Stability of finite difference schemes for capillary thin films

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Introduction: laminar roll waves in laboratory

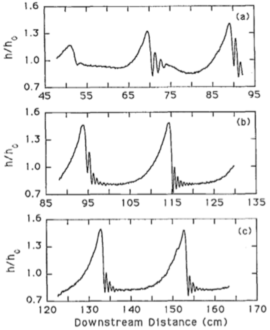


FIG. 3. The evolution of solitary waves forced at $f=1.5$ Hz with $\beta=6.4°$, $R=29$, and Weber number $W=35$. Three wave profiles are measured at increasing distances from the inlet to show the spatial evolution. Phase-sensitive averaging is employed here to reduce imaging noise.

Liu and Gollub experience
(Phys of Fluids 94)

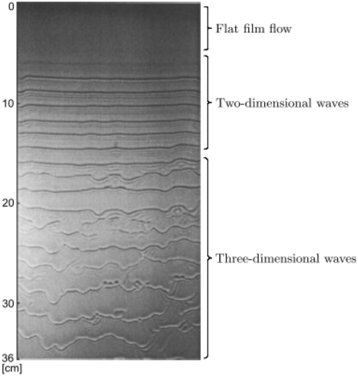
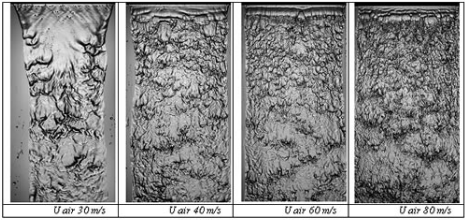
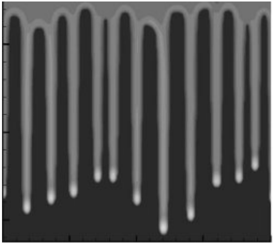


Photo of 2-d roll-waves
(Park et Nosoko AIChE, 2003)

Introduction: falling film in laboratory



Free surface instabilities
for different gas velocities (ONERA)



Fingering instabilities
and formation of drops

Outline of the talk

1. Modeling of thin film flows
   - Shallow water equations with surface tension
   - Related models: phase transition

2. Stability of difference approximations for shallow water eqs
   - Von Neumann (linearized) stability
   - Entropy stability (Schrödinger type formulation)
   - Two dimensional extension
   - Implicit strategies

3. Numerical simulations
   - Entropy stability: numerical comparison
   - Roll-waves: Liu Gollub experiment
   - Drops: wet/dry fronts

Thin films flow: shallow water equations I

- **General model:** Navier-Stokes (NS) equations with a free surface
  - **Unknowns:** velocity $\vec{u} = (u, w) \in \mathbf{R}^2$, pressure p, fluid domain $\Omega_t = \{(x, z), x \in \mathbf{R}^n, 0 \leqslant z \leqslant h(x, t)\}$
  - **Main issues:**: presence of a free surface, study of non linear waves (free surface instabilities)

- **Methodology**: under suitable assumptions, derive simpler models
  - **Aspect ratio**: $\varepsilon = H/L$, (characteristic fluid height/characteristic horizontal wavelengthl).
  - **Reynolds Number**: $Re = \rho HU/\mu$
  - **Froude Number**: $F^2 = U^2/gH$
  - **Weber Number**: $We = \rho U^2 H/\sigma$

Thin film flows: shallow water equations II

Definition (consistent models) Let $(\vec{u}_\varepsilon, p_\varepsilon, h_\varepsilon)$ be an exact solution of Navier-Stokes equations: $NS_\varepsilon(\vec{u}_\varepsilon, p_\varepsilon, h_\varepsilon) = 0$. Define $q_\varepsilon = \int_0^{h_\varepsilon} u_\varepsilon(.,z)dz$.

