87E-002	PF	PF	PF
4.80E-002	PF	PF	4.25E-002
2.24E-002	PF	4.27E-002	4.68E-002
1.05E-002	3.01E-002	4.18E-002	4.43E-002
5.34E-003	3.49E-002		

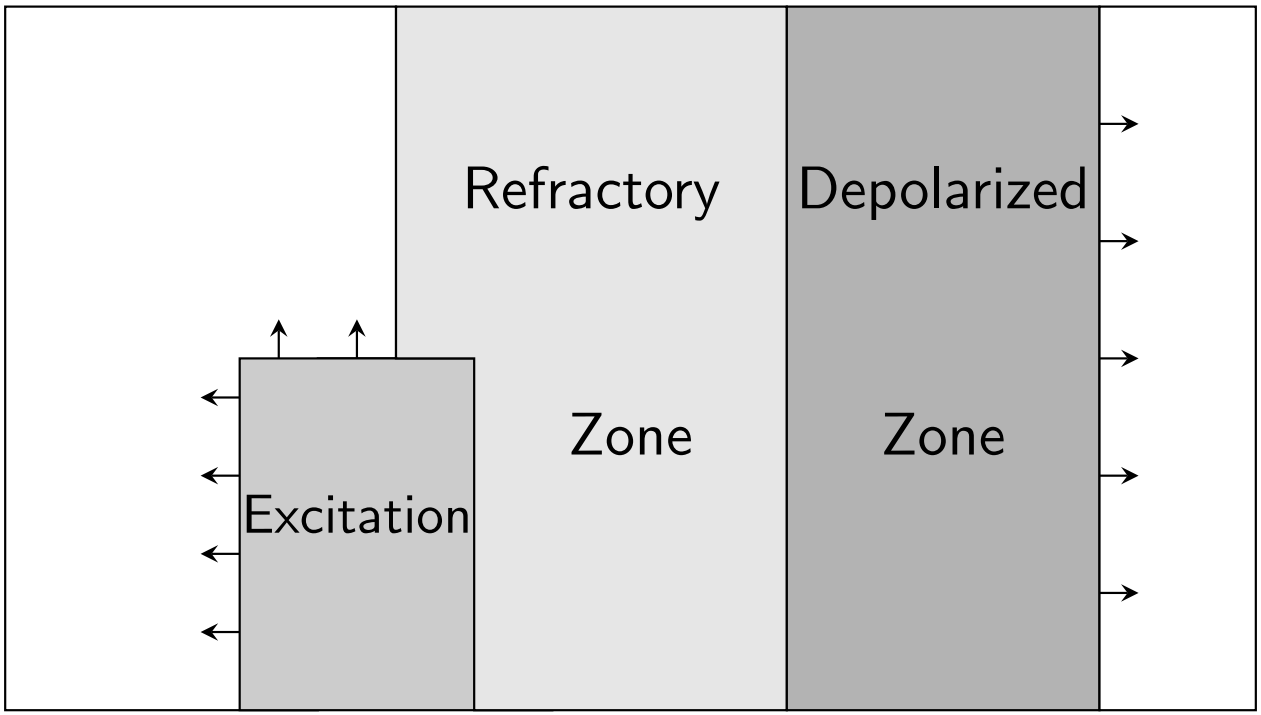
TABLE : Aliev-Panfilov : D =1.E-3, c =4.43E-2 $cm.ms^{-1}$.

Planar wave - summary

- 2nd vs 4th order : 6 to 8 times more expensive,
- dividing h by 2 : 3 to 20 times more expensive,
- AP : the higher order, the better,
- Realistic models : efficiency of high order is reduced (stiffness $\rightarrow$ limitation).
  $\hookrightarrow$ 4th order seems to be optimal.

Spiral waves

Setup :



