VMS finite element for MHD and RMHD. Applications to magnetic confinement fusion

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Tokamaks

- Tokamak stands for: Toroidal chamber with magnetic coils (Toroidal'naya Kamera s Magnitnymi Katushhami).
- JET (UK), Asdex-Upgrade (Germany), Tore-Supra (France) and DIIID (USA)

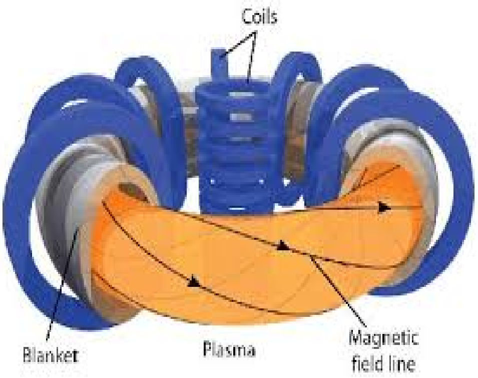
Amplification factor

$$Q = \frac{P_{fus}}{P_{inj}}$$

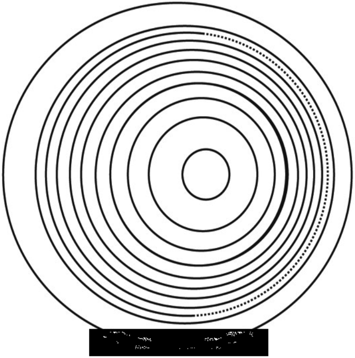
JET : $Q \approx 1.$

ITER : $Q = 10.$

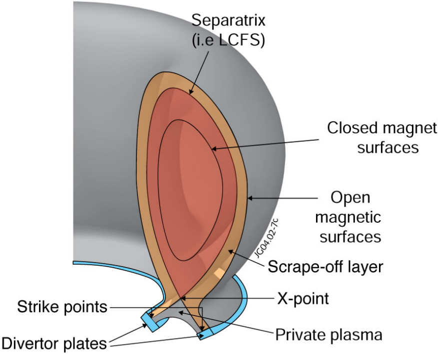
DEMO? : $Q = 50.$



Tokamaks configurations



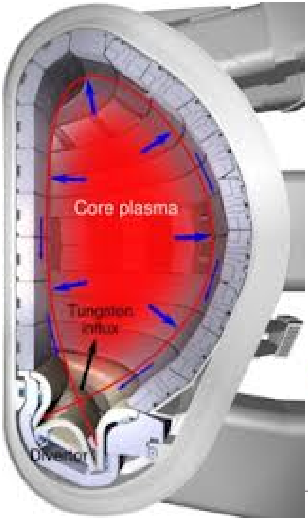
Limiter Configuration

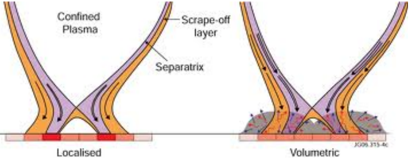


Divertor

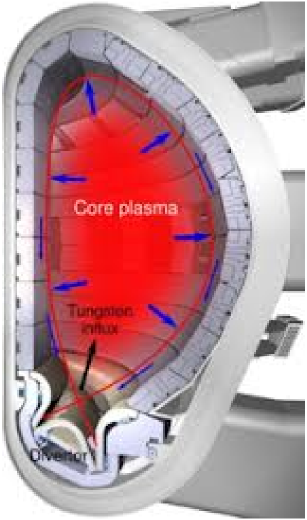
X-point configuration

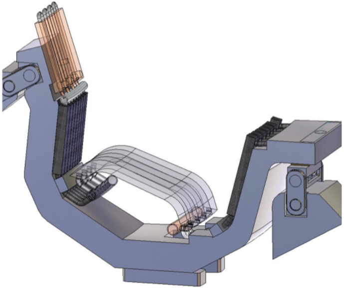
X-point configuration

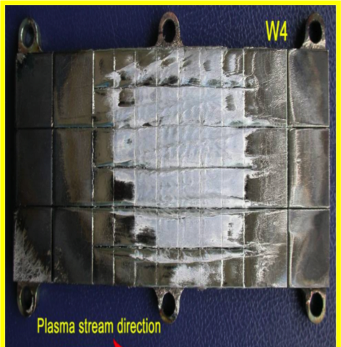




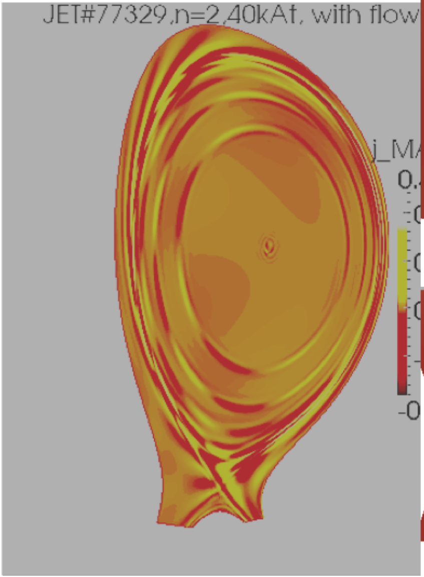
Context

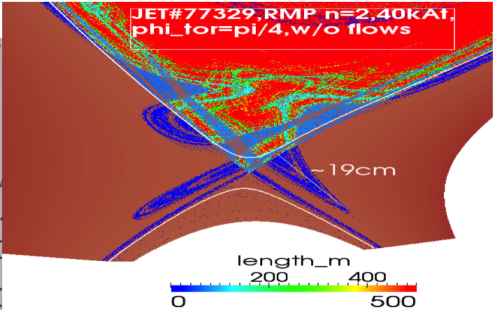

Core plasma
Tungsten influx
Divertor

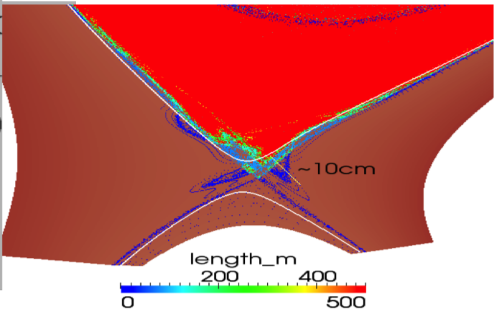



W4
Plasma stream direction

Jorek : Reduced MHD $\mathbf{B} = F_{eq}\nabla\phi + \nabla \times (\psi\nabla\phi)$







Fluid description of plasma : Assumptions

- Macroscopic velocity is small compared to the sound speed or magnetic speed (Alfen)

$$[v] \ll \left[\sqrt{\frac{\gamma p}{\rho}}\right] \quad [v] \ll \left[\frac{\mathbf{B}}{\sqrt{\rho}}\right]$$

- Deby length is small compared to characteristic system length: local electrical quasi-neutrality

$$\sqrt{\frac{T_e}{n_e e^2}} \ll [x]$$

- Magnetic speed is small compared to light speed : negligible displacement current

$$\left[\frac{\mathbf{B}}{\sqrt{\rho}}\right] \ll c_l$$

- $\nu^\star = \frac{\nu_{ei}}{\omega_b} \gg 1$ where $\omega_b = \sqrt{\frac{T_e}{m_e}}\frac{1}{q(r)R}$

Fluid description of plasma : MHD equations

Rationalized Gaussian unit (c=1)

- Conservation of particles
- Conservation of momentum
- Conservation of energy
- Faraday's law

