The MOOD method for steady-state Euler equations

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MOTIVATION

- to solve steady-state hyperbolic equations using finite volume schemes
- prototypes: Burgers' equation (scalar case) and Euler's equation (vectorial case)
- regular solutions: high accuracy
- solutions with a shock: stability (no oscillations)
- approach: MOOD (Multidimensional Optimal Order Detection)

inviscid Burgers' equation

- we seek the velocity function $\phi = \phi(x)$, solution of the 1D steady-state inviscid Burgers' equation

$$\frac{\mathrm{d}F(\phi)}{\mathrm{d}x} = f, \text{ in } \Omega = (0, 1)$$

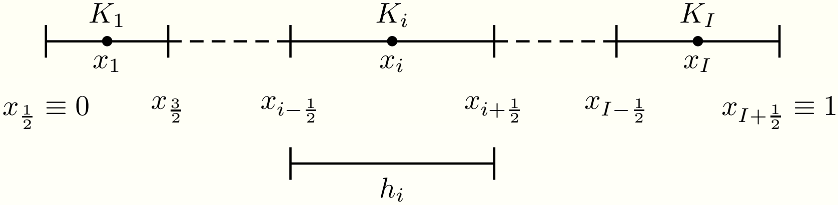
- with Dirichlet boundary conditions

$$\phi = \phi_{\mathrm{lf}}, \text{ on } x = 0$$
$$\phi = \phi_{\mathrm{rg}}, \text{ on } x = 1$$

- with

$$F(\phi) = \frac{\phi^2}{2}$$
$$f = f(x)$$

NOTATION



- K_i — cell i
- I — number of cells
- $x_{i-\frac{1}{2}}$, $x_{i+\frac{1}{2}}$ — boundary points of cell i
- h_i — length of cell i
- x_i — centroid of cell i

FV scheme (I)

- integrating equation $\frac{\mathrm{d}F(\phi)}{\mathrm{d}x} = f$ over cell K_i results in

  $$\frac{1}{h_i}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) - \bar{f}_i = 0$$

  with

  $$F_{i+\frac{1}{2}} = F(\phi(x_{i+\frac{1}{2}}))$$
  $$\bar{f}_i = \frac{1}{h_i}\int_{K_i} f(\xi)\,\mathrm{d}\xi$$

- let

  $$\mathcal{F}_{i+\frac{1}{2}} \approx F_{i+\frac{1}{2}}$$
  $$f_i \approx \bar{f}_i$$

  the residual at cell K_i

  $$\mathcal{G}_i = \frac{1}{h_i}\left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}\right) - f_i$$

FV scheme (II)

- goal — to compute an approximation of the mean value of ϕ in each cell of the mesh represented by ϕ_i
- the approximation to the mean value of f over cell K_i, f_i, will be computed by gaussian quadrature
- the way that the numerical fluxes $\mathcal{F}_{i+\frac{1}{2}}$ are computed characterize the FV scheme
  - first-order finite volume scheme

    $$\mathcal{F}_{i+\frac{1}{2}} = \frac{1}{2}\left(\frac{\phi_i^2}{2} + \frac{\phi_{i+1}^2}{2}\right) - \frac{\max(|\phi_{i+1}|, |\phi_i|)}{2}(\phi_{i+1} - \phi_i)$$

  - high-order finite volume scheme

high-order FV scheme (i)

- to achieve high-order numerical approximations, we introduce local polynomial reconstructions of the underlying solutions
- stencil: the stencil S_i of cell K_i is composed of the $\mathsf{d_i}+1$ closest neighbour cells excluding cell K_i
- reconstruction: the polynomial $\widehat{\boldsymbol{\phi}}_i(x;\mathsf{d_i})$ is based on the data associated to the stencil under a least-square technique
- consistency: if the underlying solution is a polynomial of degree d, the reconstruction of degree d is exact

HIGH-ORDER FV SCHEME (II)

- define the polynomial reconstructions based on the data of the associated stencil

$$\boldsymbol{\phi}_i(x;\mathsf{d_i}) = \phi_i + \sum_{\alpha=1}^{\mathsf{d_i}} R_{i,\alpha}\left[(x-x_i)^\alpha - M_{i,\alpha}\right]$$

with

$$M_{i,\alpha} = \frac{1}{h_i}\int_{K_i}(x-x_i)^\alpha\,\mathrm{d}x \Rightarrow \frac{1}{h_i}\int_{K_i}\boldsymbol{\phi}_i(x)\,\mathrm{d}x = \phi_i$$

- for a given stencil S_i, we consider the quadratic functional

$$\widehat{R}_i = \arg\min\left(\widehat{E}_i(R_i) = \sum_{j\in S_i}\left[\frac{1}{h_j}\int_{K_j}\boldsymbol{\phi}_i(x;\mathsf{d_i})\,\mathrm{d}x - \phi_j\right]^2\right)$$

- we set $\widehat{\boldsymbol{\phi}}_i$ the associated polynomial that corresponds to the best approximation in the least squares sense of the data of the stencil

HIGH-ORDER FV SCHEME (III)

- to construct a generic high-order scheme one has to substitute the left and right states in by states evaluated through high-order polynomial reconstructions
- let us assume that a cell polynomial degree map is given $\mathcal{M} = (\mathsf{d_1}, \ldots, \mathsf{d_I})^T$ with its associated stencil map $S = (S_1, \ldots, S_I)^T$
- $$\widehat{\phi}_i^- = \widehat{\boldsymbol{\phi}}_i(x_{i-\frac{1}{2}}; \min(\mathsf{d_{i-1}}, \mathsf{d_i})) \quad \text{and} \quad \widehat{\phi}_i^+ = \widehat{\boldsymbol{\phi}}_i(x_{i+\frac{1}{2}}; \min(\mathsf{d_i}, \mathsf{d_{i+1}}))$$
- $$\mathcal{F}_{i+\frac{1}{2}}(\Phi) = \frac{1}{2}\left(\frac{\left(\widehat{\phi}_i^+\right)^2}{2} + \frac{\left(\widehat{\phi}_{i+1}^-\right)^2}{2}\right) - \frac{\max\left(\left|\widehat{\phi}_i^+\right|, \left|\widehat{\phi}_{i+1}^-\right|\right)}{2}\left(\widehat{\phi}_{i+1}^- - \widehat{\phi}_i^+\right)$$
- note that the minimal polynomial degree $\min(\mathsf{d_i}, \mathsf{d_{i+1}})$ at interface $x_{i+\frac{1}{2}}$ is mandatory to ensure that the cell is updated with the first-rder scheme when $\mathsf{d_i} = 0$ or $\mathsf{d_{i+1}} = 0$

RESIDUAL FORM

- algorithm

$$
\begin{array}{ccccc}
f & \rightarrow & f_i & & \\
\Phi = (\phi)_{1=1,\ldots,I} & \rightarrow & \widehat{\boldsymbol{\phi}}_i(x;\mathsf{d_i}) & \rightarrow & \mathcal{F}_{i+\frac{1}{2}} \\
f_i + \mathcal{F}_{i+\frac{1}{2}} & \rightarrow & \mathcal{G}_i = \dfrac{1}{h_i}\left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}\right) - f_i & &
\end{array}
$$

- the very-high order scheme explicitly depends on the map $\mathcal{M}$ and the stencil S

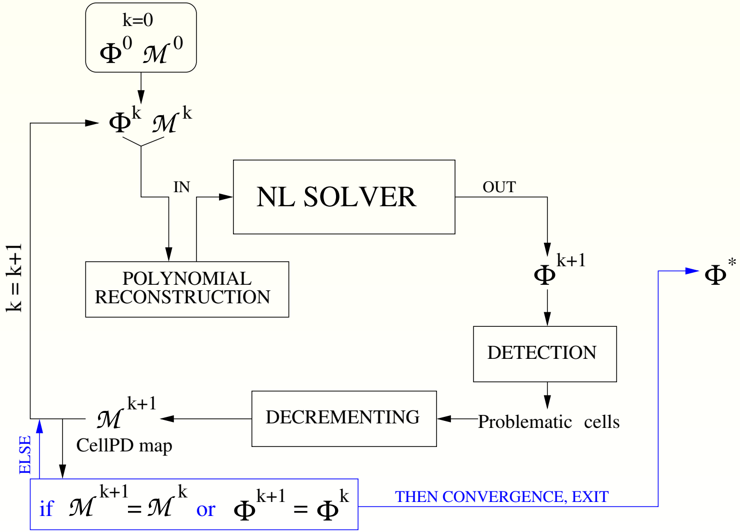
  $\mathcal{G} \equiv \mathcal{G}(\Phi; \mathcal{M}, S)$

- the numerical solution is given by vector $\Phi^\dagger = (\phi_i^\dagger)_{i=1,\ldots,I}$ which is the solution of the nonlinear problem

THE MOOD LOOP (I)

