SHARK-FV

A comparison between the second-order MOOD and MUSCL methods for the 1D shallow water problem in the presence of dry/wet interfaces

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- Presentation outline

- Motivation
- Generic second-order FV scheme
- MOOD vs. MUSCL
- Numerical results
- Conclusions and future work

- Motivation

- The SW problem with varying bathymetry has a wide number of applications: tsunami, flooding, coastal erosion…

- A large number of numerical schemes exists, in particular very high-order FV schemes.

- Still, new 2^{nd}-order methods are interesting because these techniques are widely used in engineering and environmental applications due to their simplicity and computational efficiency but need to be improved.

- The MUSCL method is often used in this context, but recently a new method, MOOD, involving a different limiting approach has been proposed and tested, for example, for the Euler system and for the 2D-SW problem with varying bathymetry (up to 6^{th}-order of accuracy on unstructured meshes).

- MOOD and MUSCL differ essentially in the limiting procedure used to prevent oscillations in the vicinity of discontinuities (*a priori* for MUSCL, *a posteriori* for MOOD).

- Motivation

- **Aim**: compare the "performance" of MOOD vs. MUSCL for the 1D-SW system with varying bathymetry.

Assess:
- accuracy (order of convergence for smooth solutions)
- shock capturing ability (sharpness and oscillations)
- behaviour of the numerical solution near dry/wet interfaces

⇩

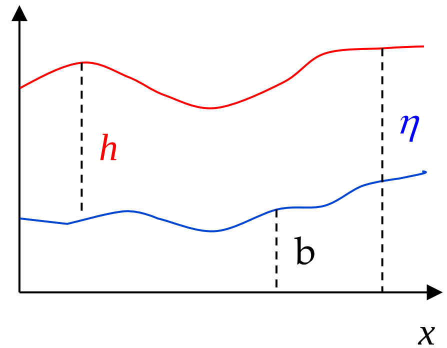
set of numerical simulations of different nature

- Generic 2nd-order FV scheme

❖ Non-linear 1D-SW system

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t (hu) + \partial_x \left(hu^2 + \dfrac{g}{2} h^2 \right) = -gh\partial_x b, \end{cases}$$

hydrostatic condition: $\eta = h + b$



❖ Discretisation: time

$$[0,T] \;\rightarrow\; 0 = t^0 < t^1 < \ldots < t^n < \ldots < t^N = T$$

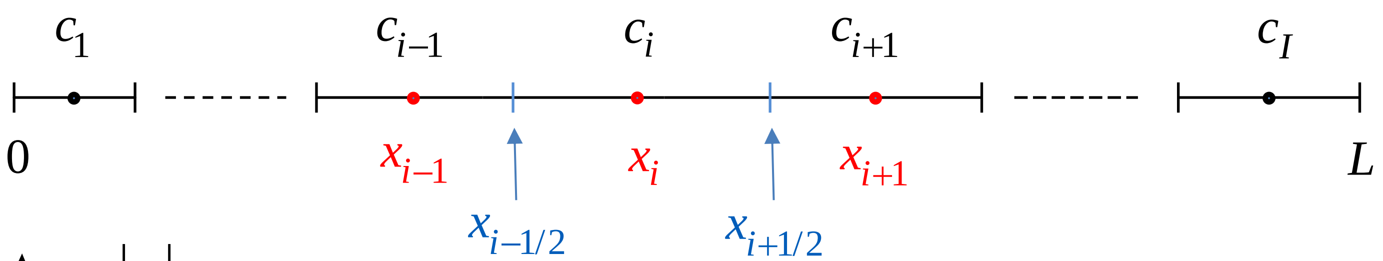
time step:

$$\Delta \mathrm{t}_k = t^k - t^{k-1}, \;\; \mathrm{k} = 1, \ldots, N. \qquad \Delta \mathrm{t}_k = \Delta \mathrm{t} = \frac{T}{N} \;\; \text{if regular subdivision}$$

- Generic 2nd-order FV scheme

❖ Discretisation: space

$$\Omega = [0, L] \;\rightarrow\; \text{I non-overlapping cells:}$$



$$\Delta x_i = |c_i|$$

For a function ϕ at t^n: $\quad \phi_i^n \approx \phi(x_i)$

$$\begin{cases} \phi_{i+1/2,L}^n \approx \phi(x_{i+1/2}^-) \\ \phi_{i+1/2,R}^n \approx \phi(x_{i+1/2}^+) \end{cases}$$

