A high-order, admissibility and asymptotic-preserving finite volume scheme on 2D unstructured meshes

UNIVERSITÉ DE NANTES

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UMR 6629 - Nantes

SHARK-FV

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Outline

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Problematic

Hyperbolic systems of conservation laws with source term:

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

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$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

- $\mathcal{A}$: set of admissible states,
- $\mathbf{W} \in \mathcal{A} \subset \mathbb{R}^N$,
- $\mathbf{F}$: physical flux,
- $\gamma > 0$: controls the stiffness,
- $\mathbf{R}$: $\mathcal{A} \to \mathcal{A}$; smooth function with some compatibility conditions (Berthon, LeFloch, and Turpault [BLT13]).

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Hyperbolic systems of conservation laws with source term:

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

Under compatibility conditions on $\mathbf{R}$, when $\gamma t \to \infty$, (1) degenerates into a diffusion equation:

$$\partial_t w - \mathrm{div}\big(f(w)\nabla w\big) = 0 \tag{2}$$

- $w \in \mathbb{R}$, linked to $\mathbf{W}$,
- $f(w) > 0$.

Example: isentropic Euler equations with friction

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0 \\ \partial_t \rho \mathbf{u} + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = -\kappa \rho \mathbf{u} \end{cases}, \text{ with: } p'(\rho) > 0$$

$$\mathcal{A} = \{(\rho, \rho \mathbf{u})^T \in \mathbb{R}^3 / \rho > 0\}$$

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Formalism of (1):

- $\mathbf{W} = (\rho \quad \rho\mathbf{u})^T$
- $\mathbf{R}(\mathbf{W}) = (\rho \quad 0)^T$
- $\mathbf{F}(\mathbf{W}) = \left(\rho\mathbf{u} \quad \rho\mathbf{u} \otimes \mathbf{u} + p\mathbf{I}\right)^T$
- $\gamma(\mathbf{W}) = \kappa > 0$

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- $\gamma(\mathbf{W}) = \kappa > 0$

Limit diffusion equation:

$$\partial_t \rho - \mathrm{div}\left(\frac{p'(\rho)}{\kappa} \nabla \rho\right) = 0$$

Goal of an AP scheme

Conservation laws (1):

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W})$$

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Goal of an AP scheme

Conservation laws (1):
$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \xrightarrow{\gamma t \to \infty}$$

Diffusion equation (2):
$$\partial_t w - \mathrm{div}\big(f(w)\nabla w\big) = 0$$

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Conservation laws (1):

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consistent:

$\Delta t, \Delta x \to 0$

Numerical scheme

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$$
\begin{array}{ccc}
\text{Conservation laws (1):} & & \text{Diffusion equation (2):} \\
\partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) & \xrightarrow{\gamma t \to \infty} & \partial_t w - \operatorname{div}\big(f(w)\nabla w\big) = 0 \\
\Big\uparrow \text{consistent: } \Delta t, \Delta x \to 0 & & \\
\text{Numerical scheme} & \xrightarrow[\gamma t \to \infty]{} & \text{Limit scheme}
\end{array}
$$

Goal of an AP scheme

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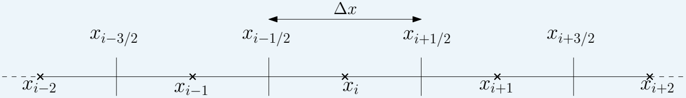
\begin{tikzcd}[column sep=large, row sep=huge]
\begin{array}{c}\text{Conservation laws (1):}\\ \partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W})\end{array} \arrow[r, "\gamma t \to \infty"] & \begin{array}{c}\text{Diffusion equation (2):}\\ \partial_t w - \operatorname{div}\big(f(w)\nabla w\big) = 0\end{array} \\
\text{Numerical scheme} \arrow[u, "\begin{array}{c}\text{consistent:}\\ \Delta t, \Delta x \to 0\end{array}"] \arrow[r, "\gamma t \to \infty"'] & \text{Limit scheme} \arrow[u, dashed, "\text{consistent?}"']
\end{tikzcd}
$$

Example of a non AP scheme in 1D



Naïve scheme in 1D:

$$\frac{\mathbf{W}_i^{n+1} - \mathbf{W}_i^n}{\Delta t} = -\frac{1}{\Delta x}\left(\boldsymbol{\mathcal{F}}_{i+1/2} - \boldsymbol{\mathcal{F}}_{i-1/2}\right) + \gamma(\mathbf{W}_i^n)(\mathbf{R}(\mathbf{W}_i^n) - \mathbf{W}_i^n)$$

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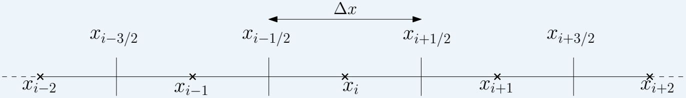