- $\widehat{\boldsymbol{\phi}}_i(x; 0)$: stable/low accuracy
- $\widehat{\boldsymbol{\phi}}_i(x; 6)$: oscillations if there is a discontinuity/high accuracy
- compromise: to locally reduce the polynomial degree in domains where the solution is discontinuous while preserving the optimal order in the smooth function domain
- MOOD: *a posterior* limiting procedure — the limitation is applied after the computation of the fluxes
- MOOD idea: to consider the maximum polynomial degree for the reconstruction, to evaluate the flux, to compute the candidate solution and then make some corrections only if it is really necessary
- the detection criteria: a first filter detects all extrema (called ED) and a second filter (called u2) applied on detected extrema splits them into regular or non-regular ones
- for non-regular extrema, one decrements the cell polynomial degree

THE MOOD LOOP (II)



Burgers: manufactured regular solution (i)

- manufactured regular solution

  $$u(x) = \sin(3\pi x)\exp(x) + 2$$

  then

  $$f(x) = (\exp(x)\sin(3\pi x) + 2)(\exp(x)\sin(3\pi x) + 3\pi\exp(x)\cos(3\pi x))$$
  $$u_{\mathrm{lf}} = 2$$
  $$u_{\mathrm{rg}} = 2$$

- errors evaluation

  $$E_\infty(I) = \max_{i=1}^{I} |u_i - \bar{u}_i|$$
  $$O_\infty(I_1, I_2) = \frac{|\log(E_\infty(I_1)/E_\infty(I_2))|}{|\log(I_1/I_2)|}$$

  $\bar{u}_i$: the exact mean value of u over cell K_i

- initial approximation: $\Phi^0 = (2, \ldots, 2)^T$

Burgers: manufactured regular solution (ii)

	I	E_∞	$\mathcal{O}_\infty$	# bad cells
$\mathbb{P}_0$	40	3.2E−01	—	0
	80	1.6E−01	1.0	0
	160	8.2E−02	1.0	0
	320	4.1E−02	1.0	0
$\mathbb{P}_1$	40	2.9E−02	—	0
	80	5.4E−03	2.4	0
	160	1.1E−03	2.3	0
	320	2.5E−04	2.2	0
$\mathbb{P}_5$	40	2.9E−05	—	0
	80	3.8E−07	6.2	0
	160	8.5E−09	5.5	0
	320	1.6E−10	5.7	0

Burgers: solution with a shock

- data

$$f(x) = -\pi\cos(\pi x)u(x)$$
$$u_{\mathrm{lf}} = 1$$
$$u_{\mathrm{rg}} = -0.1$$

- analytical solution

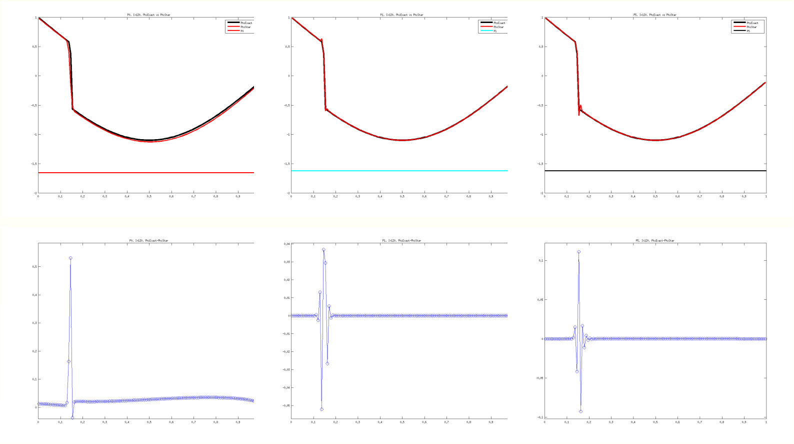
$$u(x) = \begin{cases} 1 - \sin(\pi x) & \text{if } 0 \leqslant x \leqslant x_{\mathrm{s}} \\ -0.1 - \sin(\pi x) & \text{if } x_{\mathrm{s}} \leqslant x \leqslant 1 \end{cases}$$

where x_{s} is the location of the shock ($x_{\mathrm{s}} = 0.1486$)

- initial approximation: : $U^0 = (u_i^0)_{i=1,\dots,I}$

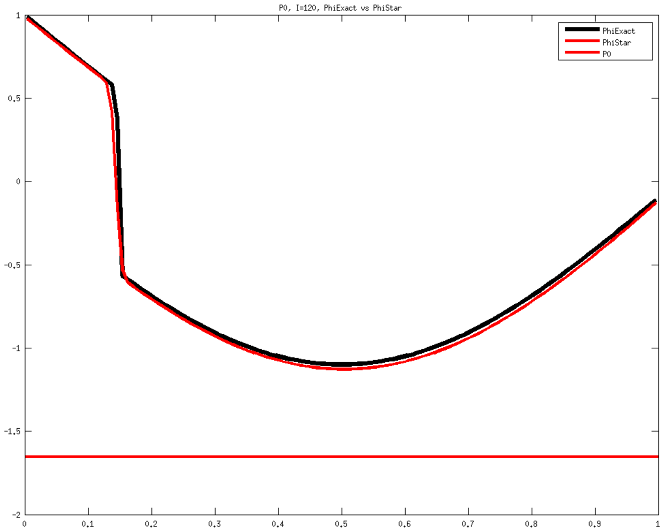
$$u_i^0 = \begin{cases} 1 & \text{if } 0 \leqslant x_i \leqslant \frac{1}{4} \\ -0.1 & \text{if } \frac{1}{4} \leqslant x_i \leqslant 1 \end{cases}$$

Burgers: solution with a shock/unlimited (i)

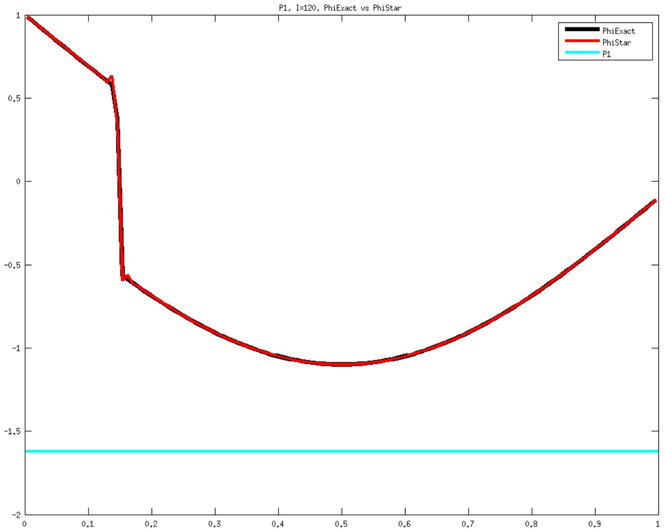


- the $\mathbb{P}_0$ scheme solution is rather diffused
- the $\mathbb{P}_1$ and $\mathbb{P}_5$ scheme solutions seem sharper and more accurate but with numerical oscillations as expected

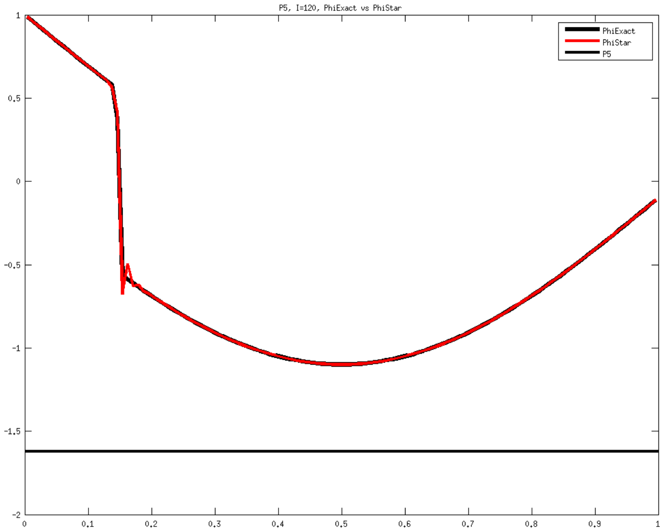
Burgers: solution with a shock/unlimited (ii)



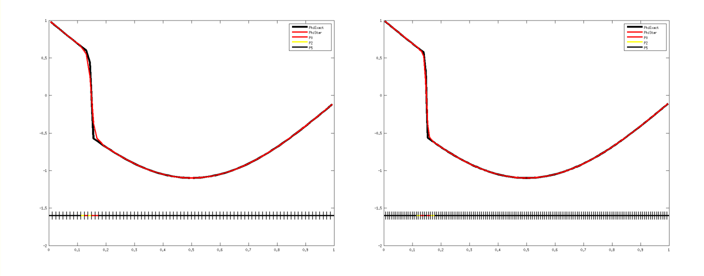
Burgers: solution with a shock/unlimited (iii)



