SHARK-FV 2016 Conference

SHARING HIGHER-ORDER ADVANCED RESEARCH KNOW-HOW on FINITE VOLUME

Great White

São Félix, Portugal
May 23 - 27, 2016

High-order finite volume method for curved boundaries and non-matching domain-mesh problems

May 23-27, 2016, São Félix, Portugal

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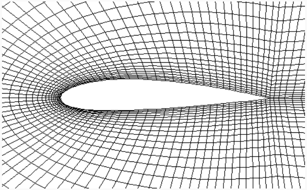
Université de Toulouse

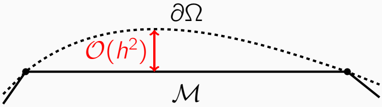
Universidade do Minho

Outline

Motivation and Background

Replacing curved boundaries by polygonal edges associated to the mesh provides at most 2nd-order accuracy





- Isoparametric elements are widely applied for FEM and DG
- Very few methods have been developed in the context of HO-FVM

State-of-art in HO-FVM

- Ghost cells approach: add extra cells between the geometric boundary and the computational domain

[1] C.F. Ollivier-Gooch and M. Van Altena, A high-order accurate unstructured mesh finite-volume scheme for the advection-diffusion equation, Journal of Computational Physics, (2002).

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- Ghost cells approach: add extra cells between the geometric boundary and the computational domain
- Ollivier-Gooch approach[1]: enforce the BC by constraining the LSM associated to the PR
  - can be very time consuming if the LSM matrix has to be updated (moving boundaries/interfaces, tracking interfaces/discontinuities problems)

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New approach: detach the BC conservation from the LSM matrix

  - Easier, flexible, more efficient, more elegant

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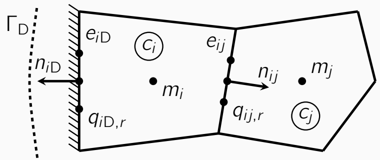
Problem Formulation

- Poisson's equation and BCs:

$$\begin{aligned} \nabla^2\phi &= f, && \text{in } \Omega \\ \phi &= \phi_D, && \text{on } \Gamma_D \\ \nabla\phi \cdot n &= g_N, && \text{on } \Gamma_N \\ \alpha\phi + \beta\nabla\phi \cdot n &= g_R, && \text{on } \Gamma_R \end{aligned}$$

  - Ω, $\partial\Omega = \{\Gamma_D \cup \Gamma_N \cup \Gamma_R\}$ – real domain and its boundary
  - n – unit normal vector to $\partial\Omega$
  - ϕ_D – Dirichlet BC
  - g_N – Neumann BC
  - g_R – Robin BC with coefficients α and β

Generic FV Scheme



- c_i, e_{ij} – cell, edge
- q_i, q_{ij} –quadrature points
- n_{ij} – normal vector

$$\int_{\partial c_i} (\nabla \phi) \cdot n_i \, \mathrm{d}s = \int_{c_i} f \, \mathrm{d}x \Rightarrow \sum_{j \in \nu(i)} |e_{ij}| \left[\sum_{r=1}^{R} \zeta_r \mathbb{F}_{ij,r} \right] - f_i |c_i| = \mathcal{O}(h_i^{2R})$$

- Physical fluxes:

$$\mathbb{F}_{ij,r} = \nabla \phi(q_{ij,r}) \cdot n_{ij},$$

- Source term:

$$f_i = \sum_{s=1}^{S} \zeta_s f(q_{i,s})$$

Polynomial Reconstruction Machinery

SHARK-FV Non-conservative PR for inner edges e_{ij}

$$\varphi_{ij}(x) = \sum_{0 \leq |\alpha| \leq d} \mathcal{R}_{ij}^{\alpha}(x - m_{ij})^{\alpha}$$

- $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2}$
- $\mathcal{R}_{ij} = (\mathcal{R}_{ij}^{\alpha})_{0 \leq |\alpha| \leq d}$ – coefficients (to be determined)