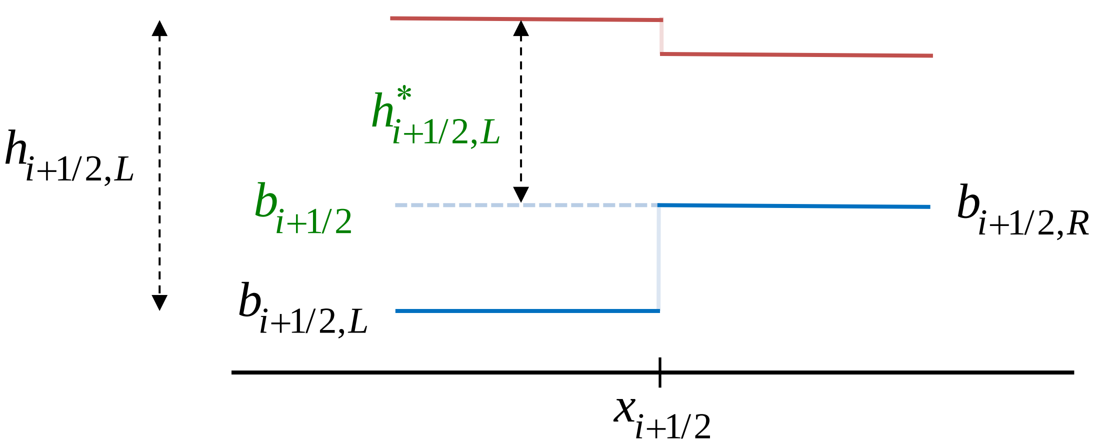
- Generic 2nd-order FV scheme

❖ Hydrostatic reconstruction (Audusse et al., 2004)

Define: $b_{i+1/2} = \max(b_{i+1/2,L}, b_{i+1/2,R})$

Set: $h^*_{i+1/2,K} = \max(0, h_{i+1/2,K} + b_{i+1/2,K} - b_{i+1/2}),$

$$\eta^*_{i+1/2,K} = h^*_{i+1/2,K} + b_{i+1/2}, \quad u^*_{i+1/2,K} = u_{i+1/2,K}, \quad \text{with } K = L, R.$$



- Generic 2nd-order FV scheme

❖ Generic second-order scheme (Audusse et al. methodology)

Setting $U = (h, hu, b)$, the discretisation of the 1D-SW system leads to:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x_i}\left(\mathcal{F}_{i+1/2}^n + \varepsilon_{i+1/2,L}^n - \mathcal{F}_{i-1/2}^n - \varepsilon_{i-1/2,R}^n\right) - \Delta t\, S_i^n,$$

where

$$\mathcal{F}_{i-1/2}^n = \mathrm{F}\left(U_{i-1/2,L}^{*,n}, U_{i-1/2,R}^{*,n}\right),$$

$$\varepsilon_{i-1/2,K}^n = \frac{g}{2}\left[(h_{i-1/2,K}^{*,n})^2 - (h_{i-1/2,K}^n)^2\right], \quad \text{with } K = L, R,$$

$$S_i^n = -g\,\frac{h_{i+1/2,L}^n + h_{i-1/2,R}^n}{2}\,\frac{b_{i+1/2,L}^n - b_{i-1/2,R}^n}{\Delta x_i}.$$

- MOOD vs. MUSCL

- Local second-order reconstruction

Local linear reconstruction required $\rightarrow$ approximation of first derivatives:

$$\text{interfaces: } p_{i-1/2}(\phi) = \frac{\phi_i - \phi_{i-1}}{\Delta x}, \quad p_{i+1/2}(\phi) = \frac{\phi_{i+1} - \phi_i}{\Delta x}.$$

$$\text{centroid: } p_i(\phi) = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}.$$

But non-limited linear reconstruction will lead to **oscillations in the vicinity of discontinuities** and even in less extreme cases.

$\Downarrow$

Need to **implement limiting procedures** $\rightarrow$ loss of accuracy!

- MOOD vs. MUSCL

- Limitation strategy

MUSCL **(Monotonic Upstream-Centred Scheme for Conservation Laws; van Leer 1979)**:

slopes are reduced in order to verify some stability criterion; corrections to slopes are made before computing the solution (***a priori***).

MOOD **(Multi-dimensional Optimal Order Detection; Clain, Diot, Loubère 2011)**:

assumes solution is smooth enough and the candidate solution is computed without changing the slopes; corrections to slopes are made if the candidate solution is “problematic” (***a posteriori***).