Spiral waves

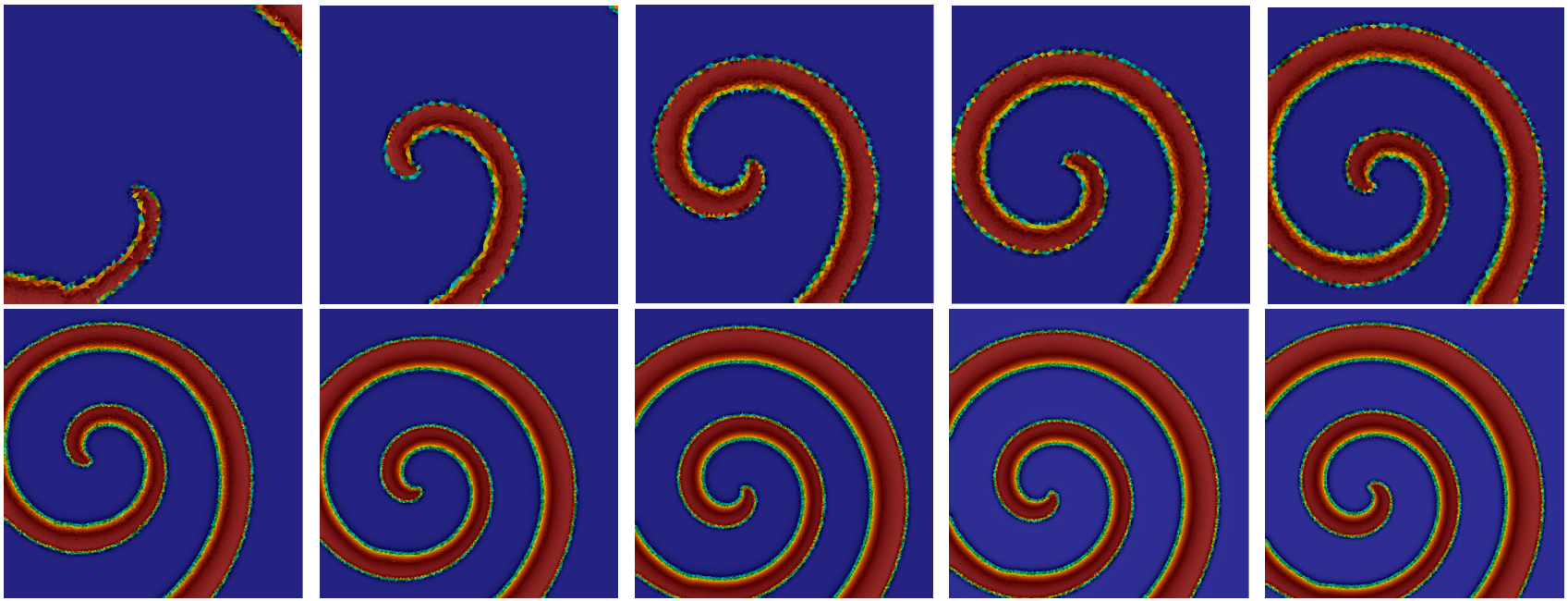


FIGURE : Spiral wave (AP model) obtained on a moderately coarse mesh with the schemes from order 2 (left) to 6 (right), $t = 100$ms.