A shallow water model is

- consistent if $SV_\varepsilon(q_\varepsilon, h_\varepsilon) = R_\varepsilon$ and $\lim_{\varepsilon\to 0} \|R_\varepsilon\| = 0$,
- of order k if $\|R_\varepsilon\| = O(\varepsilon^k)$: order 1 (1998), order 2 (2001)!

Exemple of first order consistent model (P.N., J.-P. Vila)

$$\partial_t h + \partial_x q = 0,$$
$$\partial_t q + \partial_x\left(\frac{q^2}{h} + \frac{h^5}{45} + \frac{5\mathrm{cotan}(\theta)h^2}{12\,R_e}\right) = \frac{5}{6\varepsilon R_e}\left(h - \frac{3q}{h^2}\right) + \varepsilon^2 We\, h\partial_{xxx}h. \quad (1)$$

Remark: in Liu-Gollub experiments, $\varepsilon^2 We = O(1)$!

Related models I: Euler Korteweg equations

Remark: neglecting source term, shallow water equations with surface tension are a particular case of Euler Korteweg equations

Euler-Korteweg equations in conservative variables

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x \left(\rho u^2 + P(\rho)\right) = \partial_x \left(\rho\kappa(\rho)\partial_{xx}\rho + (\rho\kappa'(\rho) - \kappa(\rho))\frac{(\partial_x\rho)^2}{2}\right),$$

- $\kappa(\rho) = \mathit{constant}$, $P(\rho) = a\rho^\gamma$: shallow water type equations
- $\kappa(\rho) = \mathit{constant}/\rho$: quantum hydrodynamic (=NLS)
- $\kappa(\rho) = \mathit{constant}$, $P(\rho) = \dfrac{\gamma\rho}{1-\rho} - \rho^2$: Van der Waals gas (phase transition)

Additional Energy equation

$$\partial_t \left(\rho\frac{u^2}{2} + F(\rho) + \kappa(\rho)\frac{(\partial_x\rho)^2}{2}\right) + \partial_x\mathcal{F}(\rho, u, \partial_x\rho, \partial_x u) = 0$$

Related models II: water waves

- **General model**: Euler equations with a free surface (incompressible, irrotational)
- **Unknowns**: velocity $\vec{u} = (u, w) \in \mathbf{R}^2$, pressure p, fluid domain $\Omega_t = \{(x, z), x \in \mathbf{R}^n, -h(x) \leqslant z \leqslant \eta(t, x)\}$
- **Main issues**: presence of a free surface, no regularization effects
- **Non dimensional numbers**: $\sigma = \dfrac{H}{\lambda}$ (dispersion), $\varepsilon = \dfrac{a}{H}$ (nonlinearity)

Boussinesq equations

$$\partial_t \eta + \partial_x((h + \varepsilon\eta)) = 0,$$

$$\partial_t \bar{u} + \varepsilon \bar{u} \partial_x \bar{u} + \partial_x \eta + \sigma^2 \left(\frac{h^2}{6} \partial_x^2 (\partial_t \bar{u}) - \frac{h}{2} \partial_x^2 (h \partial_t \bar{u}) \right) = O(\varepsilon\sigma^2 + \sigma^4).$$

Stability of difference schemes: von Neumann stability

- **Remark**: due to the presence of the third order derivative, the energy equation is hardly satisfied in the original formulation
- **A simplified problem**: we check stability for linearized shallow water equations (=Fourier analysis)
- **Interest:** provides necessary and, in practice, sufficient condition of stability

Linearized equations (conservative variables: $v = (h, q)^T$)

$$\partial_t v + A\partial_x v = B\partial_{xxx} v, \quad A = \begin{pmatrix} 0 & 1 \\ \bar{c}^2 - \bar{u}^2 & 2\bar{u} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \bar{\sigma} & 0 \end{pmatrix}.$$

- **Dispersion relation:** $s(k) = \bar{u} \pm \sqrt{\bar{c}^2 + \bar{\sigma} k^2}$
- **Heuristic CFL condition** $s(k)\frac{\delta t}{\delta x} \leqslant 1$. Here $s(k) \sim K/\delta x$ then CFL condition: $\delta t = O(\delta x^2)$.