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$$E_{ij}(\mathcal{R}_{ij}) = \sum_{q \in S_{ij}} \omega_{ij,q} \left[\frac{1}{|c_q|} \int_{c_q} \varphi_{ij}(x) \, \mathrm{d}x - \phi_q \right]^2$$

- $\widetilde{\varphi}_{ij}(x)$ defined with $\widetilde{\mathcal{R}}_{ij} = \underset{\mathcal{R}_{ij}}{\arg\min} \left[E_{ij}(\mathcal{R}_{ij}) \right]$

Polynomial Reconstruction Machinery

SHARK-FV Conservative PR for Dirichlet boundary edges $e_{i\mathrm{D}}$

$$\varphi_{i\mathrm{D}}(x) = \phi_{i\mathrm{D}} + \sum_{1 \le |\alpha| \le d} \mathcal{R}_{i\mathrm{D}}^{\alpha} \left[(x - m_{i\mathrm{D}})^{\alpha} - M_{i\mathrm{D}}^{\alpha} \right]$$

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- $\mathcal{R}_{i\mathrm{D}} = (\mathcal{R}_{i\mathrm{D}}^{\alpha})_{1\le|\alpha|\le d}$ – coefficients (to be determined)

$$E_{i\mathrm{D}}(\mathcal{R}_{i\mathrm{D}}) = \sum_{q\in S_{i\mathrm{D}}} \omega_{i\mathrm{D},q} \left[\frac{1}{|c_q|}\int_{c_q} \varphi_{i\mathrm{D}}(x)\,\mathrm{d}x - \phi_q\right]^2$$

- $\widehat{\varphi}_{i\mathrm{D}}(x)$ defined with $\widehat{\mathcal{R}}_{i\mathrm{D}} = \underset{\mathcal{R}_{i\mathrm{D}}}{\arg\min}\left[E_{i\mathrm{D}}(\mathcal{R}_{i\mathrm{D}})\right]$

Polynomial Reconstruction Machinery

- Naive mean-value conservation – 2nd-order!

$$\phi_{i\mathrm{D}} = \frac{1}{|e_{i\mathrm{D}}|}\int_{e_{i\mathrm{D}}} \phi_{\mathrm{D}}(x)\,\mathrm{d}s, \qquad \frac{1}{|e_{i\mathrm{D}}|}\int_{e_{i\mathrm{D}}} \varphi_{i\mathrm{D}}(x)\,\mathrm{d}s = \phi_{i\mathrm{D}}$$

$$M^{\alpha}_{i\mathrm{D}} = \frac{1}{|e_{i\mathrm{D}}|}\int_{e_{i\mathrm{D}}} (x - m_{i\mathrm{D}})^{\alpha}\,\mathrm{d}x$$

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$$M_{i\mathrm{D}}^{\alpha} = \frac{1}{|e_{i\mathrm{D}}|}\int_{e_{i\mathrm{D}}} (x - m_{i\mathrm{D}})^{\alpha}\,\mathrm{d}x$$

- Wise mean-value conservation – Quadrature points on $\partial\Omega$!

$$\phi_{i\mathrm{D}} = \frac{1}{|\widehat{e}_{i\mathrm{D}}|}\int_{\widehat{e}_{i\mathrm{D}}} \phi_{\mathrm{D}}(x)\,\mathrm{d}s, \qquad \frac{1}{|\widehat{e}_{i\mathrm{D}}|}\int_{\widehat{e}_{i\mathrm{D}}} \boldsymbol{\varphi}_{i\mathrm{D}}(x)\,\mathrm{d}s = \phi_{i\mathrm{D}}$$

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$$M^{\alpha}_{i\mathrm{D}} = \frac{1}{|\widehat{e}_{i\mathrm{D}}|}\int_{\widehat{e}_{i\mathrm{D}}} (x - m_{i\mathrm{D}})^{\alpha}\,\mathrm{d}x$$

- Wise point-value conservation:

$$\phi_{i\mathrm{D}} = \phi_{\mathrm{D}}(p_{i\mathrm{D}}),\ p_{i\mathrm{D}} \in \widehat{e}_{i\mathrm{D}}, \qquad \varphi_{i\mathrm{D}}(p_{i\mathrm{D}}) = \phi_{i\mathrm{D}}$$

$$M^{\alpha}_{i\mathrm{D}} = (p_{i\mathrm{D}} - m_{i\mathrm{D}})^{\alpha}$$

Polynomial Reconstruction Machinery

- Conservation of Dirichlet BC only
- Coefficients $M^{\alpha}_{i\mathrm{D}}$ are boundary dependent

☞ LSM matrix has to be updated for...