$$\left\{\begin{array}{ll} \partial_t \rho + \nabla \cdot \boldsymbol{m} & = \nabla \cdot \boldsymbol{m}' \\ \partial_t \boldsymbol{m} + \nabla \cdot (\boldsymbol{m} \otimes \boldsymbol{v}) + \nabla \cdot (p\mathbb{I} + \pi\mathbb{I} - \mathbf{B} \otimes \mathbf{B}) & = \nabla \cdot \left(\underline{\boldsymbol{\tau}} + \underline{\boldsymbol{\tau}}'\right) \\ \partial_t \mathcal{E} + \nabla \cdot ((\mathcal{E} + p - \pi)\, \boldsymbol{v} + \mathbf{E} \times \mathbf{B}) + \nabla \cdot \boldsymbol{q} & = \nabla \cdot \left(\left(\underline{\boldsymbol{\tau}} + \underline{\boldsymbol{\tau}}'\right) \boldsymbol{v}\right) \\ \partial_t \mathbf{B} + \nabla \times \mathbf{E} & = 0 \end{array}\right.$$

- Conservation of elect. momentum (generalized Ohm's Law)
- Ampere's law

$$\left\{\begin{array}{lll} \mathbf{E} & = & -\boldsymbol{v} \times \mathbf{B} + \eta \mathbf{J} + \widetilde{\mathrm{E}} \\ \mathbf{J} & = & \nabla \times \mathbf{B} \end{array}\right.$$

- No magnetic mono-pole : $\nabla \cdot \mathbf{B} = 0$
- Divergence free current : $\nabla \cdot \mathbf{J} = 0$

MHD : Fluid description of plasma

Conservative formulation :

$$\begin{cases} \partial_t \rho + \nabla \cdot \boldsymbol{m} & = \nabla \cdot \boldsymbol{m}' \\ \partial_t \boldsymbol{m} + \nabla \cdot (\boldsymbol{m} \otimes \boldsymbol{v}) + \nabla \cdot (p\mathbb{I} + \pi\mathbb{I} - \mathbf{B} \otimes \mathbf{B}) & = \nabla \cdot (\underline{\boldsymbol{\tau}} + \underline{\boldsymbol{\tau}}') \\ \partial_t \mathcal{E} + \nabla \cdot ((\mathcal{E} + p - \pi)\boldsymbol{v} + \mathbf{E} \times \mathbf{B}) + \nabla \cdot \boldsymbol{q} & = \nabla \cdot ((\underline{\boldsymbol{\tau}} + \underline{\boldsymbol{\tau}}')\,\boldsymbol{v}) \\ \partial_t \mathbf{B} + \nabla \times \mathbf{E} & = 0 \end{cases}$$

where

$$\boldsymbol{m} = \rho \boldsymbol{v}, \quad \mathcal{E} = \rho\varepsilon + \rho\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2} + \frac{\mathbf{B}\cdot\mathbf{B}}{2}, \quad \pi = \frac{\mathbf{B}\cdot\mathbf{B}}{2}$$