Von Neumann stability I: formulation of the problem

Stability of difference approximation in the form

$$v_i^{n+1} - v_i^n + \lambda_1 \left(f_{i+\frac{1}{2}}^{n+\theta} - f_{i-\frac{1}{2}}^{n+\theta} \right) = \lambda_3 B \left(v_{i+2}^{n+\theta} - 2v_{i+1}^{n+\theta} + 2v_{i-1}^{n+\theta} - v_{i-2}^{n+\theta} \right). \quad (2)$$

with $\lambda_k = \delta t / \delta x^k$, and $v_i^{n+\theta} = (1-\theta) v_i^n + \theta v_i^{n+1}$.

- Lax-Friedrichs scheme: $f_{i+\frac{1}{2}}^n = \dfrac{Av_i^n + Av_{i+1}^n}{2} - \dfrac{1}{2\lambda_1}(v_{i+1}^n - v_i^n)$
- Rusanov scheme: $f_{i+\frac{1}{2}}^n = \dfrac{Av_i^n + Av_{i+1}^n}{2} - \dfrac{\rho(A)}{2}(v_{i+1}^n - v_i^n)$
- Roe scheme: $f_{i+\frac{1}{2}}^n = \dfrac{Av_i^n + Av_{i+1}^n}{2} - \dfrac{|A|}{2}(v_{i+1}^n - v_i^n)$

Von Neumann stability II: first order accurate schemes

Definition

We search for solutions of (2) in the form $v_k^n = \xi^n e^{-ik\theta}$: a scheme is stable in the sense of Von Neumann if $|\xi| \leqslant 1$ for all $\theta \in [0, 2\pi]$

- **Instability of Roe scheme:** The scheme (2) with Roe type flux and $\theta = 0$ (forward Euler time discretization: FE), $\theta = 1$ (backward Euler time discretization: BE) is always unstable: **the equivalent system of PDEs is ill posed** (bad interaction between numerical viscosity and third order terms).

- **Stability of Lax-Friedrichs scheme:**
  - FE time discretization ($\theta = 0$): stable under cfl condition $\delta t = O(\delta x^2)$
  - BE time discretization ($\theta \geqslant 1/2$): inconditionally stable

- **Stability of Rusanov scheme:**
  - FE time discretization ($\theta = 0$): stable under cfl condition $\delta t = O(\delta x^3)$
  - BE time discretization ($\theta \geqslant 1/2$): inconditionally stable

Von Neumann stability III: second order accurate schemes

We use a MUSCL type scheme for space discretization:

$$\begin{aligned} \frac{dv_j}{dt} &- \frac{A}{8\delta x}(v_{j+2} - 6v_{j+1} + 6v_{j-1} - v_{j-2}) \\ &+ \frac{\nu_n}{8\delta x^2}\left(v_{j+2} - 4v_{j+1} + 6v_j - 4v_{j-1} + v_{j-2}\right) \\ &= \frac{B}{\delta x^3}\left(v_{j+2} - 2v_{j+1} + 2v_{j-1} - v_{j-2}\right). \end{aligned}$$

Remark: ν_n is the numerical viscosity (L-F: $\nu_n = \delta x^2/2\delta t$, Ru: $\nu_n = \rho(A)\delta x$)

- **Stability of Lax-Friedrichs scheme:**
  - Runge Kutta 2 : stable under CFL condition $\delta t = O(\delta x^2)$
  - Crank Nicolson ($\theta = 1/2$): inconditionally stable
- **Stability of Rusanov scheme:**
  - Runge Kutta 2 ($\theta = 0$): stable under CFL condition $\delta t = O(\delta x^{7/3})$
  - Crank Nicolson ($\theta = 1/2$): inconditionally stable