  - ... moving boundaries/interfaces or dynamic BC in time-dependent and unsteady problems
  - ... optimization problems
  - ... tracking interfaces/discontinuities problems
  - ... etc.

- Ollivier-Gooch method consists in an augmented LSM matrix by constraints rows (equivalent to the wise conservation)

ARCH

Adaptive **R**econstruction for **C**onservation of **H**igh-order

- The aim of ARCH is to improve the boundary treatment
- The main ideas are...
  - ... conserve the HO accuracy
  - ... detach the boundary from the LSM matrix
  - ... easy handling of moving boundaries
  - ... generic treatment of Dirichlet, Neumann and Robin BCs

SHARK-FV ARCH for boundary edges e_{iB}

$$\varphi_{iB}(x;\psi_{iB}) = \psi_{iB} + \varphi_{iB}(x)$$

- $\varphi_{iB}(x)$ – non-conservative/naive conservative/wise conservative PR
- ψ_{iB} – free parameter (to be determined)

$$E_{iB}(\mathcal{R}_{iB};\psi_{iB}) = \sum_{q\in S_{iB}} \omega_{iB,q} \left[\psi_{iB} + \frac{1}{|c_q|}\int_{c_q} \varphi_{iB}(x;\psi_{iB})\,\mathrm{d}x - \phi_q\right]^2$$

- $\breve{\varphi}_{iB}(x;\psi_{iB})$ defined with $\breve{\mathcal{R}}_{iB} = \underset{\mathcal{R}_{iB}}{\arg\min}\,[E_{iB}(\mathcal{R}_{iB};\psi_{iB})]$
- ψ_{iB} is a RHS – the LSM matrix remains the same :D

ARCH Method

- Generic BC on $\partial\Omega$ to satisfy:

$$g(x;\alpha,\beta) = \alpha(x)\phi(x) + \beta(x)\nabla\phi \cdot n$$

  - Dirichlet BC: $\alpha = 1$, $\beta = 0$
  - Neumann BC: $\alpha = 0$, $\beta \neq 0$
  - Robin BC: $\alpha \neq 0$, $\beta \neq 0$

ARCH Method

- Generic BC on $\partial\Omega$ to satisfy:

$$g(x;\alpha,\beta) = \alpha(x)\phi(x) + \beta(x)\nabla\phi \cdot n$$

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  - Robin BC: $\alpha \neq 0$, $\beta \neq 0$

- For a given $p_{i\mathrm{B}} \in \partial\Omega$, the parameter $\psi_{i\mathrm{B}}$ is prescribed such that

$$\alpha(p_{i\mathrm{B}})\boldsymbol{\varphi}_{i\mathrm{B}}(p_{i\mathrm{B}};\breve{\psi}_{i\mathrm{B}}) + \beta(p_{i\mathrm{B}})\nabla\boldsymbol{\varphi}_{i\mathrm{B}}(p_{i\mathrm{B}};\breve{\psi}_{i\mathrm{B}}) \cdot n = g(p_{i\mathrm{B}};\alpha,\beta)$$

$$\breve{\psi}_{i\mathrm{B}} = \psi^0_{i\mathrm{B}} - \frac{\mathcal{B}(p_{i\mathrm{B}};\alpha,\beta,\psi^0_{i\mathrm{B}})(\psi^0_{i\mathrm{B}} - \psi^1_{i\mathrm{B}})}{\mathcal{B}(p_{i\mathrm{B}};\alpha,\beta,\psi^0_{i\mathrm{B}}) - \mathcal{B}(p_{i\mathrm{B}};\alpha,\beta,\psi^0_{i\mathrm{B}} + \varphi^1_{i\mathrm{B}})}$$