Spiral waves

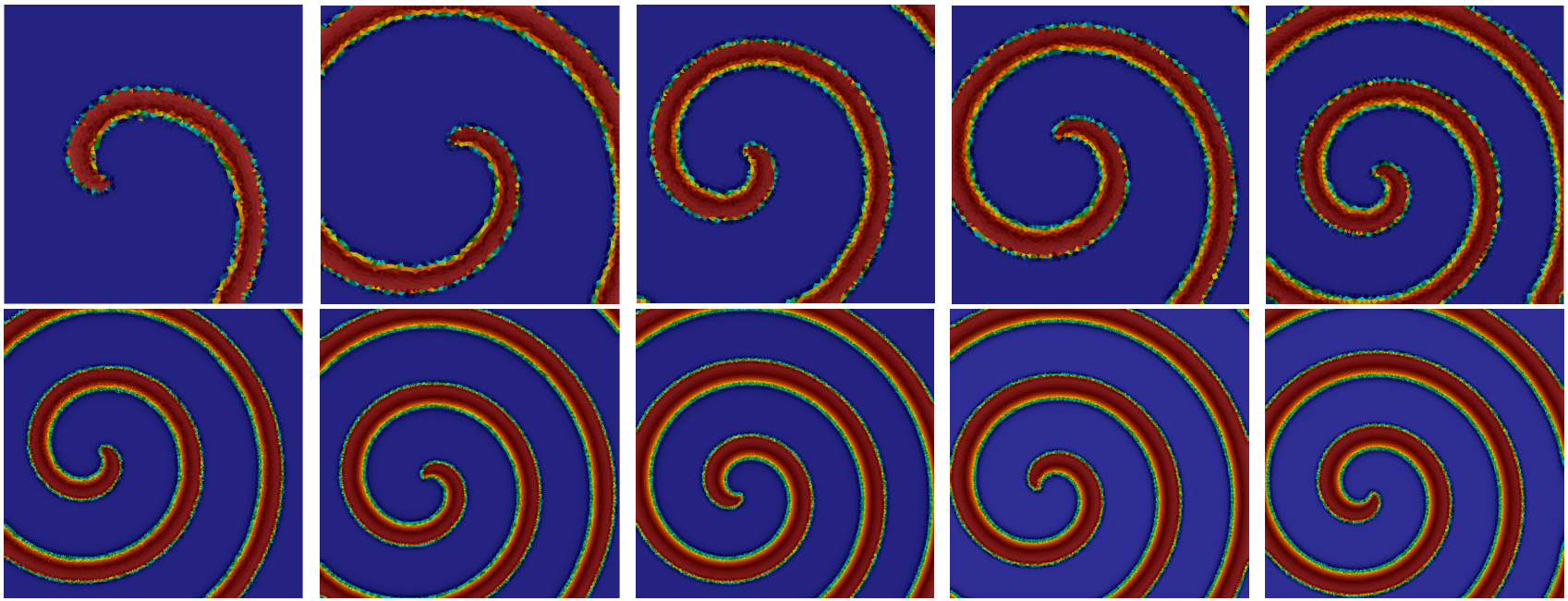


FIGURE : Spiral wave (AP model) obtained on a moderately coarse mesh with the schemes from order 2 (left) to 6 (right), $t = 150$ms.

Spiral waves

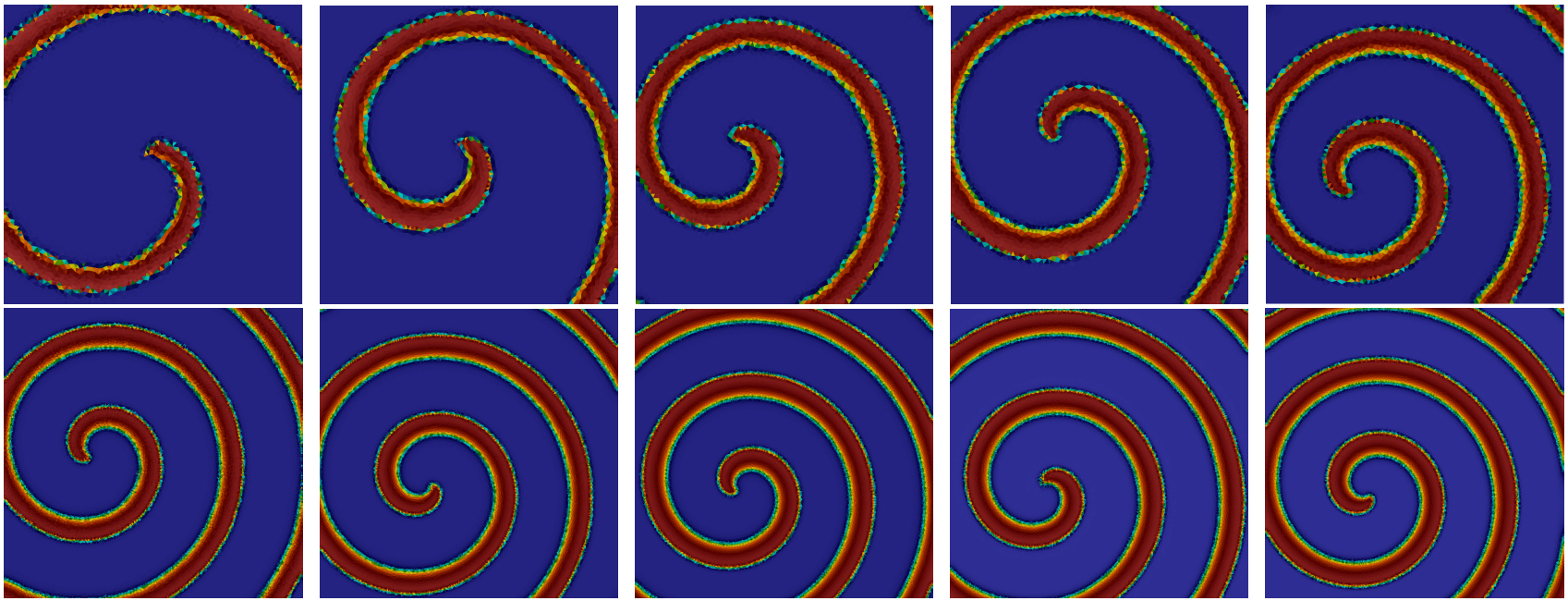


FIGURE : Spiral wave (AP model) obtained on a moderately coarse mesh with the schemes from order 2 (left) to 6 (right), $t = 200$ms.

Conclusion and perspectives

- High-order is very interesting in this context,
- For realistic models, 4th order seems optimal,
- Flexible scheme (order, dimension,...),
- Drawback : 2 reconstructions,
- Parallelization ?

Thanks for your attention !