Well posedness of Equivalent Equations I

Modified or equivalent equation

$$v_t + Av_x = Qv_{xx} + Bv_{xxx} \tag{3}$$

Well posedness with initial data in L^2_p via Fourier Analysis :
$v(x,t) = e^{-ix\xi}\hat{v}(t), \quad \frac{d\hat{v}}{dt} = i\xi\left(A + i\xi Q + \xi^2 B\right)\hat{v}$

- well posedness requires eigenvalues X of $\left(A + i\xi Q + \xi^2 B\right)$ satisfies $\xi Im(X) \geqslant 0 \ \forall \xi \in \mathbb{R}$
- Scalar continuous case : $\hat{v}(t) = e^{i\xi\left(a+\xi^2\bar{\sigma}\right)t}e^{-\xi^2 qt}\hat{v}(0)$ The problem is well posed for initial data in L^2_p iff $q > 0$
- System case

$$A = \begin{pmatrix} 0 & 1 \\ \bar{c}^2 - \bar{u}^2 & 2\bar{u} \end{pmatrix},\ Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ \bar{\sigma} & 0 \end{pmatrix},$$

Theorem

Viscosity matrices are admissible (ie (3) is well posed with initial data in L^2_p) iff $q_{12} = 0$ and $c^2 \geqslant \frac{\left(\bar{u}(q_{11}-q_{22})-q_{21}\right)^2}{\left(q_{11}+q_{22}\right)^2},\ q_{11} + q_{22} \geqslant 0,\ \ \bar{\sigma} + q_{11}q_{22} \geqslant 0$

Well posedness of equivalent Equations II

- Remark : Continuous case with $q_{11} = 0$. The problem is ill posed unless $q_{12} = 0$ and $c^2 \geqslant \frac{(\bar{u}q_{22}+q_{21})^2}{q_{22}^2}$, $q_{22} \geqslant 0$, and $\bar{\sigma} \geqslant 0$
- Proof relies on explicit formulae for Eigenvalues X : for large ξ (high frequ.) X satisfies

$$
\begin{aligned}
2X &= (2\bar{u} + i\xi\,(q_{11} + q_{22})) \\
&\pm\sqrt{4iq_{12}\bar{\sigma}}\left(\xi^{\frac{3}{2}} + \frac{-i\sqrt{\xi}}{2\xi q_{12}\bar{\sigma}}\,(q_{11}q_{22} + \bar{\sigma} - q_{12}q_{21}) + O\left(\xi^{-\frac{1}{2}}\right)\right)
\end{aligned}
$$

- Modified equation for Godunov/Roe scheme $Q = |A|$

$$
|A| = \begin{bmatrix} \dfrac{|u-c|\,(u+c) - |u+c|\,(u-c)}{2c} & \dfrac{|u+c| - |-u+c|}{2c} \\ \left(c^2 - u^2\right)\dfrac{|u+c| - |-u+c|}{2c} & \dfrac{|u+c|\,(u+c) - |u-c|\,(u-c)}{2c} \end{bmatrix}
$$

Entropy stability of difference schemes: new formulation 1

"Entropy" of the Euler-Korteweg system

$$U(\rho, u, \partial_x \rho) = \int \rho \frac{u^2}{2} + F(\rho) + \kappa(\rho) \frac{(\partial_x \rho)^2}{2}$$

- Not an usual entropy (presence of $\partial_x \rho$): reduction of order needed (see C.W. Shu for KdV type equations with DG methods)
- A natural new variable: $w = \sqrt{\frac{\kappa(\rho)}{\rho}} \partial_x \rho$
- The "entropy" U now reads $U(\rho, u, w) = \int \rho \frac{u^2 + w^2}{2} + F(\rho).$
- **Remark:** a strategy used for compressible Navier-Stokes equations ("Bresch-Desjardins" entropy) to define new weak solutions.