- $\psi^0_{i\mathrm{B}} \neq \psi^1_{i\mathrm{B}}$

Finite Volume Scheme

- Numerical fluxes:
  - inner edge e_{ij}: $\mathcal{F}_{ij,r} = \nabla\widetilde{\varphi}_{ij}(q_{ij,r}) \cdot n_{ij}$
  - boundary edge $e_{i\mathrm{B}}$: $\mathcal{F}_{i\mathrm{B},r} = \nabla\breve{\varphi}_{i\mathrm{B}}(q_{i\mathrm{D},r}; \psi_{i\mathrm{B}}) \cdot n_{i\mathrm{B}}$
- PR and fluxes are linear identities
- Linear residual operator $\Phi \to G_i(\Phi)$ for vector $\Phi \in \mathbb{R}^I$

$$\mathcal{G}_i(\Phi) = \sum_{j\in\nu(i)} |e_{ij}| \left[\sum_{r=1}^{R} \zeta_r \mathcal{F}_{ij,r}\right] - f_i|c_i|,$$

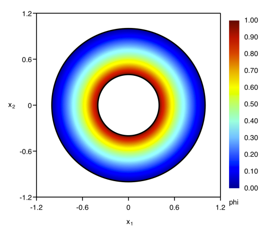
$$\mathcal{G}(\Phi) = (\mathcal{G}_i(\Phi))_{i=1,\dots,I}$$

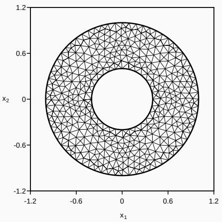
- Free matrix method
- Compute vector $\Phi^\star \in \mathbb{R}^I$ solution of $\mathcal{G}(\Phi) = 0$

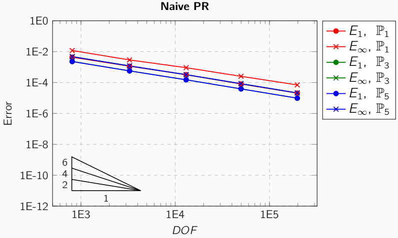
Numerical Benchmark

Numerical Benchmark

- Manufactured solution:
  $\phi(r) = a + b\ln(r)$
  - a, b such that
    $\phi \in [0, 1]$







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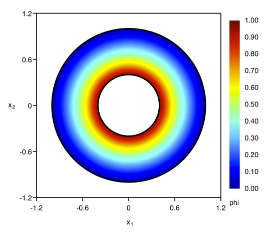

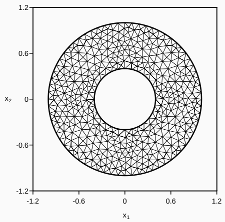

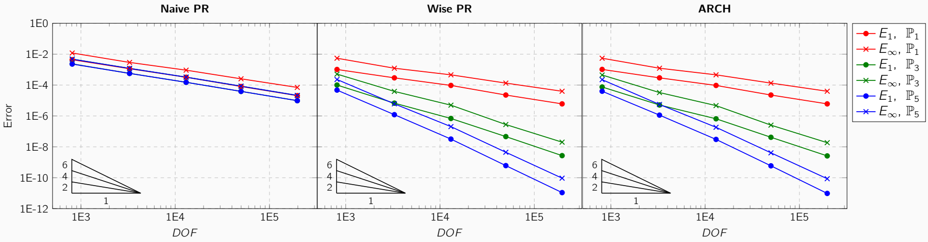

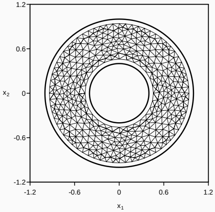
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-0.6
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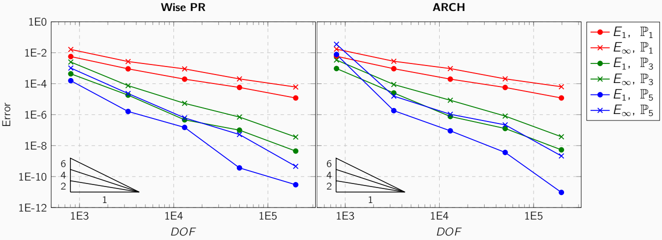
1.2
x_2
x_1