Closure relations :

1. $p = (\gamma - 1)\rho\varepsilon$
2. $\mathbf{E} = -\boldsymbol{v} \times \mathbf{B} + \eta\mathbf{J} + \widetilde{\mathrm{E}}$
3. $\mathbf{J} = \nabla \times \mathbf{B} \rightarrow \nabla \cdot \mathbf{J} = 0.$

Constraint

- $\nabla \cdot \mathbf{B} = 0 \leftarrow \mathbf{B} = \nabla \times \mathbf{A}$

$\widetilde{\mathrm{E}}$ contains the inductive applied electric field.

Compact formulation : Ideal MHD

$$\boldsymbol{w} = \begin{pmatrix} \rho \\ \boldsymbol{m} \\ \mathcal{E} \\ \mathbf{B} \end{pmatrix}, \qquad \underline{\mathbf{f}} = \begin{pmatrix} \boldsymbol{m}^{\mathrm{T}} \\ \boldsymbol{v} \otimes \boldsymbol{m} + \mathrm{P}\mathbb{I} - \mathbf{B} \otimes \mathbf{B} \\ \mathcal{H}\boldsymbol{m}^{\mathrm{T}} - \left(\mathbf{B} \cdot \boldsymbol{v}\right)\mathbf{B}^{\mathrm{T}} \\ \boldsymbol{v} \otimes \mathbf{B} - \mathbf{B} \otimes \boldsymbol{v} \end{pmatrix}$$

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \underline{\mathbf{f}} = 0 \tag{1}$$

It is useful to reformulate this system in quasi-linear form as

$$\frac{\partial \boldsymbol{w}}{\partial t} + \underline{\mathbf{A}}\left(\boldsymbol{w}, \boldsymbol{\partial}\right)\boldsymbol{w} = 0 \tag{2}$$

For any direction $\boldsymbol{n}$, $\underline{\mathbf{A}}\left(\boldsymbol{w}, \boldsymbol{n}\right)$ is a matrix with eigenvalues

$$\boldsymbol{v} \cdot \boldsymbol{n}, \quad \boldsymbol{v} \cdot \boldsymbol{n} \pm c_a, \quad \boldsymbol{v} \cdot \boldsymbol{n} \pm c_s, \quad \boldsymbol{v} \cdot \boldsymbol{n} \pm c_f$$

Finite Volume and DG approximations

Our recent work in the current context:

1. J Vides, B Nkonga, E Audit. *A simple two-dimensional extension of the HLL Riemann solver for hyperbolic systems of conservation laws.* Journal of Computational Physics 280, 643-675, 2015
2. DS Balsara, J Vides, K Gurski, B Nkonga, M Dumbser, S Garain, E Audit. *A two-dimensional Riemann solver with self-similar sub-structure-Alternative formulation based on least squares projection.* Journal of Computational Physics 304, 138-161, 2016.

Finite Element : weak formulation

The system to be solve :

$$\mathrm{R}\left(\boldsymbol{w}\right) = 0 \tag{3}$$

where the residual is

$$\mathrm{R}\left(\boldsymbol{w}\right) := \frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \underline{\mathbf{f}} - \nabla \cdot \underline{\mathbf{g}}$$

Weak formulation

$$\boxed{\int_{\boldsymbol{\Omega}_{x,h}} \mathrm{R}\left(\boldsymbol{w}\right) \cdot \boldsymbol{w}^{\star} = 0, \qquad \forall \boldsymbol{w}^{\star} \in \overrightarrow{\mathcal{W}}_{\boldsymbol{h}}} \tag{4}$$

where

- $\overrightarrow{\mathcal{W}}_{\boldsymbol{h}} \equiv \overrightarrow{\mathcal{W}}_{\boldsymbol{h}}\left(\boldsymbol{\Omega}_{x,h}\right)$ is an approximated finite element space.
- $\boldsymbol{w}^{\star}$ is any test function in $\overrightarrow{\mathcal{W}}_{\boldsymbol{h}}\left(\boldsymbol{\Omega}_{x,h}\right)$.