Entropy stability of difference schemes: new formulation 2

Euler-Korteweg equations: "Schrodinger type formulation"

$$\partial_t v + \partial_x f(v) = \partial_x(B(\rho)\partial_x(\rho^{-1}v)), \quad B(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu(\rho) \\ 0 & -\mu(\rho) & 0 \end{pmatrix} \quad (4)$$

with $v = (\rho, \rho u, \rho w)^T$, $f(v) = (\rho u, \rho u^2 + P(\rho), \rho u w)^T$.

- The Schrodinger formulation is obtained by setting $\psi = \rho u + i\rho w$ (useful for well posedness: see Benzoni-Danchin-Descombes 2006)
- Setting $U(v) = \rho\frac{u^2+w^2}{2} + F(v)$ and $G(v) = u(U(v) + P(\rho))$:

Energy equation in the new formulation (classic energy estimate)

$$\partial_t U(v) + \partial_x G(v) = \partial_x \left(\mu(\rho)(u\partial_x w - w\partial_x u)\right). \quad (5)$$

Entropy stability of difference scheme: definition

We consider the following semi discretized system (setting $z = \rho^{-1} v$)

$$\frac{d}{dt} v_j(t) + \frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{\delta x} = \frac{B(\rho_{j+\frac{1}{2}})\,(z_{j+1} - z_j) - B(\rho_{j-\frac{1}{2}})\,(z_j - z_{j-1})}{\delta x^2}. \qquad (6)$$

Definition

The semi-discretized scheme (6) is entropy stable if there exists a numerical flux $\mathcal{G}_{j+\frac{1}{2}}$, consistent with the entropy flux in (5), so that

$$\frac{d}{dt} U(v_j(t)) + \frac{\mathcal{G}_{j+\frac{1}{2}} - \mathcal{G}_{j-\frac{1}{2}}}{\delta x} \leqslant 0.$$

E. Tadmor *Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems* Acta Numerica (2003)

P.G. LeFloch, J.M. Mercier, C. Rohde *Fully discrete, entropy conservative schemes of arbitrary order*, SIAM J. Numer. Anal. 40 (2002)

C. Chalons, P.G. LeFloch *High-Order Entropy-Conservative Schemes and Kinetic Relations for van der Waals Fluids*, JCP 168 (2001).

Entropy stability: semi-discrete schemes

Theorem

Consider the entropy stable scheme

$$\frac{d}{dt}v_j(t) + \frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{dx} = 0, \tag{7}$$

which is a difference approximation of (4) *with* $B = 0$, *then the associated difference scheme* (6) *is an entropy stable difference scheme.*

Proof The scheme is entropy stable:

$$U_v(v_j(t))^T(f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}) = \mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j+\frac{1}{2}} + \mathcal{R}_j, \quad \mathcal{R}_j \geqslant 0$$

Moreover, setting $\mathcal{K}_j = \langle U_v(v_j(t)); \text{r.h.s of } (5)\rangle$ one has

$$dx^2\mathcal{K}_j = \mu_{j+\frac{1}{2}}(u_j w_{j+1} - u_{j+1}w_j) - \mu_{j-\frac{1}{2}}(u_{j-1}w_j - u_j w_{j-1})$$

Entropy stability: fully-discrete schemes I

We consider only first order accurate schemes

Backward Euler time discretization

$$v_j^{n+1} - v_j^n + \lambda_1 \left(f_{j+\frac{1}{2}}^{n+1} - f_{j-\frac{1}{2}}^{n+1} \right) = \lambda_2 \left(B(\rho_{j+\frac{1}{2}}^{n+1}) \left(z_{j+1}^{n+1} - z_j^{n+1} \right) - B(\rho_{j-\frac{1}{2}}^{n+1}) \left(z_j^{n+1} - z_{j-1}^{n+1} \right) \right). \quad (8)$$

Forward Euler time discretization

$$v_j^{n+1} - v_j^n + \lambda_1 \left(f_{j+\frac{1}{2}}^{n} - f_{j-\frac{1}{2}}^{n} \right) = \lambda_2 \left(B(\rho_{j+\frac{1}{2}}^{n}) \left(z_{j+1}^{n} - z_j^{n} \right) - B(\rho_{j-\frac{1}{2}}^{n}) \left(z_j^{n} - z_{j-1}^{n} \right) \right). \quad (9)$$

with $f_{j+\frac{1}{2}}$ corresponding to a semi discrete entropy stable scheme.