Burgers: solution with a shock/unlimited (iv)

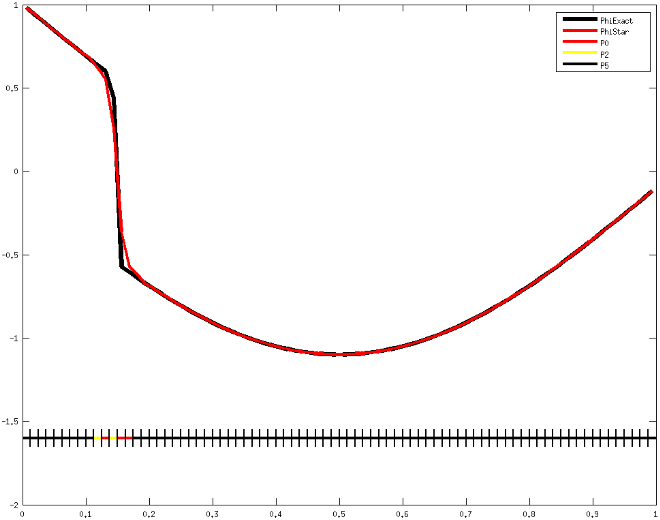


Burgers: solution with a shock/MOOD

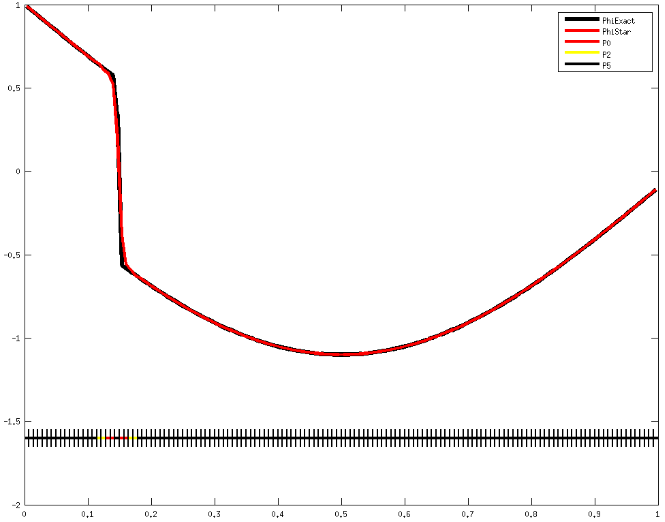


- we now trigger the MOOD loop to eliminate the non-physical oscillations with $\mathsf{d}^{\mathsf{max}} = 5$
- the MOOD method provides a stable and sharp solution without oscillations
- when the mesh is refined, the solution gets better without any oscillations ($I = 80$ and $I = 140$)
- only cells around the shock were decremented

Burgers: solution with a shock/MOOD ($I = 80$)



Burgers: solution with a shock/MOOD ($I = 140$)



Burgers — solution with a shock/MOOD

	I	[0; 0.1]				[0.3; 1]				
		E_1	$\mathcal{O}_1$	E_∞	$\mathcal{O}_\infty$	E_1	$\mathcal{O}_1$	E_∞	$\mathcal{O}_\infty$	# bad cells
$\mathbb{P}_0$	80	1.7E−03	—	1.9E−02	—	3.1E−02	—	5.3E−02	—	0
$\mathbb{P}_1$	80	3.0E−05	—	1.8E−03	—	4.9E−05	—	1.0E−04	—	2
$\mathbb{P}_5$	80	1.1E−04	—	4.6E−03	—	4.8E−05	—	9.8E−05	—	2
	80	1.7E−04	—	6.8E−03	—	4.8E−05	—	9.8E−05	—	0
MOOD	100	4.2E−05	6.3	2.4E−03	4.7	3.1E−05	2.0	6.3E−05	2.0	0
$\mathsf{d}^{\mathsf{max}} = 5$	120	2.4E−05	3.1	1.7E−03	2.0	2.1E−05	2.0	4.3E−05	2.0	0
	140	3.0E−06	13.5	2.4E−04	12.5	1.5E−05	2.1	3.2E−05	2.0	0

Euler's equations

$$\frac{\mathrm{d}\mathbb{F}(U)}{\mathrm{d}x} = \frac{\mathrm{d}S}{\mathrm{d}x}, \text{ in } \Omega = (0,1)$$

where the conservative variable U is given by

$$U = (\rho, \rho u, E)^T$$

the flux is given by

$$\mathbb{F}(U) = (\rho u, \rho u^2 + p, u(E+p))^T,$$

and the source term $\frac{\mathrm{d}S}{\mathrm{d}x}$, $S = (D, F, H)^T = (D(x), F(x), H(x))^T$

Euler: manufactured regular solution (i)

- manufactured regular solution

$$\rho(x) = \exp(x) + \exp(x)\sin(3\pi x) + 2$$
$$u(x) = \sin(\alpha\pi x) + \exp(x)$$
$$p(x) = \exp(x)$$

then

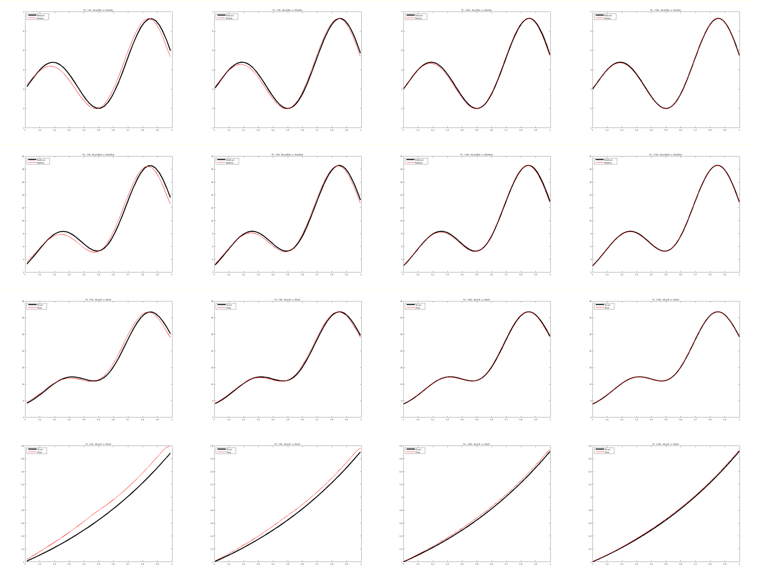
$$\frac{\mathrm{d}S}{\mathrm{d}x} = (\ldots, \ldots, \ldots)^T$$
$$U_{\mathrm{lf}} = (\ldots, \ldots, \ldots)^T$$
$$U_{\mathrm{rg}} = (\ldots, \ldots, \ldots)^T$$

- initial approximation: analytical soluton/linear combination of the boundary conditions

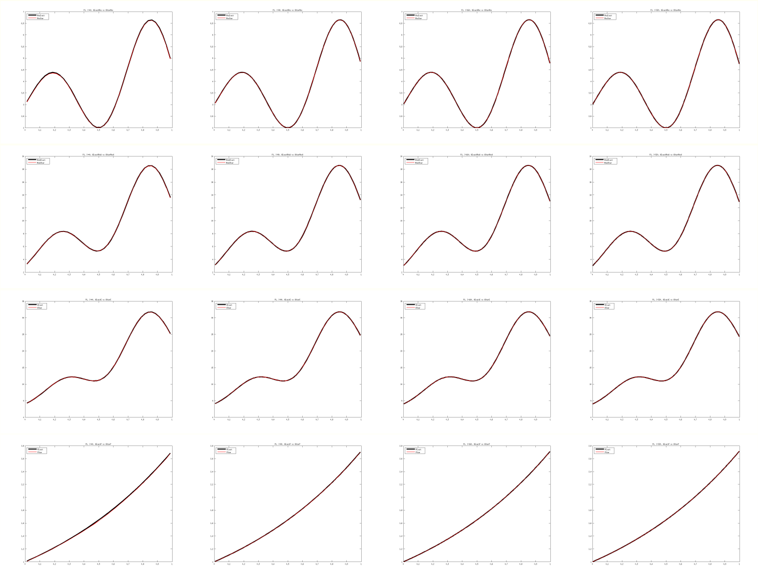
EX3, SS, X0: EXACT, RUSANOV, UNLIMITED, $\alpha = 1$