⇩

MOOD generally **less intrusive**, providing **better accuracy**

- MOOD vs. MUSCL

- MUSCL in brief

Reconstructed values at the left- and right-hand side of interfaces:

$$\phi_{i-1/2,R} = \phi_i - q_i(\phi)\frac{\Delta x}{2}, \quad \phi_{i+1/2,L} = \phi_i + q_i(\phi)\frac{\Delta x}{2}, \quad \text{with } \phi = \eta, h, hu,$$

$$b_{i-1/2,R} = \eta_{i-1/2,R} - h_{i-1/2,R}, \quad b_{i+1/2,L} = \eta_{i+1/2,L} - h_{i+1/2,L},$$

where

$$q_i(\phi) = \theta\left(p_{i-1/2,R}(\phi), p_{i+1/2,L}(\phi)\right) \quad \left\{ \begin{array}{l} \text{van-Alabada} \\ \text{min-mod} \\ \text{van-Leer} \end{array} \right.$$

- MOOD vs. MUSCL

- MOOD in brief

  - **Key concepts**: **Cell/Edge Polynomial Degree** (CPD/EPD)

EPD $\longrightarrow$ polynomial degree used to evaluate the reconstruction at both sides of each edge:

$$\mathrm{EPD}_{i+1/2}(\phi) = \min\left(\mathrm{CPD}_i(\phi), \mathrm{CPD}_{i+1}(\phi)\right)$$

$$\mathrm{CPD}_i(\phi) = \begin{cases} 1 & \rightarrow \quad \text{full slope used (second-order)} \\ 0 & \rightarrow \quad \text{null slope used (first-order)} \end{cases}$$

- MOOD vs. MUSCL

- MOOD in brief

- **MOOD loop:**

know approximation $U^n = (U_i^n)_{i=1,\dots,I}$ at time t^n

initialise $\mathrm{CPD} = 1$ for each cell and function

compute candidate solution U^* from CPD/EDP map

check if each U_i^* satisfies a set of conditions (detectors)

if all cells are valid then $U^{n+1} = U^*$, otherwise reduce CPD and

next time step

- MOOD vs. MUSCL

- MOOD basic detectors

- **Aim**: determine if U_i^* is **eligible** (physically ok, no spurious oscillations)

Physical Admissible Detector: $h_i^* \geq 0$
(PAD)

Maximum Principle Detector: $\min(\phi_{i-1}^n, \phi_i^n, \phi_{i+1}^n) \leq \phi_i^* \leq \max(\phi_{i-1}^n, \phi_i^n, \phi_{i+1}^n)$
(MPD)

Extrema Detector: $\min(\phi_{i-1}^*, \phi_{i+1}^*) \leq \phi_i^* \leq \max(\phi_{i-1}^*, \phi_{i+1}^*)$
(ED)

- MOOD vs. MUSCL

- MOOD relaxation detectors

  - **Aim**: distinguish real (smooth) extrema from extrema caused by local spurious oscillations (Gibbs phenomenon)

Local curvature indicator

$$C_i(\gamma) = \frac{\gamma_{i-1} - 2\gamma_i + \gamma_{i+1}}{(\Delta x)^2}, \quad i = 2, \dots, I-1; \quad C_1(\gamma) = C_2(\gamma), \quad C_I(\gamma) = C_{I-1}(\gamma).$$

$$\chi_{\min,i} = \min(C_{i-1}, C_i, C_{i+1}), \quad \chi_{\max,i} = \max(C_{i-1}, C_i, C_{i+1}), \quad i = 2, \dots, I-1.$$

$$\chi_{\min,i} = \chi_{\min,i}(\phi^*), \quad \chi_{\max,i} = \chi_{\max,i}(\phi^*).$$