Finite Element : Variational Multi-Scales

Find $\boldsymbol{w}_h \in \overrightarrow{\mathcal{W}}_{\boldsymbol{h}}$ such that

$$\boxed{\int_{\boldsymbol{\Omega}_{x,h}} \mathrm{R}\left(\boldsymbol{w}_h\right) \cdot \boldsymbol{w}^\star + \int_{\boldsymbol{\Omega}_{x,h}} \boldsymbol{w}' \cdot \left(\underline{\mathbf{A}}^{\mathrm{T}}\left(\boldsymbol{w}_h, \boldsymbol{\partial}\right) \boldsymbol{w}^\star\right) = 0, \quad \forall \boldsymbol{w}^\star \in \overrightarrow{\mathcal{W}}_{\boldsymbol{h}}}$$

$\boldsymbol{w}'$ is the vector of the sub-scales

Usually

$$\boldsymbol{w}' \simeq -\left(\underline{\mathbf{A}}\left(\boldsymbol{w}, \boldsymbol{\partial}\right)\right)^{-1} \mathrm{R}\left(\boldsymbol{w}_h\right) \simeq \underline{\mathcal{T}}\,\mathrm{R}\left(\boldsymbol{w}_h\right)$$

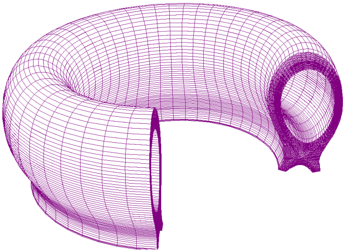
In the present context

$$\boxed{\boldsymbol{w}' \simeq \underline{\mathcal{T}}\left(\underline{\mathbf{A}}\left(\boldsymbol{w}_h, \boldsymbol{\partial}\right) \delta \boldsymbol{w}_h\right)}$$

Toroidal Geometry : $\boldsymbol{x} \equiv \boldsymbol{x}(\mathrm{R}, \phi, \mathrm{Z}) \equiv \boldsymbol{x}(\boldsymbol{\xi}, \phi)$

with $\boldsymbol{\xi} \in \boldsymbol{\Omega}_{\mathrm{2D}}$ and $\phi \in [0, 2\pi[$

$$\boldsymbol{\Omega}_x \equiv \boldsymbol{\Omega}_{\mathrm{2D}} \times [0, 2\pi[$$



Volume integrals

$$\int_{\boldsymbol{\Omega}_x} (\cdots)\, d\boldsymbol{x} \quad = \quad \int_0^{2\pi} d\phi \int_{\boldsymbol{\Omega}_{\mathrm{2D}}} (\cdots)\, \mathrm{R}\, d\boldsymbol{\xi}$$

Surface integrals

$$\int_{\partial\boldsymbol{\Omega}_x} (\cdots)\, d\mathrm{S}_x \quad = \quad \int_0^{2\pi} d\phi \int_{\partial\boldsymbol{\Omega}_{\mathrm{2D}}} (\cdots)\, \mathrm{R}\, d\mathrm{S}$$

Finite element Space : $\mathbf{B} = \nabla \times \mathbf{A}$

Interpolated Variables: $\mathbf{Y} = \begin{pmatrix} \rho \\ \boldsymbol{v} \\ T \\ \mathbf{A} \end{pmatrix}$.

Space of interpolations

$$\overrightarrow{\mathcal{W}}_{\boldsymbol{h}} := \mathcal{V}_h \times \left(\mathcal{V}_h \widehat{\mathbf{R}} + \mathcal{V}_h \widehat{\mathbf{Z}} + \mathcal{V}_h \widehat{\boldsymbol{\phi}}\right) \times \mathcal{V}_h \times \left(\nabla \mathcal{V}_h \times \widehat{\mathbf{R}} + \nabla \mathcal{V}_h \times \widehat{\mathbf{Z}} + \nabla \mathcal{V}_h \times \nabla \phi\right) \quad (5)$$

Where

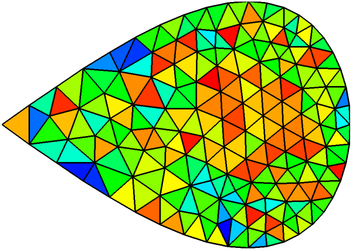
$$\mathcal{V}_h \equiv \mathcal{V}_h\left(\boldsymbol{\Omega}_{\mathrm{2D}}\right) \times \mathcal{V}_h\left([0, 2\pi[\right)$$

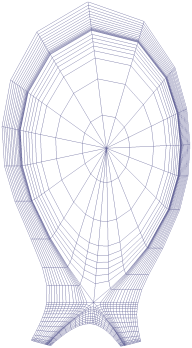
Trial functions in Toroidal Geometry : $\mathcal{N}\left(\boldsymbol{x}\right) \equiv \psi\left(\boldsymbol{\xi}\right) \mathrm{C}\left(\phi\right)$

2D Poloidal : $\boldsymbol{\Omega}_{\text{2D}} \simeq \boldsymbol{\Omega}_h \equiv (\mathcal{P}_h, \mathcal{E}_h)$

$$\mathcal{V}_h(\boldsymbol{\Omega}_{\text{2D}}) \equiv \text{SPAN}\left(\psi_{i_{\text{2D}}}(\boldsymbol{\xi})\right) \quad \text{and} \quad \mathcal{V}_h([0,2\pi[) \equiv \text{SPAN}\left(\mathrm{C}_{i_\phi}(\phi),\right)$$