	I	nIter	nGs	$E_1(\rho)$	$\mathcal{O}_1(\rho)$	$E_1(\rho u)$	$\mathcal{O}_1(\rho u)$	$E_1(E)$	$\mathcal{O}_1(E)$	$\|\mathcal{G}\|_1$	$\|\mathcal{G}(x^{(0)})\|_1$	$\mathcal{O}$
$\mathbb{P}_0$	40	6	847	1.9E−01	—	4.7E−01	—	5.9E−01	—	4.32e-13	4.9E+00	—
	80	8	1689	1.0E−01	0.9	2.4E−01	1.0	3.0E−01	1.0	8.82e-13	2.6E+00	1.0
	160	8	3369	5.2E−02	1.0	1.2E−01	1.0	1.5E−01	1.0	1.81e-12	1.3E+00	1.0
	320	8	6729	2.6E−02	1.0	6.1E−02	1.0	7.5E−02	1.0	3.32e-12	6.5E−01	1.0
$\mathbb{P}_1(2)$	40	4	605	1.5E−02	—	1.7E−02	—	2.1E−02	—	6.44e-13	2.3E−01	—
	80	4	1205	2.7E−03	2.5	3.5E−03	2.2	4.9E−03	2.1	1.23e-12	5.6E−02	2.0
	160	3	1924	5.7E−04	2.2	8.9E−04	2.0	1.3E−03	1.9	2.60e-12	1.4E−02	2.0
	320	3	3844	1.3E−04	2.1	2.3E−04	1.9	3.4E−04	1.9	5.03e-12	3.5E−03	2.0
$\mathbb{P}_3(4)$	40	3	484	7.9E−04	—	8.4E−04	—	1.6E−03	—	7.90e-13	1.6E−02	—
	80	3	964	3.6E−05	4.4	1.3E−04	2.7	1.5E−04	3.4	1.59e-12	8.9E−04	4.1
	160	3	1924	3.2E−06	3.5	10.0E−06	3.7	1.0E−05	3.9	2.73e-12	5.2E−05	4.1
	320	2	2883	2.3E−07	3.8	6.8E−07	3.9	6.7E−07	4.0	6.39e-12	3.1E−06	4.1
$\mathbb{P}_5(6)$	40	3	484	7.1E−05	—	6.5E−05	—	2.0E−04	—	7.51e-13	7.9E−04	—
	80	2	723	9.7E−07	6.2	3.3E−06	4.3	4.2E−06	5.6	1.76e-12	1.1E−05	6.2
	160	2	1443	2.3E−08	5.4	6.6E−08	5.7	6.4E−08	6.0	2.86e-12	1.5E−07	6.2
	320	2	2883	4.2E−10	5.8	1.1E−09	5.9	9.8E−10	6.0	5.88e-12	2.1E−09	6.1

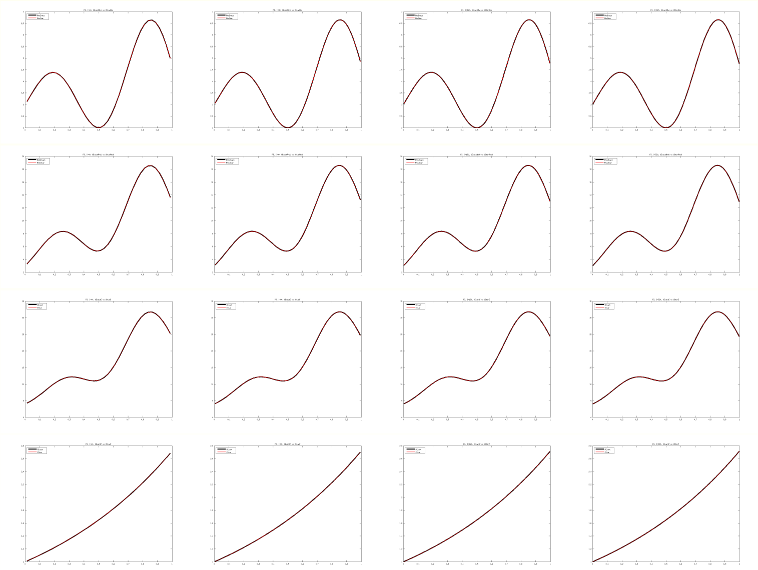
X0: EXACT, UNLIMITED, α =1, $P0$



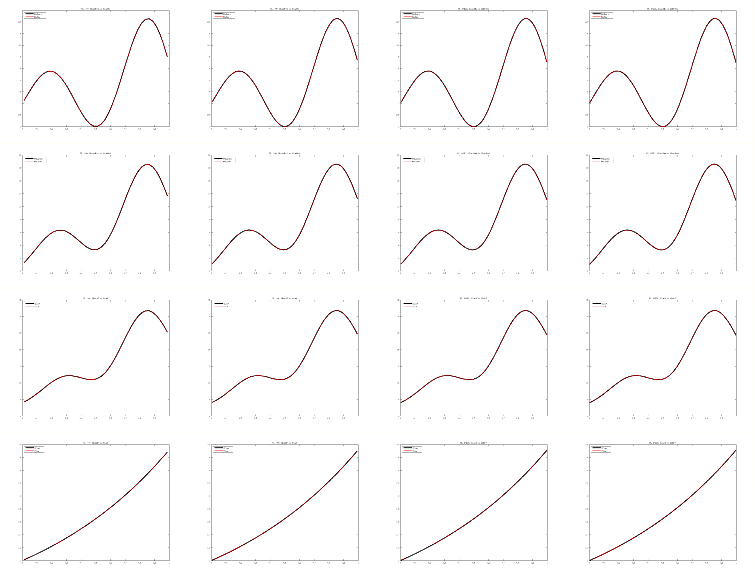
X0: EXACT, UNLIMITED, α =1, $P1$



X0: EXACT, UNLIMITED, $\alpha = 1$, $P3$



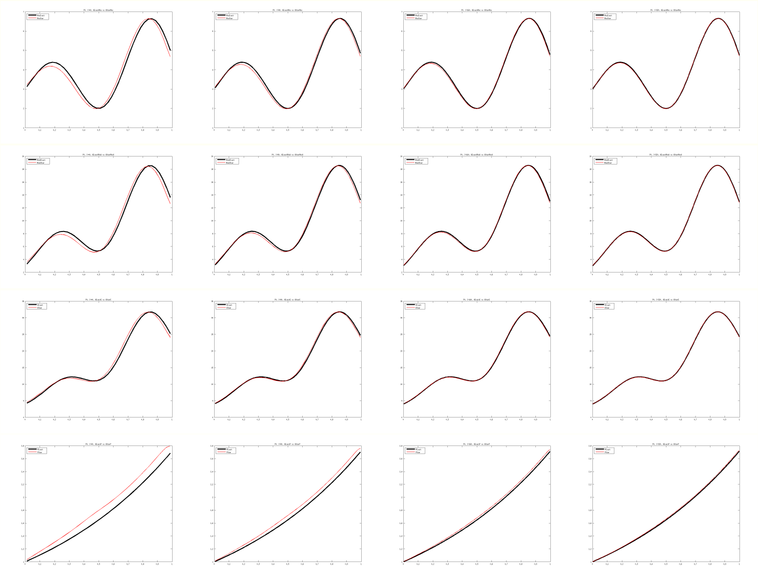
X0: EXACT, UNLIMITED, $\alpha = 1$, $P5$



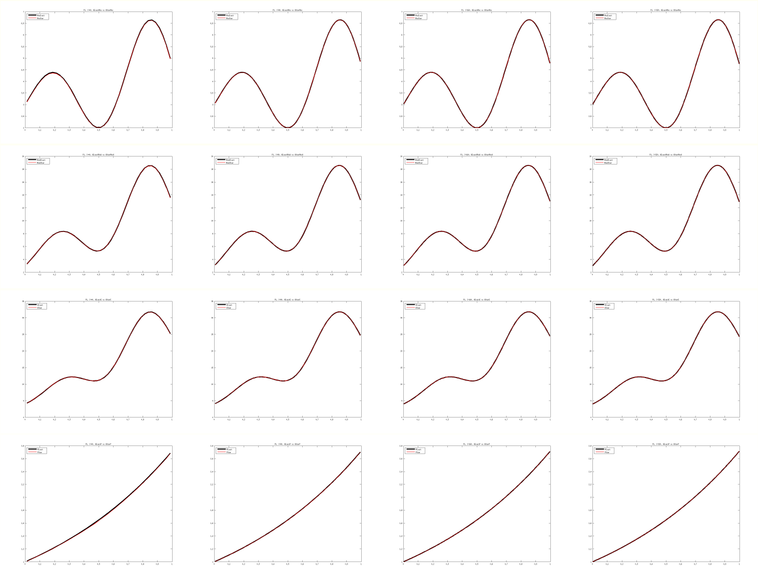
EX3, SS, X0: BCS, RUSANOV, UNLIMITED, $\alpha = 1$