Entropy stability: fully discrete scheme II

Theorem

Implicit Schemes Consider the entropy (spatially) stable semi scheme (7) *which is a difference approximation of* (4) *with* $B = 0$*, then the scheme* (8) *is (unconditionally) entropy stable. There exists* $\mathcal{G}^n_{j+\frac{1}{2}}$ *so that*

$$U(v_j^{n+1}) - U(v_j^n) + \mathcal{G}^n_{j+\frac{1}{2}} - \mathcal{G}^n_{j-\frac{1}{2}} \leqslant 0, \forall j, \quad \forall n. \tag{10}$$

Theorem

Explicit Schemes

- *Explicit scheme with Lax-Friedrichs flux is entropy stable with CFL* $\delta t \ll \delta x^2$
- *Explicit scheme with Rusanov flux is entropy stable with CFL* $\delta t \ll \delta x^3$

- **Question 1:** Two dimensional extension?
- **Question 2:** Implicit strategies (to get rid of CFL conditions $\delta t = o(\delta x^2)$)?

Two dimensional extensions I

Two-dimensional Shallow Water equations

$$\partial_t h + \mathrm{div}(h\mathbf{u}) = 0,$$

$$\partial_t(h\mathbf{u}) + \mathrm{div}\left(h\mathbf{u}\otimes\mathbf{u} + \frac{2h^5}{225}\left(\frac{g\sin(\theta)}{\nu}\right)^2 \mathbf{e}_1\otimes\mathbf{e}_1\right) + \nabla(g\cos(\theta)\frac{h^2}{2}) =$$

$$gh\sin(\theta)\mathbf{e}_1 - 3\nu\frac{\mathbf{u}}{h} + \frac{\sigma}{\rho}h\nabla\Delta h.$$

Multi-dimensional Euler-Korteweg equations

$$\partial_t\varrho + \mathrm{div}(\varrho\mathbf{u}) = 0, \qquad \partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\varrho) = \mathrm{div}\mathbf{K},$$

$$\mathbf{K} = \left(\varrho\mathrm{div}(K(\varrho)\nabla\varrho) + \frac{1}{2}(K(\varrho) - \varrho K'(\varrho))|\nabla\varrho|^2\right)\mathrm{Id}_{\mathbb{R}^n} - K(\varrho)\nabla\varrho\otimes\varrho.$$

Two dimensional extensions II: (new) extended formulation

Introduce $\mathbf{w} = \nabla\phi(\varrho)$ with $\phi'(\rho) = \sqrt{\frac{K(\varrho)}{\varrho}}$ and $F'(\varrho) = \varrho\phi'(\varrho)$.

Extended formulation of Euler Korteweg equations

$$\begin{aligned}
&\partial_t\varrho + \mathrm{div}\,(\varrho\mathbf{u}) = 0,\\
&\partial_t(\varrho\mathbf{u}) + \mathrm{div}\,(\varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathrm{I}_{\mathbf{R}^n}) = \mathrm{div}(F(\varrho)\nabla\mathbf{w}^T) - \nabla\left((F(\varrho) - \varrho F'(\varrho))\mathrm{div}(\mathbf{w})\right),\\
&\partial_t(\varrho\mathbf{w}) + \mathrm{div}\,(\varrho\mathbf{w}\otimes\mathbf{u}) = -\mathrm{div}(F(\varrho)\nabla\mathbf{u}^T) + \nabla\left((F(\varrho) - \varrho F'(\varrho))\mathrm{div}(\mathbf{u})\right).
\end{aligned}$$