- $\mathcal{P}_h = \bigcup_{p=1}^{N_p} \boldsymbol{\xi}_p$ and $\mathcal{E}_h = \bigcup_{e=1}^{N_{\boldsymbol{\tau}}} \boldsymbol{\tau}_e$
  - Triangular elements : $\boldsymbol{\tau}_e \equiv \left(\boldsymbol{\xi}_{p_1^e}, \boldsymbol{\xi}_{p_2^e}, \boldsymbol{\xi}_{p_3^e}\right)$.
  - Quadrilateral elements :$\boldsymbol{\tau}_e \equiv \left(\boldsymbol{\xi}_{p_1^e}, \boldsymbol{\xi}_{p_2^e}, \boldsymbol{\xi}_{p_3^e}, \boldsymbol{\xi}_{p_4^e}\right)$.





Isoparametric Finite element

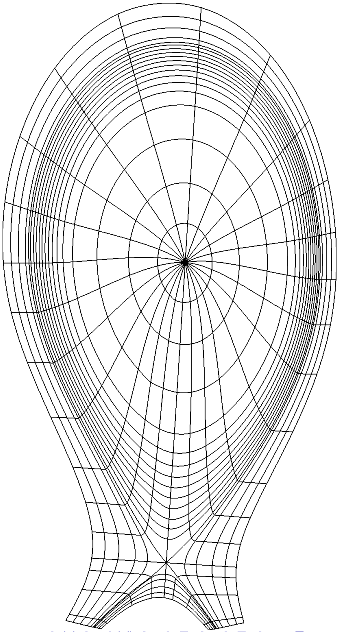
1. Parametric (S, T) to physical (R, Z)

$$\begin{cases} \mathrm{R}^{\boldsymbol{e}_{2\mathrm{D}}}(\mathrm{S}, \mathrm{T}) & = \displaystyle\sum_{\boldsymbol{j}_{2\mathrm{D}} \in \boldsymbol{e}_{2\mathrm{D}}} \mathrm{R}_{\boldsymbol{j}_{2\mathrm{D}}} \psi_{\widehat{\boldsymbol{j}}_{2\mathrm{D}}}(\mathrm{S}, \mathrm{T}) \\ \mathrm{Z}^{\boldsymbol{e}_{2\mathrm{D}}}(\mathrm{S}, \mathrm{T}) & = \displaystyle\sum_{\boldsymbol{j}_{2\mathrm{D}} \in \boldsymbol{e}_{2\mathrm{D}}} \mathrm{Z}_{\boldsymbol{j}_{2\mathrm{D}}} \psi_{\widehat{\boldsymbol{j}}_{2\mathrm{D}}}(\mathrm{S}, \mathrm{T}) \end{cases}$$

2. Test function in the physical space.

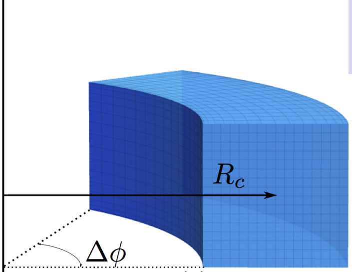
$$\psi_{\boldsymbol{j}_{2\mathrm{D}}}(\boldsymbol{\xi}) = \psi_{\widehat{\boldsymbol{j}}_{2\mathrm{D}}}\Big(\mathrm{S}(\mathrm{R}, \mathrm{Z}), \mathrm{T}(\mathrm{R}, \mathrm{Z})\Big)$$

3. Injective application : $\boldsymbol{j}_{2\mathrm{D}} \longrightarrow \widehat{\boldsymbol{j}}_{2\mathrm{D}}$.



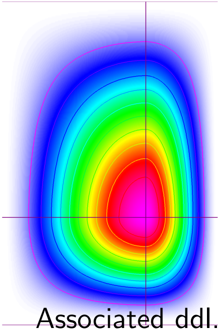
C1-Quadrangles

$$\psi_{\hat{\boldsymbol{j}}_{2\mathrm{D}}} \equiv \psi_p^{(k)} \quad \text{with} \quad \begin{cases} p = 1, 2, 3, 4 \\ k = 1, 2, 3, 4 \end{cases}$$