	I	nIter	nGs	$E_1(\rho)$	$\mathcal{O}_1(\rho)$	$E_1(\rho u)$	$\mathcal{O}_1(\rho u)$	$E_1(E)$	$\mathcal{O}_1(E)$	$\|\mathcal{G}\|_1$	$\|\mathcal{G}(x^{(0)})\|_1$	$\mathcal{O}$
$\mathbb{P}_0$	40	9	1210	1.9E−01	—	4.7E−01	—	5.9E−01	—	5.38e-13	2.6E+01	—
	80	9	2410	1.0E−01	0.9	2.4E−01	1.0	3.0E−01	1.0	1.02e-12	2.6E+01	↑
	160	9	4810	5.2E−02	1.0	1.2E−01	1.0	1.5E−01	1.0	2.08e-12	2.6E+01	0.0
	320	10	10571	2.6E−02	1.0	6.1E−02	1.0	7.5E−02	1.0	3.55e-12	2.6E+01	0.0
$\mathbb{P}_1(2)$	40	9	1210	1.5E−02	—	1.7E−02	—	2.1E−02	—	6.41e-13	2.6E+01	—
	80	9	2410	2.7E−03	2.5	3.5E−03	2.2	4.9E−03	2.1	1.27e-12	2.6E+01	↑
	160	9	4810	5.7E−04	2.2	8.9E−04	2.0	1.3E−03	1.9	2.67e-12	2.6E+01	↑
	320	10	10571	1.3E−04	2.1	2.3E−04	1.9	3.4E−04	1.9	4.92e-12	2.6E+01	↑
$\mathbb{P}_3(4)$	40	9	1210	7.9E−04	—	8.4E−04	—	1.6E−03	—	6.77e-13	2.6E+01	—
	80	9	2410	3.6E−05	4.4	1.3E−04	2.7	1.5E−04	3.4	1.60e-12	2.6E+01	↑
	160	9	4810	3.2E−06	3.5	10.0E−06	3.7	1.0E−05	3.9	3.12e-12	2.6E+01	↑
	320	10	10571	2.3E−07	3.8	6.8E−07	3.9	6.7E−07	4.0	5.82e-12	2.6E+01	↑
$\mathbb{P}_5(6)$	40	15	1576	7.1E−05	—	6.5E−05	—	2.0E−04	—	6.05e-13	2.6E+01	—
	80	9	2410	9.7E−07	6.2	3.3E−06	4.3	4.2E−06	5.6	1.69e-12	2.6E+01	↑
	160	9	4810	2.3E−08	5.4	6.6E−08	5.7	6.4E−08	6.0	3.57e-12	2.6E+01	↑
	320	10	10571	4.2E−10	5.8	1.1E−09	5.9	9.8E−10	6.0	6.40e-12	2.6E+01	↑

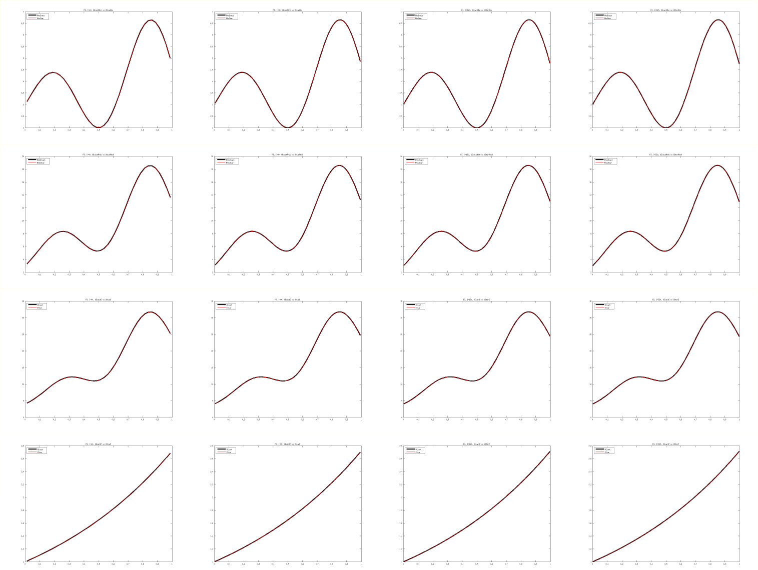
X0: LINEAR BC'S, UNLIMITED, $\alpha = 1$, $P0$



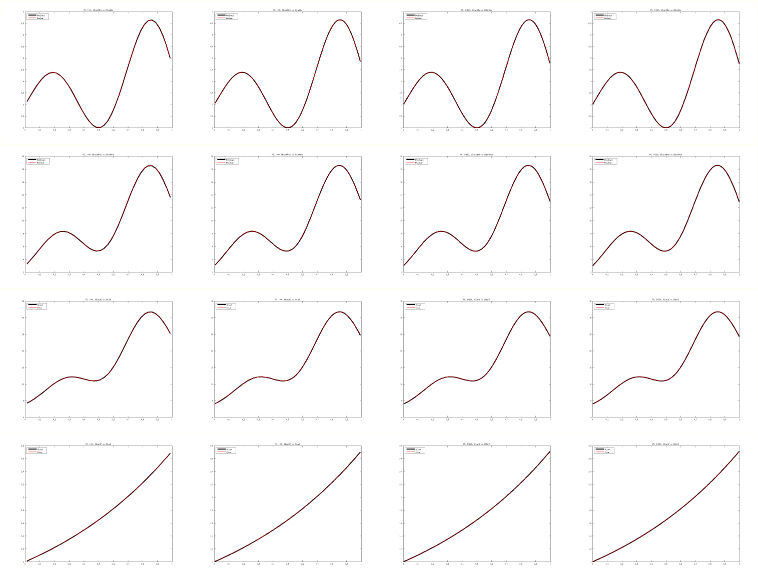
X0: LINEAR BC'S, UNLIMITED, $\alpha = 1$, $P1$



X0: LINEAR BC'S, UNLIMITED, $\alpha = 1$, $P3$



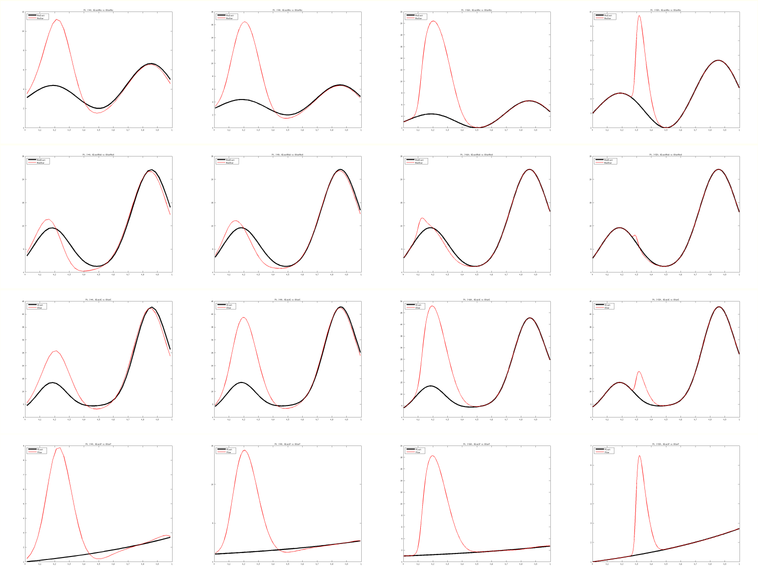
X0: LINEAR BC'S, UNLIMITED, $\alpha = 1$, $P5$



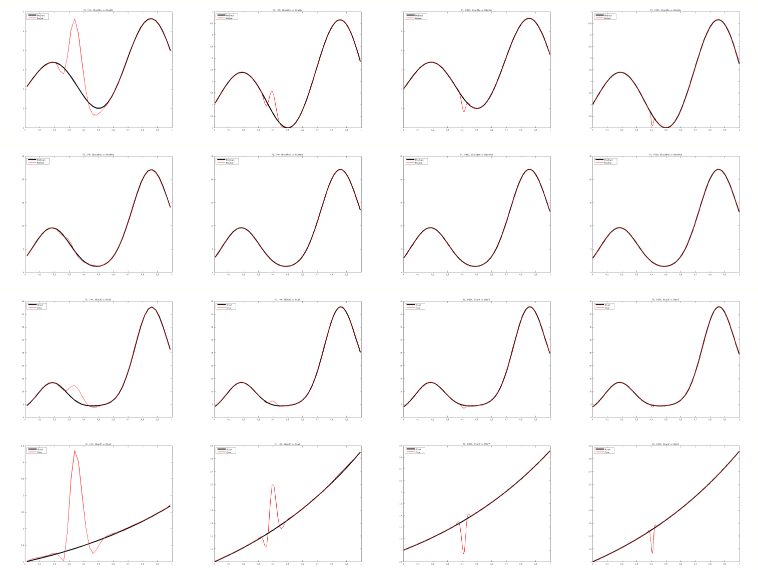
EX3, SS, X0: EXACT, RUSANOV, UNLIMITED, $\alpha = 3$