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Limit:

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} = \frac{b_{i+1/2}\Delta x(\rho_{i+1}^n-\rho_i^n) - b_{i-1/2}\Delta x(\rho_i^n-\rho_{i-1}^n)}{2\Delta x^2}$$

$$\not\to \partial_t \rho = \operatorname{div}\left(\frac{p'(\rho)}{\kappa}\nabla\rho\right)$$

Non-exhaustive state-of-the-art for AP schemes

1D meshes:

1. control of numerical diffusion:
   - telegraph equations: Gosse and Toscani [GT02],
   - M_1 model: [BD06], [BC07], [BCD07], ...
   - Euler with gravity and friction: [CCGRS10],
2. ideas of hydrostatic reconstruction to have AP properties for the Euler model with friction: [BOP07],
3. using convergence speeds and finite differences: [ABN16],
4. generalization of Gosse and Toscani: Berthon and Turpault [BT11].

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2D unstructured meshes:

1. MPFA based scheme: Buet, Després, and Franck [BDF12], with the Breil and Maire scheme [BM07] as limit,
2. SW with Manning-type friction: Duran, Marche, Turpault, and Berthon [DMTB15].
3. using the diamond scheme (Coudière, Vila, and Villedieu [CVV99]) for the limit scheme: Berthon, Moebs, Sarazin-Desbois, and Turpault [BMST16],

Outline

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What we want:

- for any 2D unstructured mesh,
- for any system of conservation laws which could be written as (1),
- under a 'hyperbolic' CFL condition:

$$\max_{K,i}\left(b_{K,i}\frac{\Delta t}{\Delta x}\right) \leq \frac{1}{2}$$

  - stability,
  - preservation of $\mathcal{A}$,
  - preservation of the asymptotic behaviour.

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Goal of the development

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How to do it:

- choose a proper limit scheme for (2),
- build a global scheme which degenerates into it,
  - extension of existing TP flux,
  - with a numerical diffusion correctly oriented.

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Choice of the limit scheme

FV scheme to discretize diffusion equations:

$$\partial_t w - \operatorname{div}(f(w)\nabla w) = 0. \quad (2)$$

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$$\partial_t w - \mathrm{div}(f(w)\nabla w) = 0. \quad (2)$$

Choice: scheme developed by Droniou and Le Potier in [DLP11]

- conservative and consistent with the diffusion equation on any mesh,
- satisfies the maximum principle and preserves $\mathcal{A}$,

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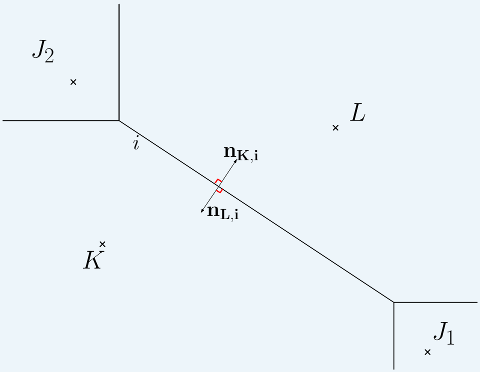
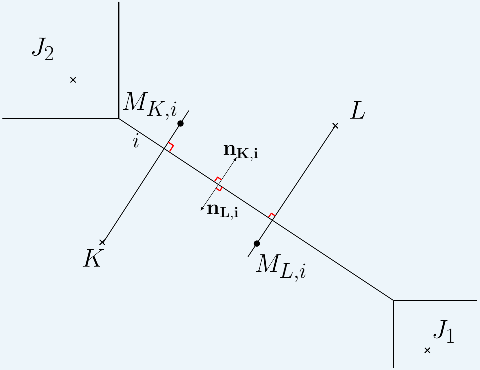
Choice: scheme developed by Droniou and Le Potier in [DLP11]

- conservative and consistent with the diffusion equation on any mesh,
- satisfies the maximum principle and preserves $\mathcal{A}$,
- nonlinear:

$$(f(w_K)\nabla_i w_K)\cdot \mathbf{n}_{K,i} \simeq \sum_{J\in\mathcal{S}_{K,i}} \overline{\nu}^J_{K,i}(w)(w_J - w_K),$$

  - $\mathcal{S}_{K,i}$ the set of points used for the reconstruction on edges i of cell K,
  - $\overline{\nu}^J_{K,i}(w) \geq 0$.