Entropy: $U(\varrho,\mathbf{u},\mathbf{w}) = \varrho F_0(\varrho) + \frac{\varrho}{2}\left(\|\mathbf{u}\|^2 + \|\mathbf{w}\|^2\right)$

$$\begin{aligned}
\partial_t U(\varrho,\mathbf{u},\mathbf{w}) \quad + \quad & \mathrm{div}\,(\mathbf{u}(U(\varrho,\mathbf{u},\mathbf{w}) + p(\varrho))) = \mathrm{div}(F(\varrho)(\nabla\mathbf{w}\mathbf{u} - \nabla\mathbf{u}\mathbf{w}))\\
& -\mathrm{div}\left((F(\varrho) - \varrho F'(\varrho))(\mathrm{div}(\mathbf{w})\mathbf{u} - \mathrm{div}(\mathbf{u})\mathbf{w})\right).
\end{aligned}$$

Remark: Under suitable compatibility conditions for the discretization of div and ∇ operators, we prove similar energy estimates than in $1d$ case.
Restriction: Entropy quadratic w.r.t. $\nabla\varrho$.

Implicit strategies

- Explicit in time discretization requires a CFL condition $\delta t = o(\delta x^2)$
- Full implicit in time schemes implies heavy computational costs (especially in 2d).

Implicit(surface tension)/Explicit (convection) time discretization

$$\begin{aligned} v_j^{n+1} - v_j^n + \lambda_1 \left(f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}} \right) \\ = \lambda_2 \left(B(\rho^{n+1}_{j+\frac{1}{2}}) \left(z^{n+1}_{j+1} - z^{n+1}_j \right) - B(\rho^{n+1}_{j-\frac{1}{2}}) \left(z^{n+1}_j - z^{n+1}_{j-1} \right) \right). \end{aligned}$$

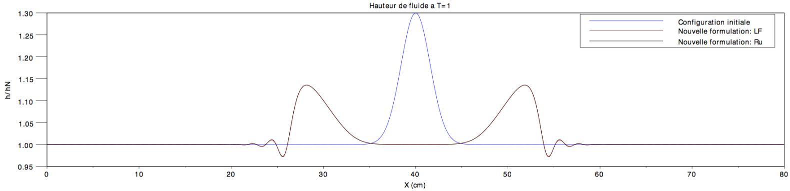
- Stable under CFL condition $\delta t = O(\delta x)$.
- Implicit steps amounts to solve (sparse) linear systems.
- Higher order time discretization: IMEx strategies

Entropy stability: numerical comparison I

- Model: shallow water equations with horizontal bottom

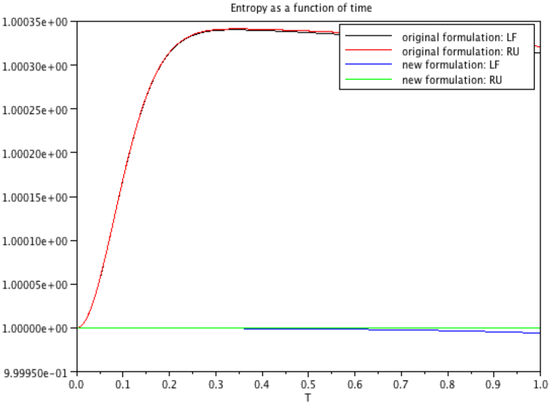
$$\partial_t h + \partial_x(hu) = 0, \quad \partial_t(hu) + \partial_x(hu^2 + g\frac{h^2}{2}) = \frac{\sigma}{\rho} h\partial_{xxx} h.$$