	I	nIter	nGs	$E_1(\rho)$	$\mathcal{O}_1(\rho)$	$E_1(\rho u)$	$\mathcal{O}_1(\rho u)$	$E_1(E)$	$\mathcal{O}_1(E)$	$\Vert\mathcal{G}\Vert_1$	$\Vert\mathcal{G}(x^{(0)})\Vert_1$	$\mathcal{O}$
$\mathbb{P}_0$	40	16	1697	1.6E+00	—	1.2E+00	—	3.4E+00	—	3.45e-13	5.5E+00	—
	80	24	5545	2.6E+00	↑	8.9E−01	0.5	5.4E+00	↑	1.12e-12	2.8E+00	1.0
	160	58	25019	3.2E+00	↑	4.2E−01	1.1	6.4E+00	↑	1.64e+01	1.4E+00	1.0
	320	30	25951	5.1E−01	2.7	7.9E−02	2.4	8.4E−01	2.9	9.41e+00	6.9E−01	1.0
$\mathbb{P}_1(2)$	40	13	1454	3.5E−01	—	9.3E−02	—	6.4E−01	—	7.16e-13	3.3E−01	—
	80	110	25071	5.1E−02	2.8	1.1E−02	3.0	8.4E−02	2.9	9.93e-01	9.8E−02	1.7
	160	55	25016	2.2E−02	1.2	2.3E−03	2.3	3.6E−02	1.2	4.44e-01	2.8E−02	1.8
	320	32	25953	9.0E−03	1.3	5.1E−04	2.2	1.5E−02	1.3	1.42e-01	7.3E−03	1.9
$\mathbb{P}_3(4)$	40	6	847	2.3E−02	—	6.0E−03	—	4.5E−02	—	8.13e-13	2.8E−02	—
	80	110	25071	1.9E−02	0.3	2.3E−03	1.4	3.1E−02	0.5	3.02e-02	2.4E−03	3.5
	160	59	25020	6.4E−03	1.6	4.9E−04	2.2	1.0E−02	1.6	3.18e-03	1.9E−04	3.7
	320	35	25956	6.6E−04	3.3	3.1E−05	4.0	1.1E−03	3.3	2.49e-04	1.3E−05	3.9
$\mathbb{P}_5(6)$	40	5	726	10.0E−03	—	2.4E−03	—	1.6E−02	—	8.32e-13	3.2E−03	—
	80	4	1205	4.8E−04	4.4	7.1E−05	5.1	8.0E−04	4.3	1.68e-12	6.8E−05	5.6
	160	59	25020	4.9E−04	↑	3.5E−05	1.0	7.9E−04	0.0	4.55e-05	1.5E−06	5.5
	320	35	25956	2.7E−05	4.2	1.1E−06	4.9	4.3E−05	4.2	5.02e-07	2.6E−08	5.9

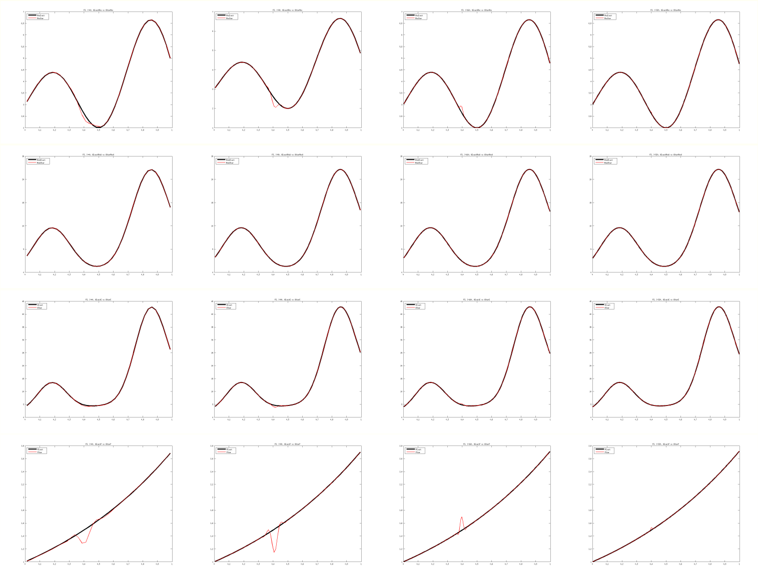
X0: EXACT, UNLIMITED, α =3, $P0$



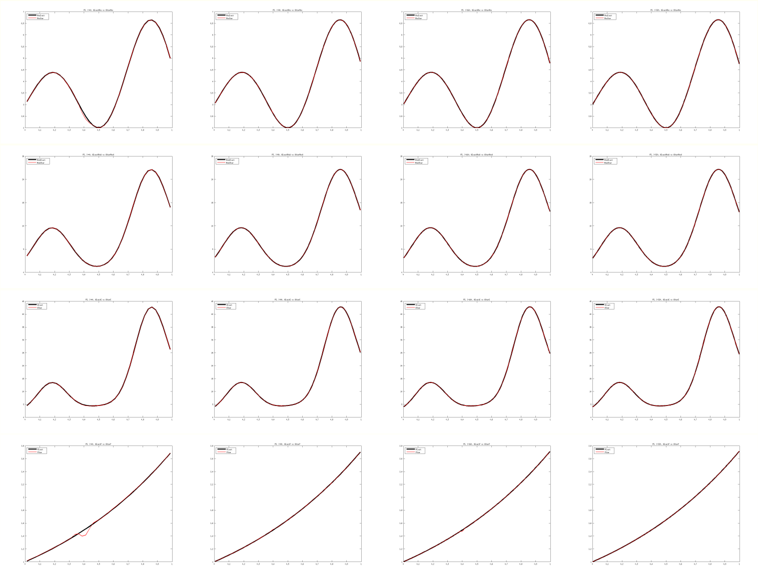
X0: EXACT, UNLIMITED, α =3, $P1$



X0: EXACT, UNLIMITED, α =3, $P3$



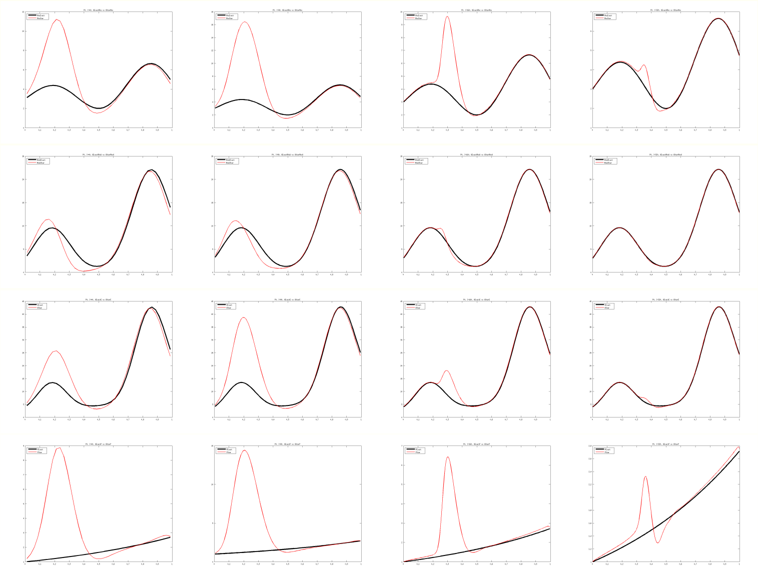
X0: EXACT, UNLIMITED, $\alpha = 3$, $P5$



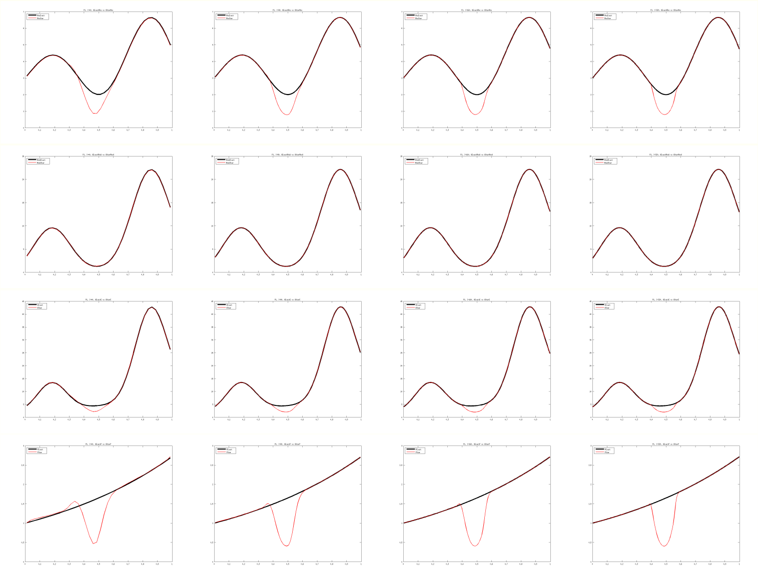
EX3, SS, X0: BCS, RUSANOV, UNLIMITED, α =3