Wise PR
ARCH
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1E-2
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Error
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DOF
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E_1, $\mathbb{P}_1$
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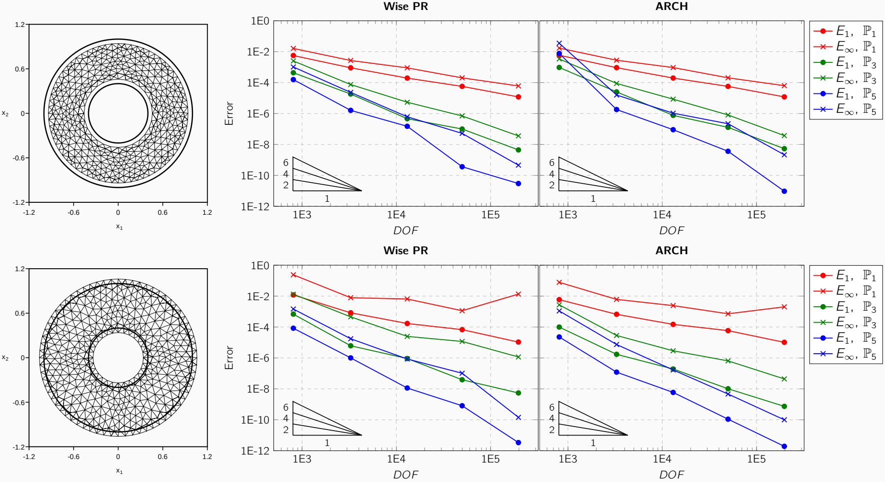
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E_1, $\mathbb{P}_1$
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E_∞, $\mathbb{P}_5$
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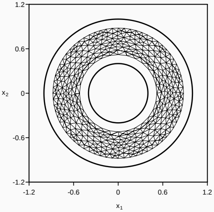
Numerical Benchmark

$DOF = 197340$

	$\mathbb{P}_1$		$\mathbb{P}_3$		$\mathbb{P}_5$	
	E_1	E_∞	E_1	E_∞	E_1	E_∞
Naive PR	2.91E−03	9.36E−03	2.91E−03	9.36E−03	2.91E−03	9.36E−03
Wise PR	1.20E−05	6.12E−05	4.45E−09	3.60E−08	2.98E−11	4.60E−10
ARCH	1.20E−05	6.37E−05	5.35E−09	3.74E−08	9.48E−12	2.15E−09