J_2
L
i
$\mathbf{n_{K,i}}$
$\mathbf{n_{L,i}}$
K
J_1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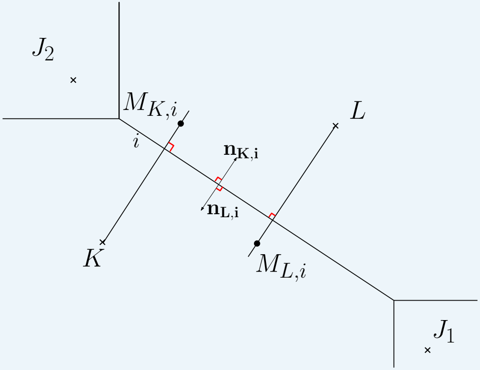
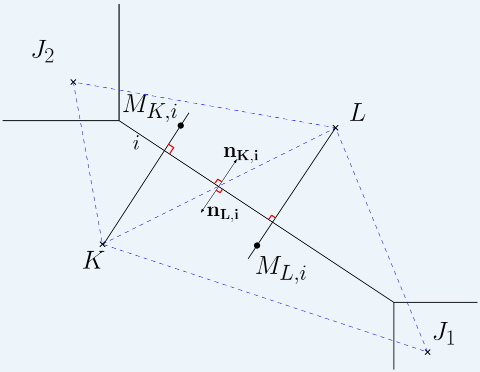


Two reconstructions:

$$\nabla_i w_K \cdot \mathbf{n}_{K,i} = \frac{w_{M_{K,i}} - w_K}{|KM_{K,i}|}$$

$$\nabla_i w_L \cdot \mathbf{n}_{L,i} = \frac{w_{M_{L,i}} - w_L}{|LM_{L,i}|}$$



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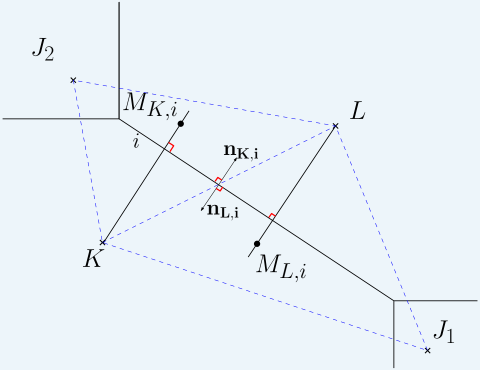

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Convex combination: $\theta_{K,i} + \theta_{L,i} = 1$, $\theta_{K,i} \geq 0$, $\theta_{L,i} \geq 0$

$$\begin{aligned}\nabla_i w_K \cdot \mathbf{n}_{K,i} &= \theta_{K,i}(w)\nabla_i w_K \cdot \mathbf{n}_{K,i} + \theta_{L,i}(w)\nabla_i w_L \cdot \mathbf{n}_{L,i} \\ &= \sum_{J \in \mathcal{S}_{K,i}} \overline{\nu}^J_{K,i}(w)(w_J - w_K), \text{ with : } \overline{\nu}^J_{K,i}(w) \geq 0\end{aligned}$$

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Outline

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Scheme for the hyperbolic part

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} \boldsymbol{\mathcal{F}}_i(\mathbf{W}_K, \mathbf{W}_L, \dots) \cdot \mathbf{n}_{K,i} \tag{3}$$

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Theorem

We assume that the conservative and consistent flux $\boldsymbol{\mathcal{F}}_i$ has the following properties:

1. *Combination:* $\exists\ \nu_{K,i}^J \geq 0,\ \boldsymbol{\mathcal{F}}_i \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{K,i}} \nu_{K,i}^J \boldsymbol{\mathcal{F}}_{KJ} \cdot \boldsymbol{\eta}_{KJ},$
2. *Technical hypothesis:* $\sum_{i \in \mathcal{E}_K} |e_i| \sum_{J \in \mathcal{S}_{K,i}} \nu_{K,i}^J \cdot \boldsymbol{\eta}_{KJ} = 0.$

Then the scheme (3) is stable, and preserves $\mathcal{A}$ under the classical following CFL condition:

$$\max_{\substack{K \in \mathscr{M} \\ J \in \overline{\mathcal{E}}_K}} \left(b_{KJ} \frac{\Delta t}{\delta_{KJ}} \right) \leq \frac{1}{2}. \tag{4}$$

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Example of fluxes (with Rusanov)

1. HLL-TP flux:

$$\boldsymbol{\mathcal{F}}_i(\mathbf{W}_K, \mathbf{W}_L) \cdot \mathbf{n}_{K,i} = \frac{\mathbf{F}(\mathbf{W}_K) + \mathbf{F}(\mathbf{W}_L)}{2} \cdot \mathbf{n}_{K,i} - b_{KL}(\mathbf{W}_L - \mathbf{W}_K)$$
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2. HLL-DLP flux:

$$\boldsymbol{\mathcal{F}}_i(\mathbf{W}) \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{K,i}} \boldsymbol{\nu}_{K,i}^J(\mathbf{W}) \boldsymbol{\mathcal{F}}_{KJ} \cdot \boldsymbol{\eta}_{KJ}$$

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But...