	I	nIter	nGs	$E_1(\rho)$	$\mathcal{O}_1(\rho)$	$E_1(\rho u)$	$\mathcal{O}_1(\rho u)$	$E_1(E)$	$\mathcal{O}_1(E)$	$\lVert\mathcal{G}\rVert_1$	$\lVert\mathcal{G}(x^{(0)})\rVert_1$	$\mathcal{O}$
$\mathbb{P}_0$	40	26	2907	1.6E+00	—	1.2E+00	—	3.4E+00	—	6.01e-12	4.5E+01	—
	80	97	20498	2.6E+00	↑	8.9E−01	0.5	5.4E+00	↑	2.30e-12	4.5E+01	0.0
	160	60	25021	6.5E−01	2.0	2.6E−01	1.7	1.2E+00	2.2	5.94e-01	4.5E+01	0.0
	320	31	25952	1.1E−01	2.5	7.1E−02	1.9	2.4E−01	2.3	2.65e-01	4.5E+01	0.0
$\mathbb{P}_1(2)$	40	15	1696	1.9E−01	—	3.5E−02	—	3.4E−01	—	6.56e-13	4.5E+01	—
	80	17	3858	1.7E−01	0.1	8.1E−03	2.1	3.2E−01	0.1	9.68e-13	4.5E+01	↑
	160	13	6734	1.6E−01	0.1	2.0E−03	2.0	3.1E−01	0.1	2.52e-12	4.5E+01	↑
	320	14	14415	1.5E−01	0.0	4.9E−04	2.0	3.0E−01	0.0	5.07e-12	4.5E+01	↑
$\mathbb{P}_3(4)$	40	14	1575	1.7E−01	—	6.8E−03	—	3.3E−01	—	7.34e-13	4.5E+01	—
	80	13	3374	1.6E−01	0.1	1.3E−03	2.4	3.1E−01	0.1	1.33e-12	4.5E+01	↑
	160	14	7215	1.5E−01	0.0	2.5E−04	2.4	3.0E−01	0.0	2.47e-12	4.5E+01	↑
	320	22	17303	1.5E−01	0.0	4.5E−05	2.5	3.0E−01	0.0	4.03e-12	4.5E+01	↑
$\mathbb{P}_5(6)$	40	12	1573	1.6E−01	—	4.7E−03	—	3.2E−01	—	8.11e-13	4.5E+01	—
	80	13	3374	1.6E−01	0.1	7.8E−04	2.6	3.0E−01	0.1	1.64e-12	4.5E+01	↑
	160	23	8664	1.5E−01	0.0	1.4E−04	2.4	3.0E−01	0.0	2.43e-12	4.5E+01	↑
	320	15	15376	1.5E−01	0.0	2.4E−05	2.6	3.0E−01	0.0	6.58e-12	4.5E+01	↑

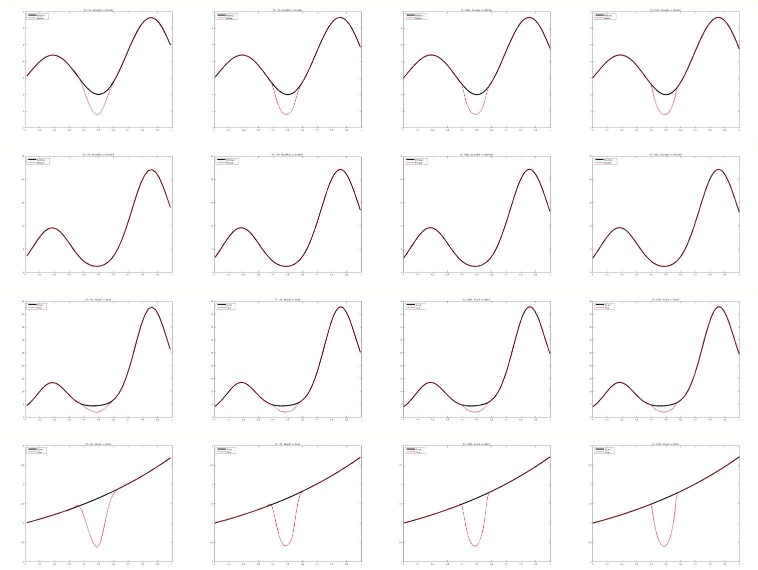
X0: LINEAR BC'S, UNLIMITED, α =3, $P0$



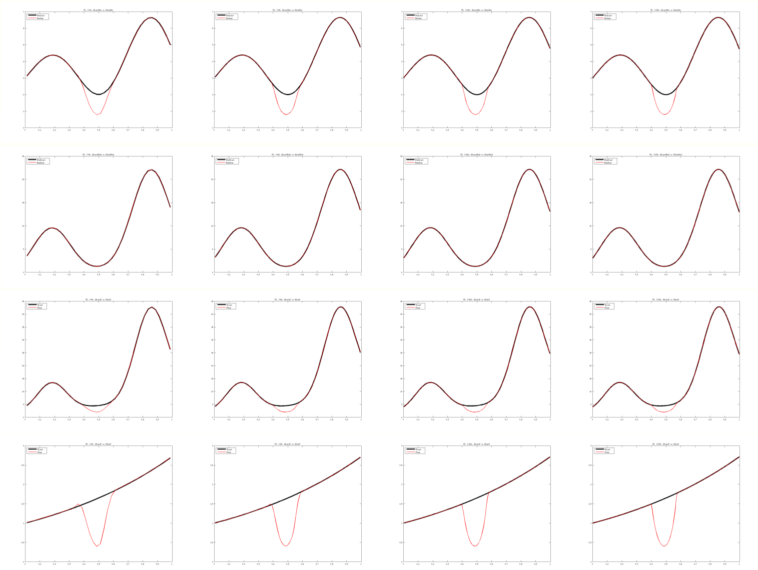
X0: LINEAR BC'S, UNLIMITED, α =3, $P1$



X0: LINEAR BC'S, UNLIMITED, $\alpha = 3$, $P3$



X0: LINEAR BC'S, UNLIMITED, α =3, $P5$



Euler's equations

From $\rho u = D$ and $\rho u^2 + p = F$, we deduce that

$$p = F - \frac{D^2}{\rho}, \tag{1}$$

and the pressure non negativity yields that $F \geqslant 0$ and

$$\rho \geqslant \frac{D^2}{F}. \tag{2}$$

For $F > 0$, we define the reference density

$$\hat{\rho}_0(D, F) = \frac{D^2}{F}, \quad \rho_0(x) = \hat{\rho}_0(D(x), F(x)) \tag{3}$$

then we have by construction $\rho(x) \geqslant \rho_0(x)$, $x \in \Omega$

Euler's equations

On the other side, from $\rho u^2 + p = F$, $u(E+p) = H$, and $e = \frac{p}{\rho(\gamma-1)}$, we deduce that

$$p = \frac{\gamma-1}{\gamma}\left(\frac{H}{u} - \frac{1}{2}\rho u^2\right) \tag{4}$$

Assuming that $D \geqslant 0$ and arguing one more time that the pressure is non-negative, we get

$$DH \geqslant \frac{D^4}{2\rho^2}$$

thus $HD \geqslant 0$. From (4), we deduce that there exists a function $\hat{F}$ with respect to ρ given by

$$\hat{F}(\rho) = \frac{\gamma+1}{2\gamma}\frac{D^2}{\rho} + \frac{\gamma-1}{\gamma}\frac{H}{D}\rho \tag{5}$$

such that $F(x) = \hat{F}(\rho(x))$ if ρ is solution of Euler's equations

Moreover, the sonic point is given by

$$\hat{\rho}_{\mathrm{son}}(D,H)=\sqrt{\frac{\gamma+1}{2(\gamma-1)}\frac{D^3}{H}},\quad \rho_{\mathrm{son}}(x)=\hat{\rho}_{\mathrm{son}}(D(x),H(x)).$$

Equation $\hat{F}(\rho)=\frac{\gamma+1}{2\gamma}\frac{D^2}{\rho}+\frac{\gamma-1}{\gamma}\frac{H}{D}\rho$ is rewritten as a quadratic polynomial in ρ of the form

$$2(\gamma-1)H\rho^2-2\gamma DF\rho+(\gamma+1)D^3=0.$$

Existence of solution requires that the following compatibility condition are satisfied:

$$F>0,\ HD\geqslant 0,\ \frac{\gamma^2}{\gamma^2-1}\geqslant 2\frac{DH}{F^2}$$

The supersonic solution $\hat{\rho}_{\text{sup}}(D, F, H)$ writes

$$\hat{\rho}_{\text{sup}}(D, F, H) = \frac{2\gamma F - \sqrt{4\gamma^2 F^2 - 4(\gamma+1)(\gamma+1)DH^2}}{4(\gamma-1)\frac{H}{D}}.$$