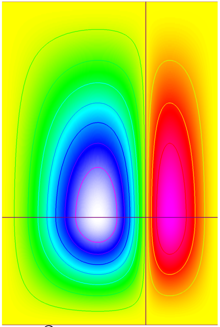


Tests functions associated to a vertex i

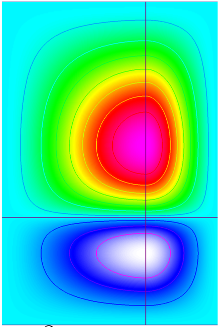
$\psi_p^{(1)}(\mathrm{S},\mathrm{T})$



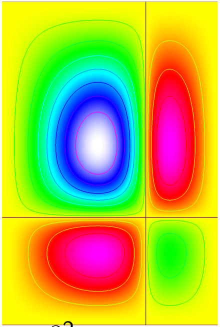
$\psi_p^{(2)}(\mathrm{S},\mathrm{T})$



$\psi_p^{(3)}(\mathrm{S},\mathrm{T})$



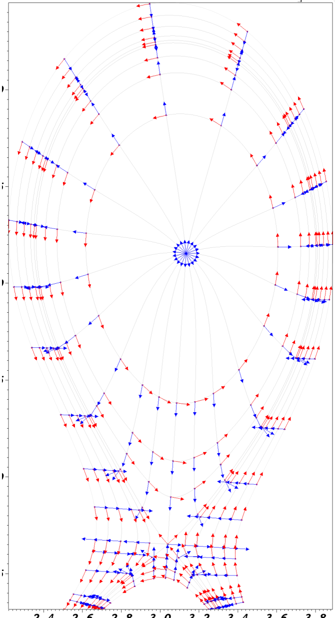
$\psi_p^{(4)}(\mathrm{S},\mathrm{T})$

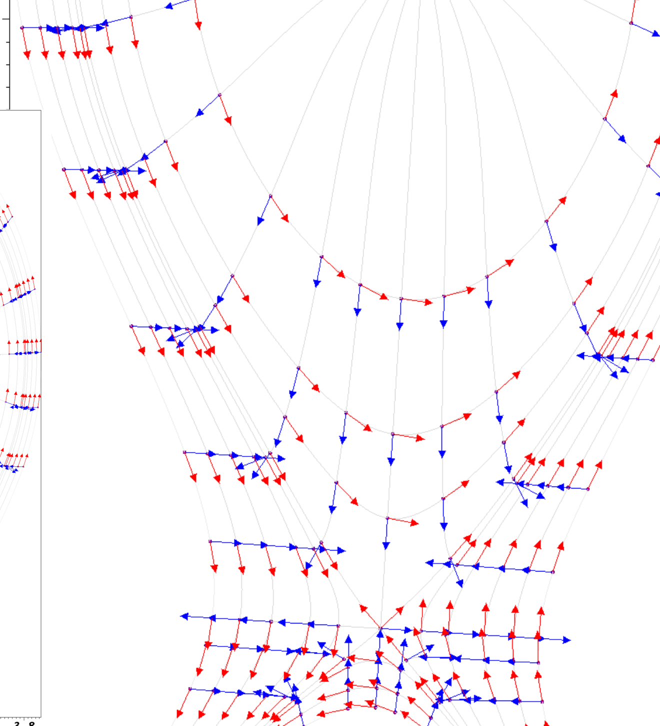


Associated ddl.

$\boldsymbol{w}(\boldsymbol{\xi}_i)$ $\quad \dfrac{\partial \boldsymbol{w}}{\partial \mathrm{R}}(\boldsymbol{\xi}_i)$ $\quad \dfrac{\partial \boldsymbol{w}}{\partial \mathrm{Z}}(\boldsymbol{\xi}_i)$ $\quad \dfrac{\partial^2 \boldsymbol{w}}{\partial \mathrm{R}\partial \mathrm{Z}}(\boldsymbol{\xi}_i)$

C1-Quadrangles





VMS for Reduced MHD

$$\frac{\partial \mathbf{w}}{\partial t} + \underline{\widetilde{\mathbf{A}}}\left(\mathbf{w}, \boldsymbol{\partial}\right)\mathbf{w} = \boldsymbol{\kappa}\left(\boldsymbol{\partial}, \cdot\right) \qquad (6)$$

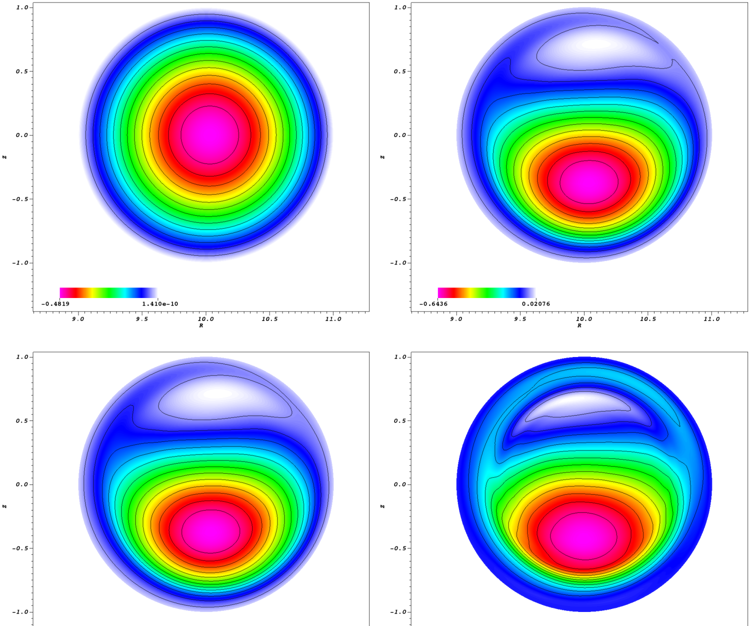
where $\mathbf{w}$ are the reduced variables.