1. HLL-TP flux:
   - Fully respects the theorem,
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But...

1. HLL-TP flux:
   - Fully respects the theorem,
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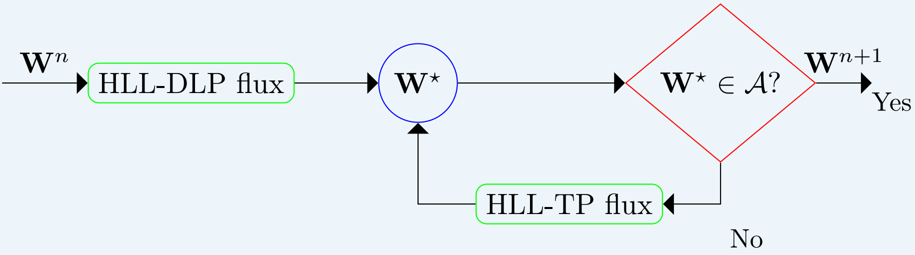
2. HLL-DLP flux:
   - Does not respect the technical hypothesis,
   - right numerical diffusion.

A posteriori procedure to preserve $\mathcal{A}$



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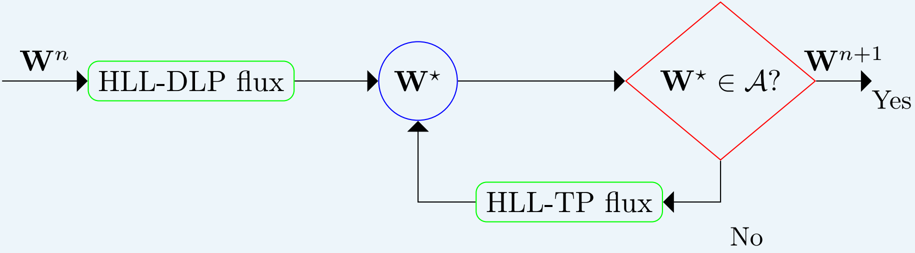

1. $\mathbf{W}^\star$ is computed with the HLL-DLP flux and with the CFL condition (4),
2. *Physical Admissiblility Detection* (PAD):
   - if $\mathbf{W}^\star \in \mathcal{A}$ then the time iterations can continue,
   - else, technical hypothesis 2 is enforced by using the HLL-TP flux on all not-admissible cells.

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Wind tunnel with step [WC84]

HLL-DLP on 1.7×10^6 cells
TP correction $< 1\%$

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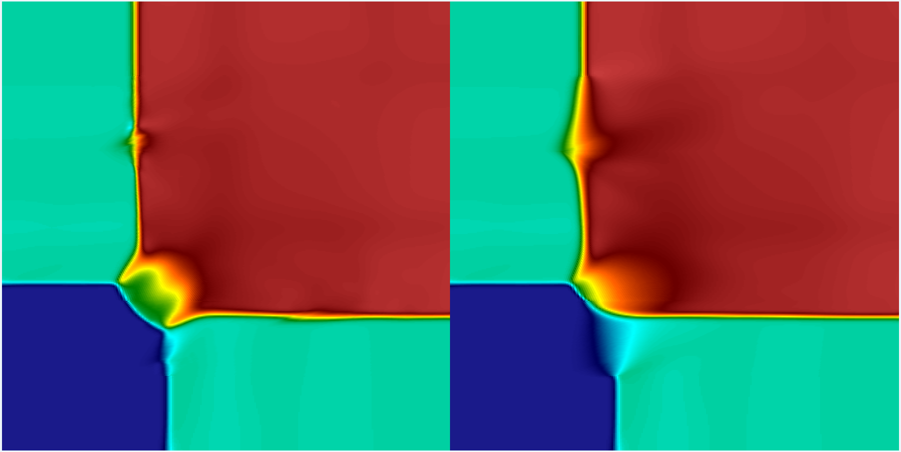
2D Riemann problems with four shocks [KT02]



HLL-DLP 1.5×10^5 cells
TP correction $< 1\%$

HLL-TP 1.5×10^5 cells

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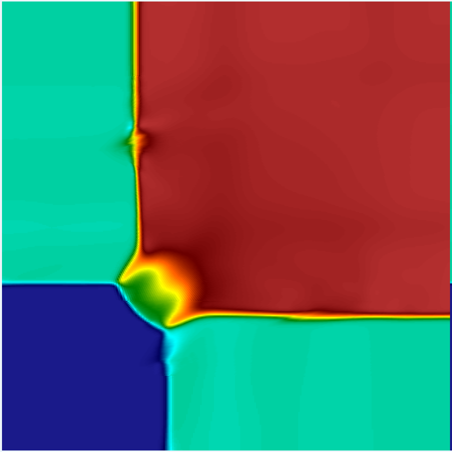