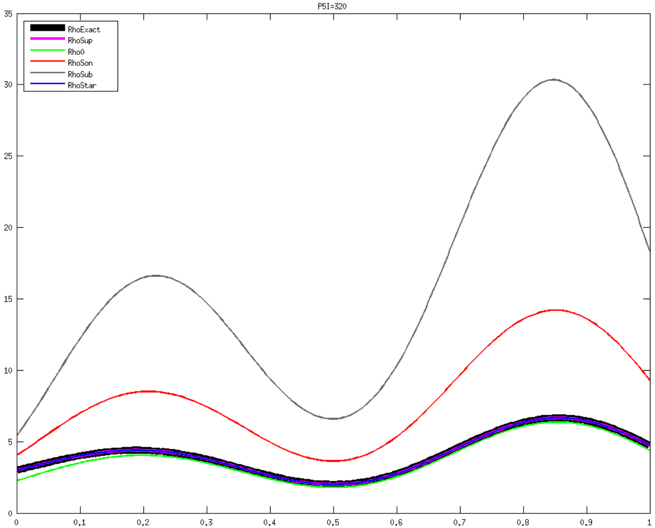
After algebraic manipulations, we also have the expression

$$\widetilde{\rho}_{\text{sup}}(\rho_0, \rho_{\text{son}}) = \frac{\rho_0}{\alpha}\left(1 - \sqrt{1-\alpha^2}\right)$$

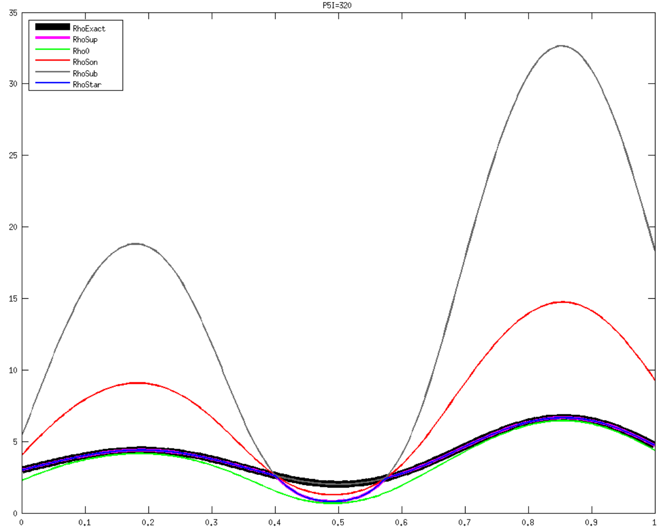
with $\alpha = \dfrac{\gamma+1}{\gamma}\dfrac{\rho_0}{\rho_{\text{son}}}$. In the same way, the subsonic solution is given by

$$\widetilde{\rho}_{\text{sub}}(\rho_0, \rho_{\text{son}}) = \frac{\rho_0}{\alpha}\left(1 + \sqrt{1-\alpha^2}\right)$$

EX3, SS, X0: BCS, UNLIMITED, $\alpha = 1$, $I = 320$



EX3, SS, X0: BCS, UNLIMITED, α =3, I = 320



Euler: solution with a shock

- data

$$f_1(x) = 0$$

$$f_2(x) = 1$$

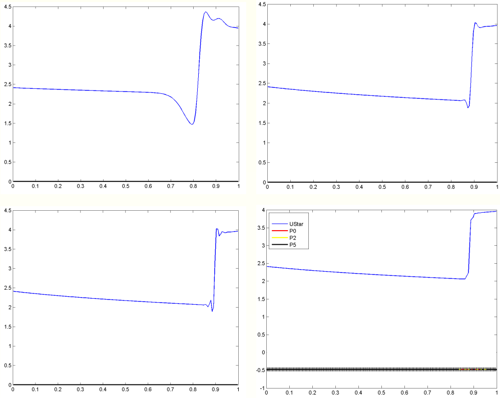
$$f_3(x) = 0$$

$$U_{\text{lf}} = (2.4142, 1.0000, 0.7058)^T : \text{ supersonic}$$

$$U_{\text{rg}} = (3.9598, 1.0000, 1.0967)^T : \text{ subsonic}$$

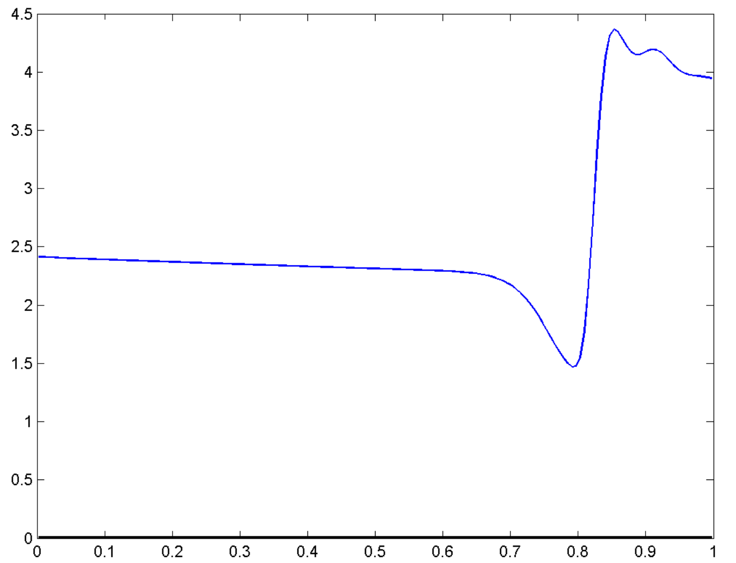
- unlimited with $\mathbb{P}_0$, $\mathbb{P}_1$, and $\mathbb{P}_5$ and $I = 200$
- MOOD with $\mathcal{M}^{(0)} = (5, \dots, 5)^T$

Euler: solution with a shock/unlimited

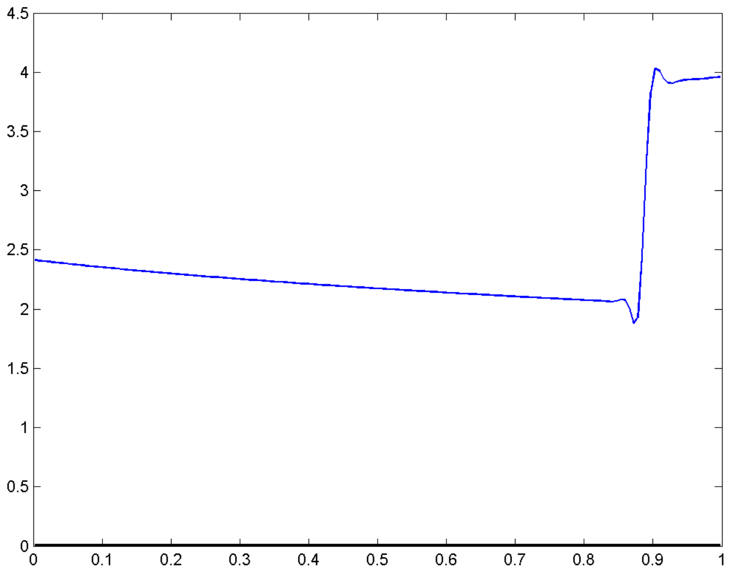


- oscillations are clearly observed and localized around the discontinuity
- with MOOD: the oscillations vanish and the cell polynomial degree map is essentially equal to 5 except in the vicinity of the shock
- the MOOD strategy succeeds in eliminating the oscillations while preserving the high degree of reconstruction in the smooth part of the

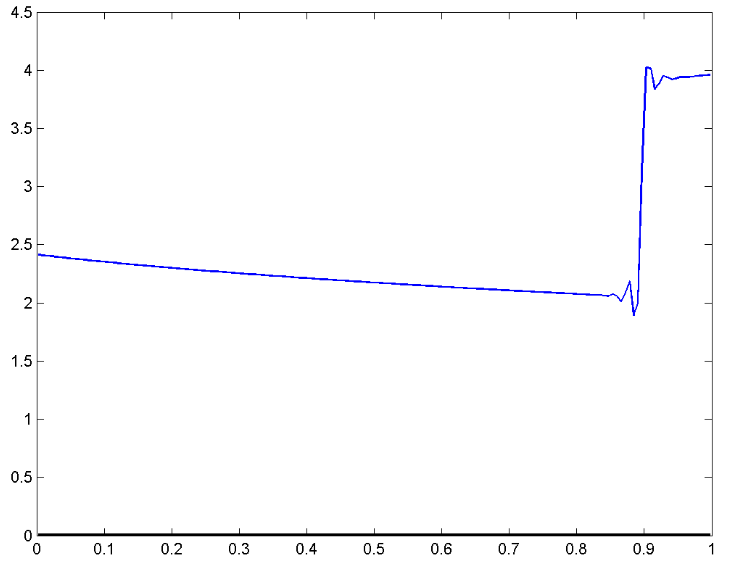
Euler: solution with a shock/unlimited $\mathbb{P}_0$



Euler: solution with a shock/unlimited $\mathbb{P}_1$



Euler: solution with a shock/unlimited $\mathbb{P}_5$



Euler: solution with a shock/MOOD