$DOF = 197340$

	$\mathbb{P}_1$		$\mathbb{P}_3$		$\mathbb{P}_5$	
	E_1	E_∞	E_1	E_∞	E_1	E_∞
Naive PR	2.87E−03	9.33E−03	2.87E−03	9.31E−03	2.87E−03	9.31E−03
Wise PR	1.07E−05	1.36E−02	5.45E−09	1.16E−06	3.37E−12	1.44E−10
ARCH	1.03E−05	2.03E−03	7.42E−10	4.40E−08	1.93E−12	1.00E−10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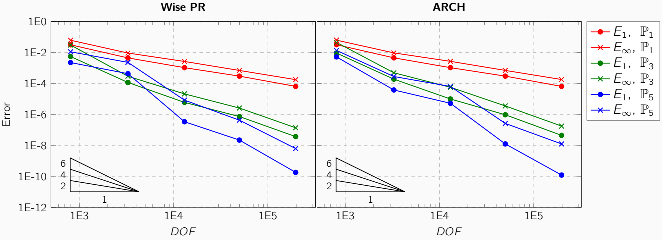
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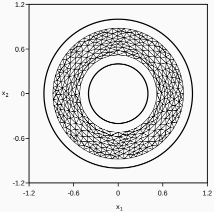
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x_2
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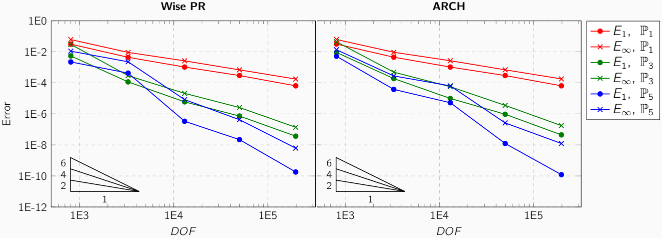
x_1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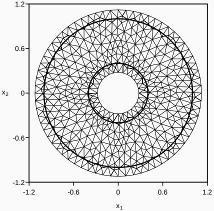
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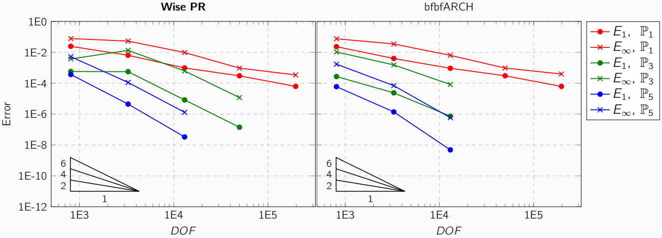
Wise PR
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DOF
E_1, $\mathbb{P}_1$
E_∞, $\mathbb{P}_1$
E_1, $\mathbb{P}_3$
E_∞, $\mathbb{P}_3$
E_1, $\mathbb{P}_5$
E_∞, $\mathbb{P}_5$

Numerical Benchmark

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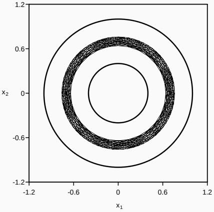
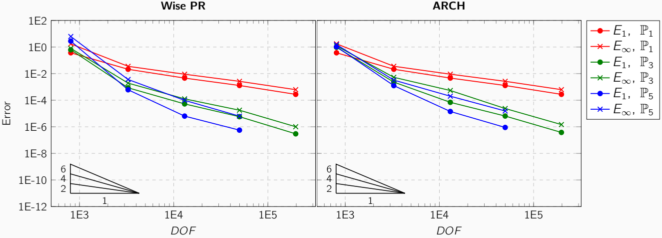
	$\mathbb{P}_1$		$\mathbb{P}_3$		$\mathbb{P}_5$	
	E_1	E_∞	E_1	E_∞	E_1	E_∞
Naive PR	5.88E−03	1.88E−02	5.88E−03	1.88E−02	5.88E−03	1.88E−02
Wise PR	6.47E−05	1.77E−04	3.68E−08	1.37E−07	1.80E−10	6.22E−09
ARCH	6.48E−05	1.80E−04	4.44E−08	1.76E−07	1.21E−10	1.24E−08

$DOF = 13056$

	$\mathbb{P}_1$		$\mathbb{P}_3$		$\mathbb{P}_5$	
	E_1	E_∞	E_1	E_∞	E_1	E_∞
Naive PR	2.13E−02	7.06E−02	2.12E−02	7.05E−02	2.12E−02	7.05E−02
Wise PR	9.72E−04	9.67E−03	8.25E−06	6.20E−04	3.31E−08	1.30E−06
ARCH	9.24E−04	6.35E−03	7.29E−07	8.27E−05	4.84E−09	5.86E−07

Numerical Benchmark





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Numerical Benchmark

- Γ_I, Γ_E – interior and exterior boundaries

$$\Gamma_\mathsf{I} : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = R_\mathsf{I}(\theta;\alpha_\mathsf{I}) \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad \Gamma_\mathsf{E} : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = R_\mathsf{E}(\theta;\alpha_\mathsf{E}) \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