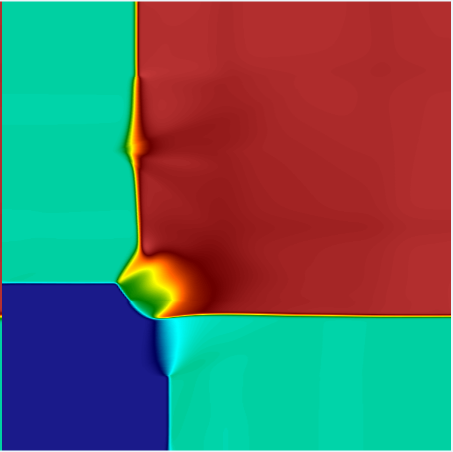

HLL-DLP 1.5×10^5 cells
TP correction $< 1\%$

HLL-TP 6×10^5 cells

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Full system:

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \quad (1)$$

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$$\partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} |e_i| \overline{\boldsymbol{\mathcal{F}}}_{K,i} \cdot \mathbf{n}_{K,i}, \tag{5}$$

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Construction of $\overline{\boldsymbol{\mathcal{F}}}_{K,i}$ with the technique of [BT11]:

$$\overline{\boldsymbol{\mathcal{F}}}_{K,i} \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{K,i}} \nu_{K,i}^J \overline{\boldsymbol{\mathcal{F}}}_{KJ} \cdot \boldsymbol{\eta}_{KJ}$$

Scheme for the full system

Full system:

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$$\begin{aligned} \overline{\boldsymbol{\mathcal{F}}}_{KJ} \cdot \boldsymbol{\eta}_{KJ} = \alpha_{KJ} \boldsymbol{\mathcal{F}}_{KJ} \cdot \boldsymbol{\eta}_{KJ} - (\alpha_{KJ} - \alpha_{KK}) \mathbf{F}(\mathbf{W}_K^n) \cdot \boldsymbol{\eta}_{KJ} \\ - (1 - \alpha_{KJ}) b_{KJ} (\mathbf{R}(\mathbf{W}_K^n) - \mathbf{W}_K^n) \end{aligned}$$

Full system:

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Construction of $\overline{\boldsymbol{\mathcal{F}}}_{K,i}$ with the technique of [BT11]:

$$\overline{\boldsymbol{\mathcal{F}}}_{K,i} \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{K,i}} \nu_{K,i}^J \overline{\boldsymbol{\mathcal{F}}}_{KJ} \cdot \boldsymbol{\eta}_{KJ}$$

$$\begin{aligned} \overline{\boldsymbol{\mathcal{F}}}_{KJ} \cdot \boldsymbol{\eta}_{KJ} = \alpha_{KJ} \boldsymbol{\mathcal{F}}_{KJ} \cdot \boldsymbol{\eta}_{KJ} - (\alpha_{KJ} - \alpha_{KK}) \mathbf{F}(\mathbf{W}_K^n) \cdot \boldsymbol{\eta}_{KJ} \\ - (1 - \alpha_{KJ}) b_{KJ} (\mathbf{R}(\mathbf{W}_K^n) - \mathbf{W}_K^n) \end{aligned}$$

$$\alpha_{KJ} = \frac{b_{KJ}}{b_{KJ} + \gamma_K \delta_{KJ}} \in [0; 1]$$

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} |e_i| \overline{\boldsymbol{\mathcal{F}}}_{K,i} \cdot \mathbf{n}_{K,i} \tag{5}$$

Theorem

The scheme (5) is consistent with the system of conservation laws (1), under the same assumptions of the previous theorem. Moreover, it preserves the set of admissible states $\mathcal{A}$ under the CFL condition:

$$\max_{\substack{K \in \mathscr{M} \\ J \in \overline{\mathcal{E}}_K}} \left(b_{KJ} \frac{\Delta t}{\delta_{KJ}} \right) \leq \frac{1}{2}. \tag{4}$$

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Scheme for the full model

- Is the scheme with the source term AP?

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Equivalent formulation:

$$\begin{aligned}
\partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) &= \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}), && (1)\\
&= \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) + (\overline{\gamma} - \overline{\gamma})\mathbf{W},\\
\partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) &= (\gamma(\mathbf{W}) + \overline{\gamma})(\overline{\mathbf{R}}(\mathbf{W}) - \mathbf{W}), && (6)
\end{aligned}$$

with: $\gamma(\mathbf{W}) + \overline{\gamma} > 0$ and $\overline{\mathbf{R}}(\mathbf{W}) := \dfrac{\gamma \mathbf{R}(\mathbf{W}) + \overline{\gamma}\mathbf{W}}{\gamma + \overline{\gamma}}$.