- Periodic boundary conditions

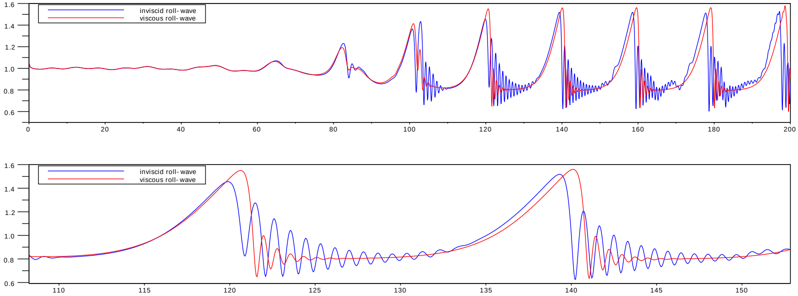


Entropy stability: numerical comparison II

- Comparison of the original formulation and the "new" formulation
- Second order schemes for numerical simulations
- Conclusion: the new formulation provides a better entropy conservation



Simulation of Liu Gollub experiment (Phys of Fluids 94)

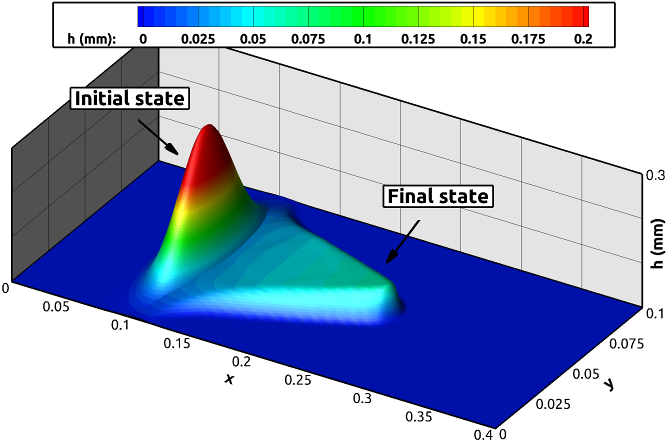


- The viscous term is heuristic
- Numerical scheme: RK2/Rusanov (2nd order) on the extended formulation.
- Reynolds number $Re = 29$, Inclination $\theta = 6.4^o$, Weber number $We = 35$.

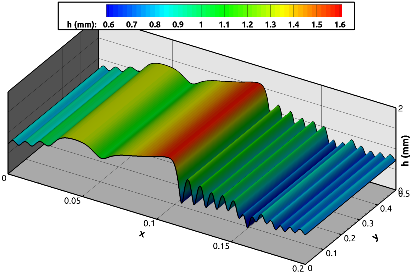
Numerical Simulations

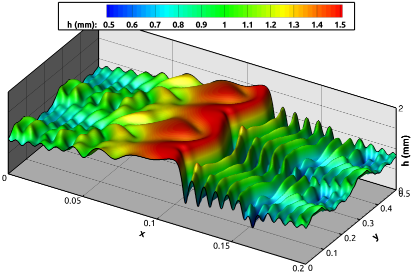
- Numerical simulation of shallow water equations (consistent models)
- 1d and 2d simulations: IMEx strategies+Extended formulations
- Falling films: roll waves and drop (wet/dry front with precursor film)
- Remark (MUSCL reconstruction): **the flux limiters does not “kill” surface tension effects.**

Simulation of drop motion



Simulation of Liu Gollub experiment (Phys of Fluids 94)





Conclusion

1. Summary
   - Proof of entropy stability with a new form of Euler-Korteweg equations
   - Numerically: extended formulation is more stable than original formulation and provides a natural implicit discretization (CFL $\delta t = O(\delta x)$).
   - Ref : Noble Vila SINUM 2014 Vol. 52, No. 6, pp. 2770 2791 and Bresch Couderc Noble Vila http://arxiv.org/abs/1503.08678
2. Open problems
   - Other dispersive models (water wave models/ bi-fluid models) ?
   - Derivation of suitable boundary conditions?
   - Higher order methods (Discontinous Galerkin methods)?