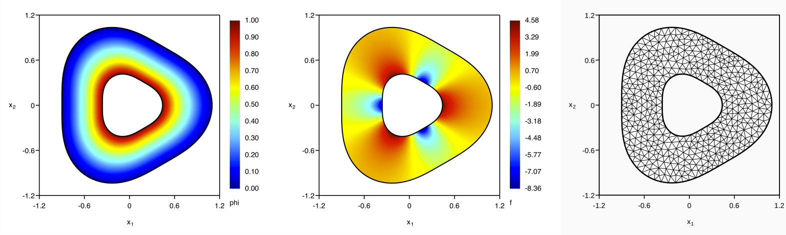
$$R_\mathsf{I}(\theta;\alpha_\mathsf{I}) = r_\mathsf{I}\left(1 + \frac{1}{10}\sin(\alpha_\mathsf{I}\theta)\right), \quad R_\mathsf{E}(\theta;\alpha_\mathsf{E}) = r_\mathsf{E}\left(1 + \frac{1}{10}\sin(\alpha_\mathsf{E}\theta)\right)$$

- Manufactured solution:

$$\phi(r,\theta) = a(\theta) + b(\theta)\ln(r)$$

- $a(\theta)$ and $b(\theta)$ are chosen such that $\phi \in [0,1]$ in Ω

Numerical Benchmark



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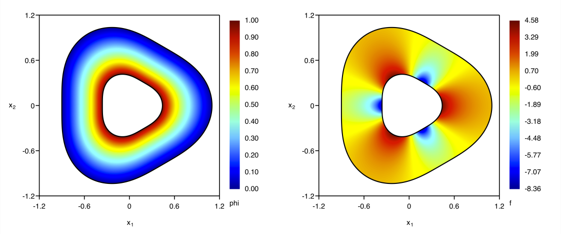
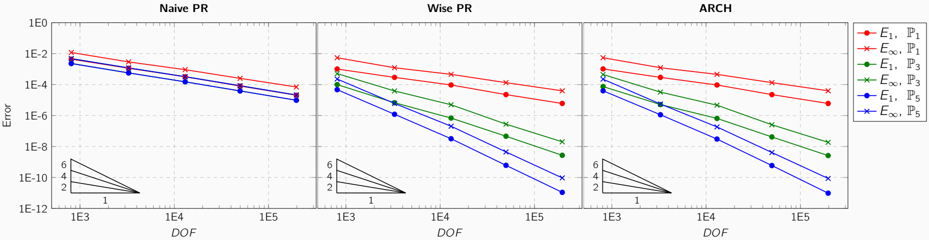
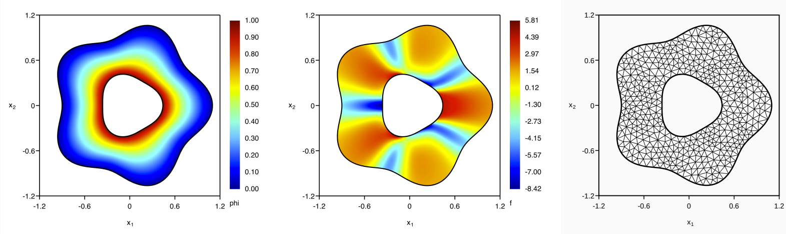
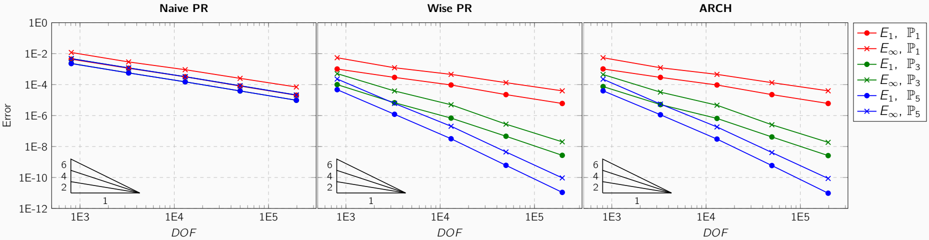