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Current limit scheme:

$$\rho_K^{n+1} = \rho_K^n + \sum_{i\in\mathcal{E}_K} \frac{\Delta t}{|K|}|e_i| \sum_{J\in\mathcal{S}_{K,i}} \nu_{K,i}^J \frac{b_{KJ}^2}{2(\kappa_K + \overline{\kappa}_{K,i}^J)\delta_{KJ}}\left(\rho_J - \rho_K\right).$$

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AP correction $\overline{\kappa}$:

$$\nu_{K,i}^J \frac{b_{KJ}^2}{2(\kappa_K + \overline{\kappa}_{K,i}^J)\delta_{KJ}} = \overline{\nu}_{K,i}^J \frac{p'(\rho)_i}{\kappa} \tag{7}$$

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Limit scheme with AP correction:

$$\rho_K^{n+1} = \rho_K^n + \sum_{i\in\mathcal{E}_K} \frac{\Delta t}{|K|}|e_i| \sum_{J\in\mathcal{S}_{K,i}} \overline{\nu}_{K,i}^J \frac{p'(\rho)_i}{\kappa}(\rho_J - \rho_K)$$

$$\longrightarrow \partial_t \rho - \operatorname{div}\left(\frac{p'(\rho)}{\kappa}\nabla\rho\right) = 0$$

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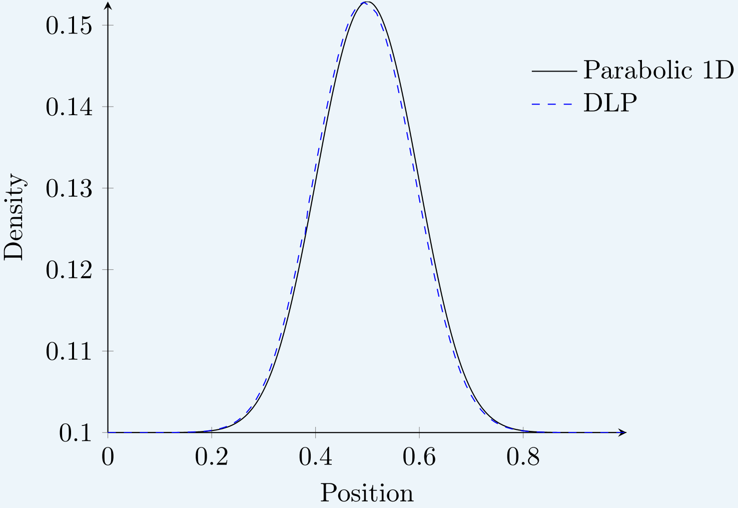
Outline

SHARK-FV

Comparison in the diffusion limit (pseudo 1D)

$$\rho_0(x,y) = 0.1\exp\left(\left(\frac{x-0.5}{0.01}\right)^2\right) + 0.1,\ \mathbf{u}=0,\ \kappa=2000,\ t_f=10,\ 1.5\times10^3 \text{ cells}$$



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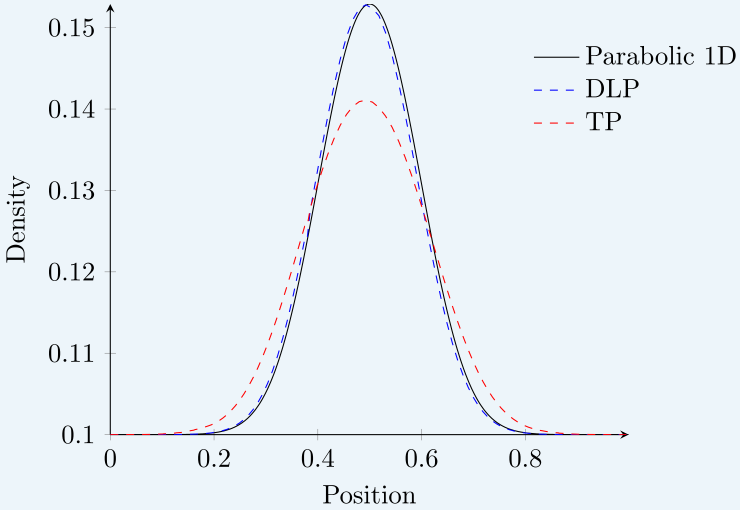

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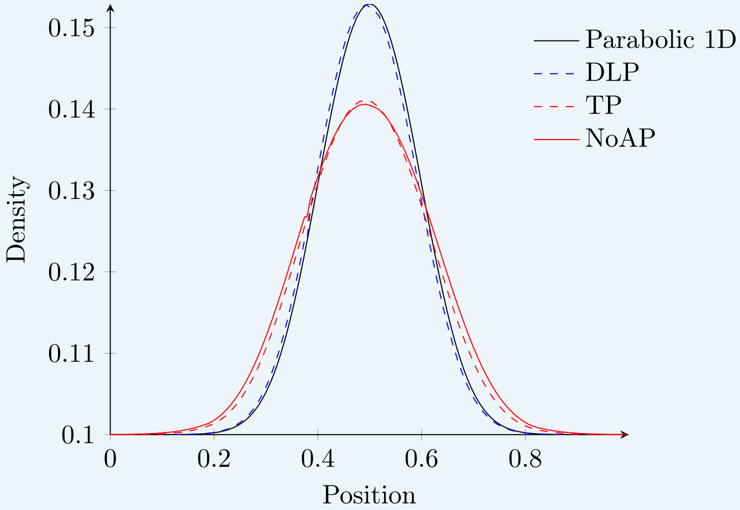