$$\mathbf{w} = \begin{pmatrix} \rho \\ p \\ \vartheta \\ \omega \\ \psi \end{pmatrix}, \quad \underline{\widetilde{\mathbf{A}}}\left(\mathbf{w}, \boldsymbol{\partial}\right) = \begin{pmatrix} \boldsymbol{v}\cdot\boldsymbol{\partial} & 0 & \rho\mathbf{B}\cdot\boldsymbol{\partial} & 0 & 0 \\ 0 & \boldsymbol{v}\cdot\boldsymbol{\partial} & \gamma p\mathbf{B}\cdot\boldsymbol{\partial} & 0 & 0 \\ 0 & \frac{\mathbf{B}\cdot\boldsymbol{\partial}}{\rho\|\mathbf{B}\|^2} & \boldsymbol{v}\cdot\boldsymbol{\partial} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{v}_\perp\cdot\boldsymbol{\partial} & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{v}_\perp\cdot\boldsymbol{\partial} \end{pmatrix}$$

For any direction $\boldsymbol{n}$, the matrix $\underline{\widetilde{\mathbf{A}}}\left(\mathbf{w}, \boldsymbol{n}\right)$ is diagonalizable and the eigenvalues are

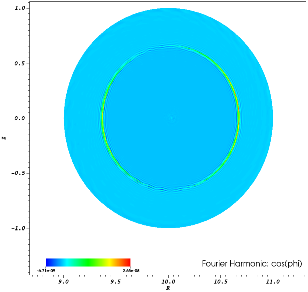
$$\boldsymbol{v}\cdot\boldsymbol{n}, \quad \boldsymbol{v}_\perp\cdot\boldsymbol{n}, \quad \boldsymbol{v}\cdot\boldsymbol{n} - \frac{|\mathbf{B}\cdot\boldsymbol{n}|}{\|\mathbf{B}\|}c \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{n} + \frac{|\mathbf{B}\cdot\boldsymbol{n}|}{\|\mathbf{B}\|}c$$

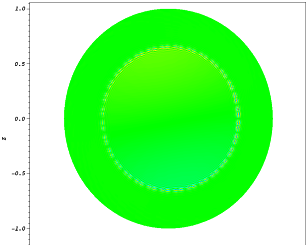
Resistive internal kink



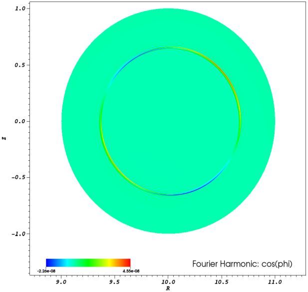
Resistive internal kink : Dynamic of the first mode

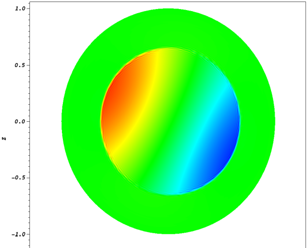
Artificial viscosity



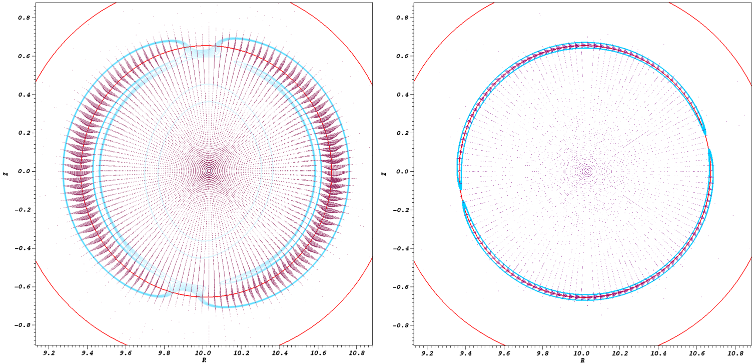


VMS-Stabilization





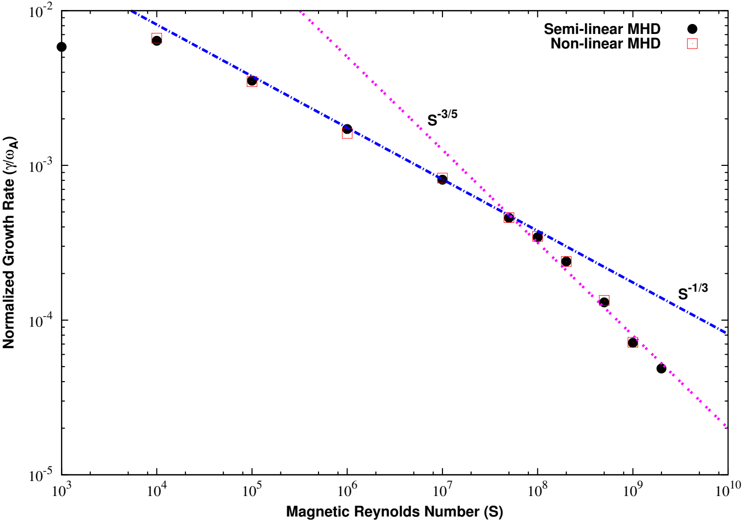
Resistive internal kink



Structure layer: $S = 10^5 \approx 10^{-1}$ and $S = 10^8 \approx 2.10^{-2}$

$S \propto \eta^{-1}$: Magnetic Reynolds number.