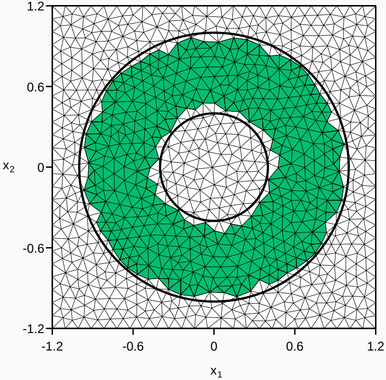


Numerical Benchmark





Numerical Benchmark | Mesh Pick-up



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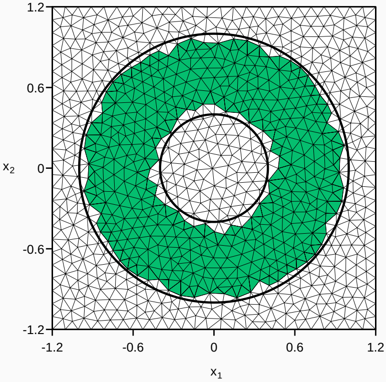

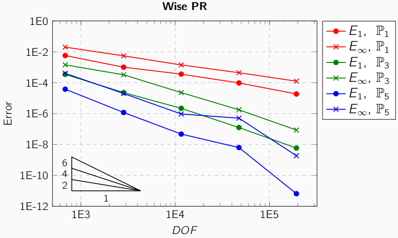

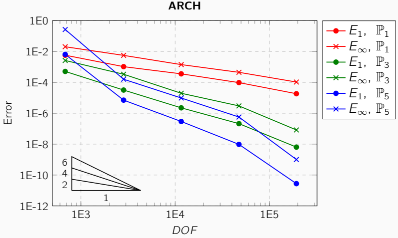

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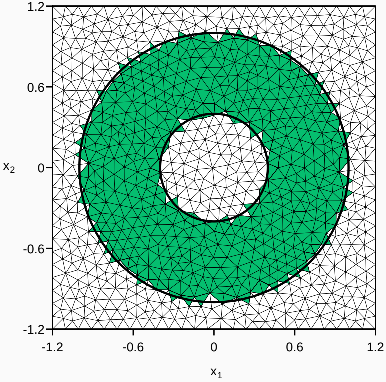

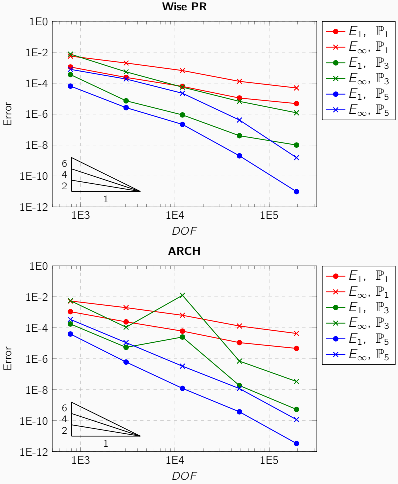

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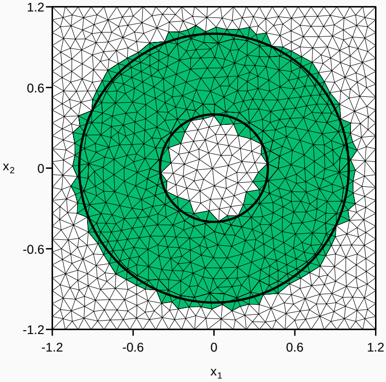

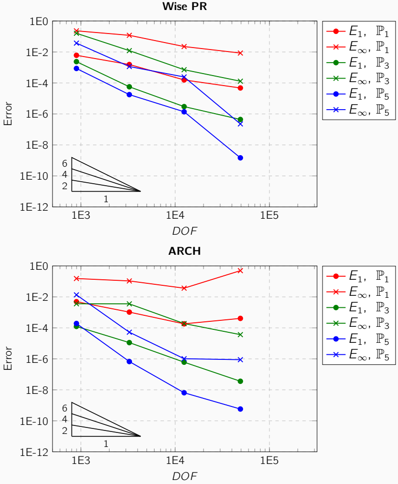

Conclusions and Final Remarks

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Obrigado