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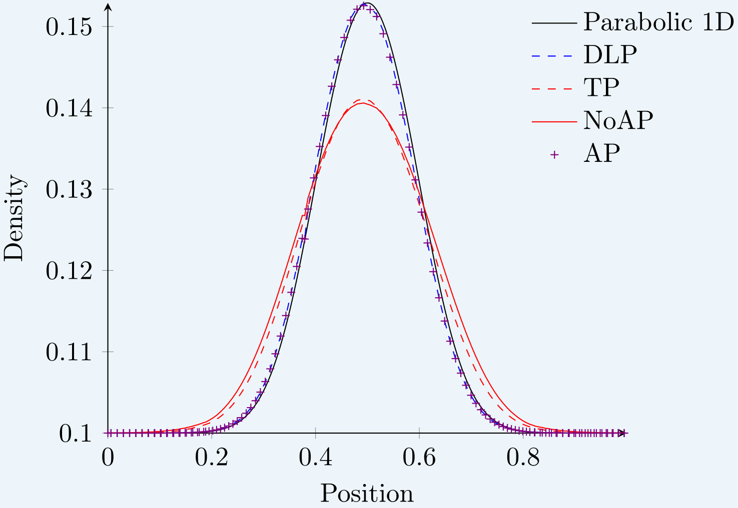


$$\rho_0(x,y) = \begin{cases} 1 & \text{if } (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 < 0.1^2 \\ 0.1 & \text{otherwise} \end{cases}, \ \mathbf{u} = 0,\ \kappa = 2000,\ t_f = 10,\ 9.4 \times 10^3 \text{ cells}$$

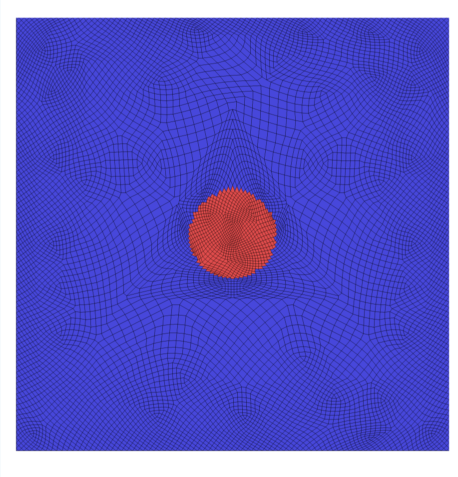
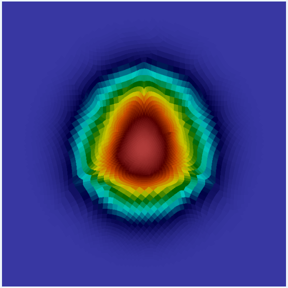
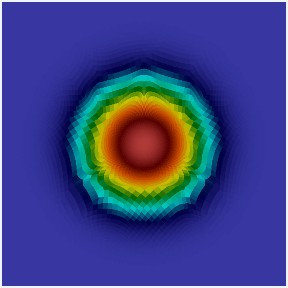


Figure: Mesh with privileged directions

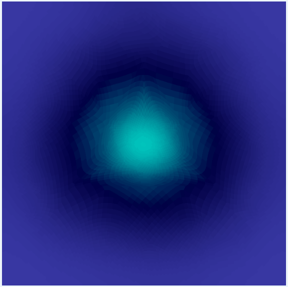
Comparison in the diffusion limit (2D)



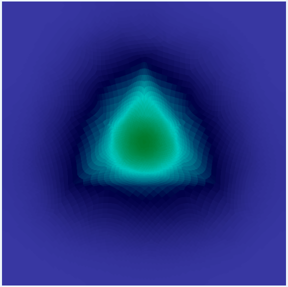
(a) HLL-DLP-AP



(b) DLP



(c) HLL-DLP-NoAP



(d) HLL-TP

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Outline

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High-order extension

Natural idea: MOOD ➔ Clain, Diot, and Loubère [CDL11]

- use a polynomial reconstruction of the solution $\widetilde{\mathbf{W}}_K(\boldsymbol{x})$,
- use the *a posteriori* limitation as in the TP flux correction.

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- α coefficients of Berthon and Turpault [BT11] ➔ limit to first order.