Resistive internal kink: growth rate



$$\gamma \propto S^{-\frac{1}{3}} \rightarrow \text{Kink}$$

$$\gamma \propto S^{-\frac{3}{5}} \rightarrow \text{Tearing}$$

Boundary Conditions: Bohm

At the divertor boundary :

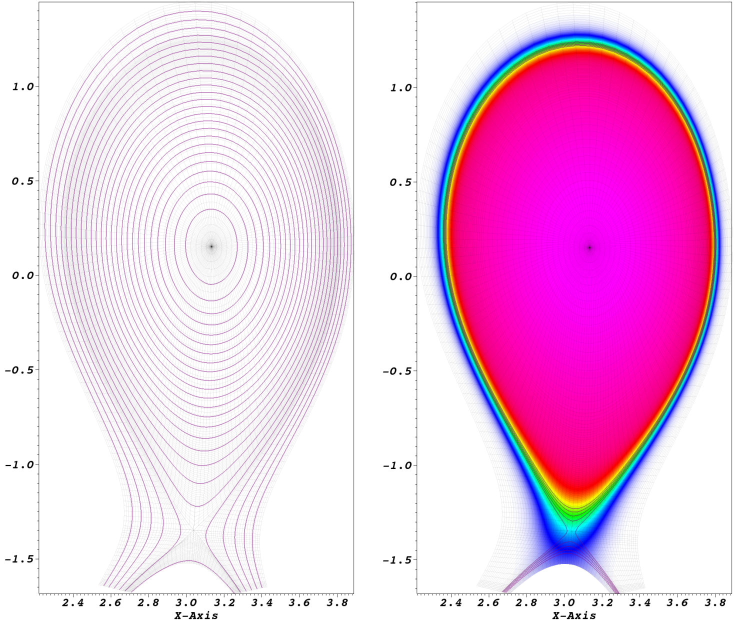
$$\boldsymbol{v} = \sqrt{\gamma T}\,\frac{\mathbf{B}\cdot\boldsymbol{n}}{|\mathbf{B}\cdot\boldsymbol{n}|}\,\frac{\mathbf{B}}{||\mathbf{B}||}$$

where $\boldsymbol{n}$ is the outward normal.

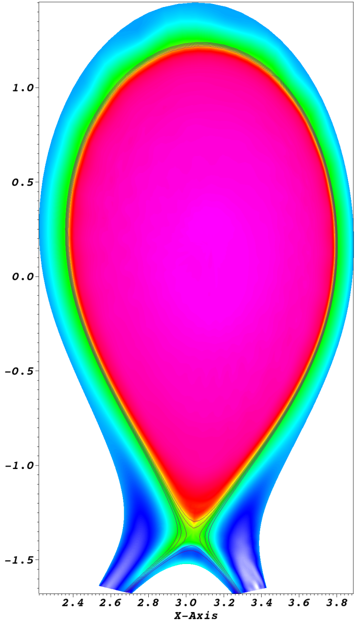
Penalization

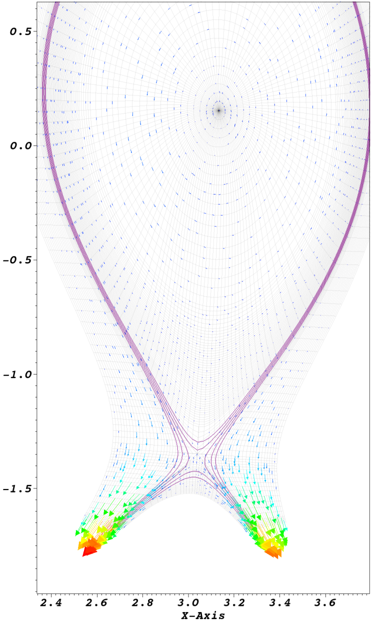
$$\frac{1}{\varepsilon}\int_{\partial\Omega_{x,h}} \rho\left(\boldsymbol{v} - \sqrt{\gamma T}\,\frac{\mathbf{B}\cdot\boldsymbol{n}}{|\mathbf{B}\cdot\boldsymbol{n}|}\,\frac{\mathbf{B}}{||\mathbf{B}||}\right)\cdot m^{\star}$$

X-point : Initial GS Equilibrium (density)



X-point : density and velocity





Perspectives

1. Simulation of ELMs using the full MHD
2. PS Finite element method (triangles)
3. Hybrid-Mesh Formulation
4. Generalized Ohm's Law
5. Ti-Te 2-fluid Modeling (non conservative)
6. MGI : Ionization/Radiation