New convex combination:

$$\overline{\mathbf{W}}_K(\boldsymbol{x}) = \beta_K \widetilde{\mathbf{W}}_K(\boldsymbol{x}) + (1 - \beta_K)\mathbf{W}_K$$

$$\beta_K = \frac{\Delta_l}{\Delta_l + \gamma_K t \Delta x_K} \in [0; 1]$$

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Cascade of schemes

High-order HLL-DLP-AP with $\overline{\mathbf{W}}(x)$

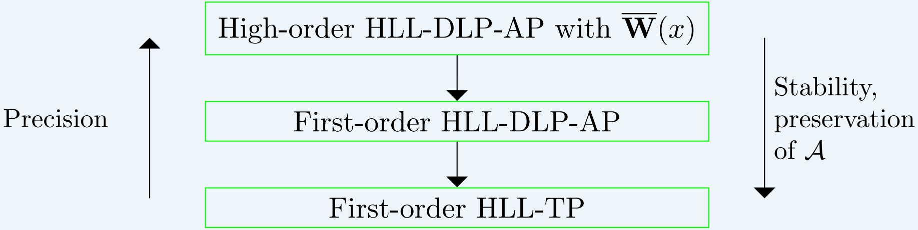
$\downarrow$

First-order HLL-DLP-AP

$\downarrow$

First-order HLL-TP

Cascade of schemes



Cascade of schemes

High-order HLL-DLP-AP with $\overline{\mathbf{W}}(x)$

$\downarrow$

First-order HLL-DLP-AP

$\downarrow$

First-order HLL-TP

Precision $\uparrow$

Stability, preservation of $\mathcal{A}$ $\downarrow$

No activation when $\gamma t \to \infty$

Step with nonlinear friction: $\kappa(\rho) = 10(\rho/7)^3$

HLL-DLP-AP-P0
on 1×10^6 cells

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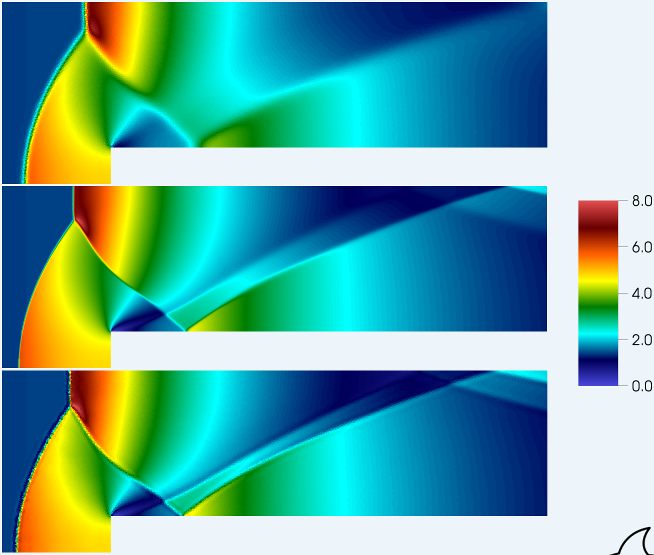
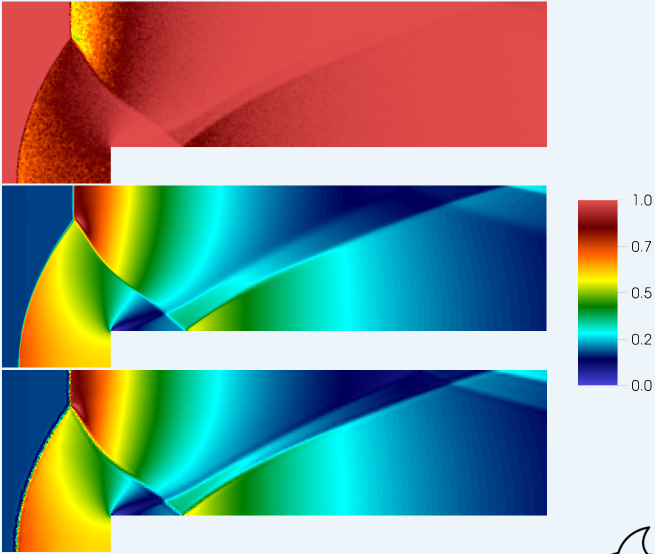
Step with nonlinear friction: $\kappa(\rho) = 10(\rho/7)^3$

HLL-DLP-AP-P0
on 4×10^4 cells

HLL-DLP-AP-P0
on 1×10^6 cells

HLL-DLP-AP-P1
on 4×10^4 cells





HLL-DLP-AP-P0
on 1×10^6 cells

HLL-DLP-AP-P1
on 4×10^4 cells

Outline

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Conclusion and perspectives

Conclusion

- generic theory for various hyperbolic problems with asymptotic behaviours,
- high-order scheme that preserve $\mathcal{A}$ and the asymptotic limit.

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- generic theory for various hyperbolic problems with asymptotic behaviours,
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Perspectives

- extend the limit scheme to take care of diffusion systems and more complex diffusion equation,
- full high-order scheme?

Thanks for your attention.

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