- MOOD vs. MUSCL

- MOOD relaxation detectors

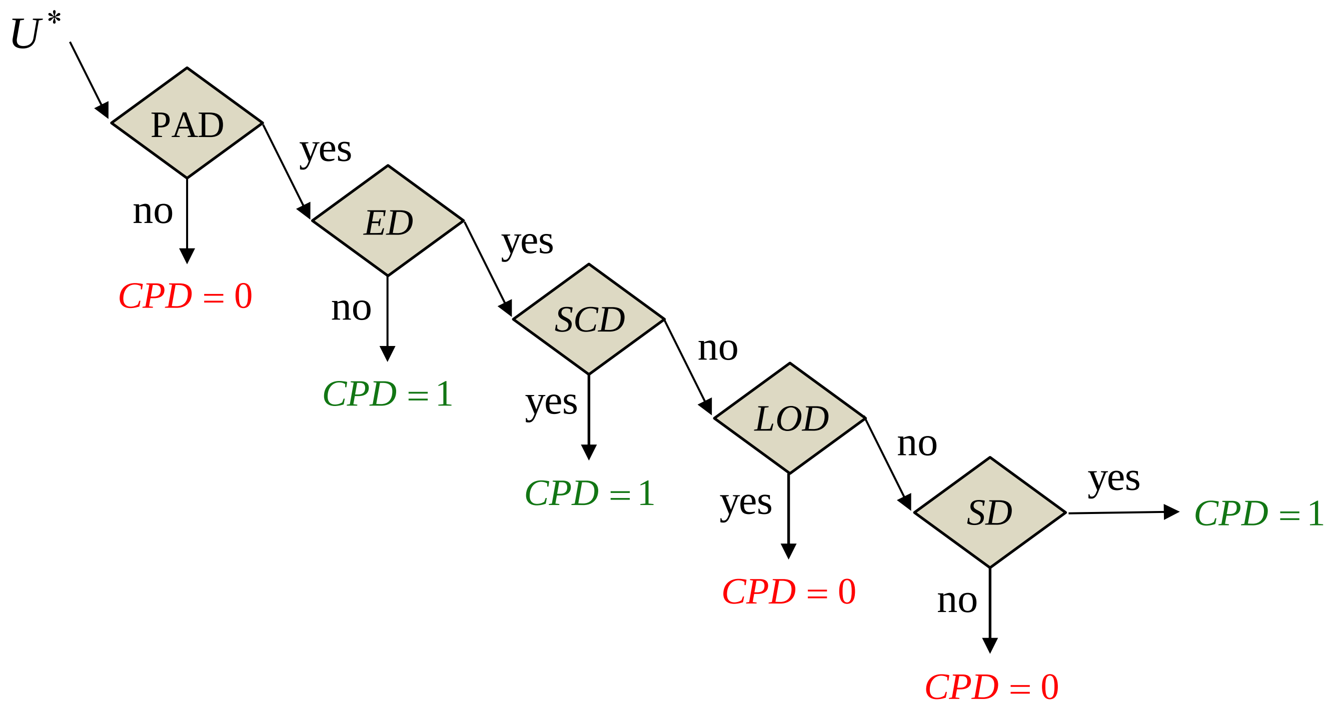
Small Curvature Detector: $\max(|\chi_{\min,i}|,|\chi_{\max,i}|) \leq \varepsilon_C.$
(SCD)
numerical solution is locally linear

Local Oscillation Detector: $\chi_{\min,i}\,\chi_{\max,i} \leq 0.$
(LOD)
local oscillation due to variation of curvature sign

Smoothness Detector: $\dfrac{\min(|\chi_{\min,i}|,|\chi_{\max,i}|)}{\max(|\chi_{\min,i}|,|\chi_{\max,i}|)} \geq 1-\varepsilon_S.$
(SD)
numerical solution is locally smooth

- MOOD vs. MUSCL

- MOOD detector algorithm



- Numerical results

- General aspects

Time scheme: TVD-RK2, $CFL = 0.4$

Conservative flux scheme: HLL

Meshes: $|c_i| = \Delta x$

Dry/wet threshold: 10^{-6}

MUSCL limiter procedure: η, h, hu

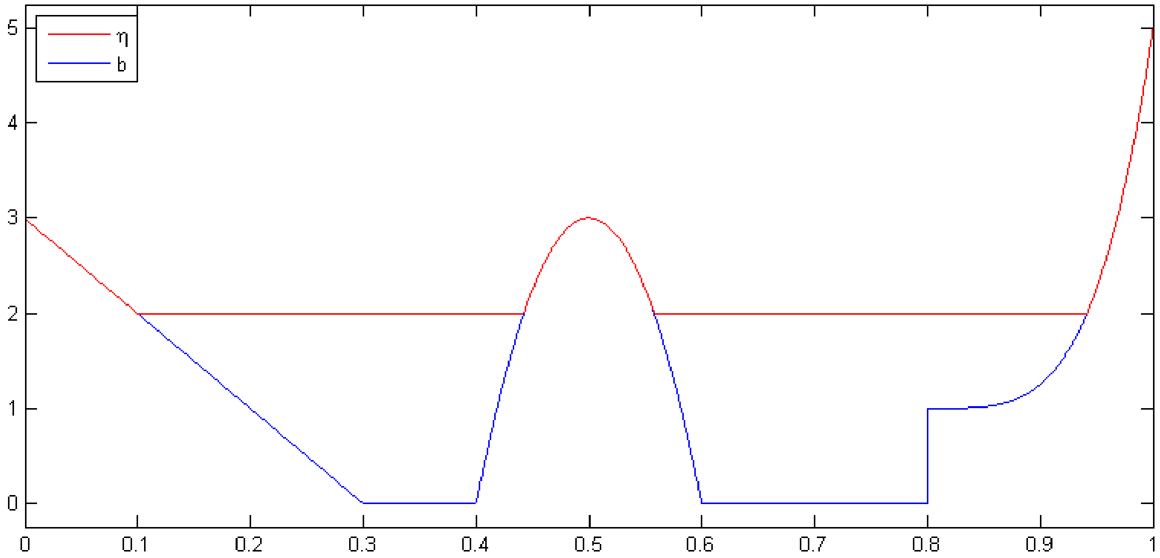
MOOD limiter procedure: h

$$\varepsilon_C = \delta^3, \text{ with } \delta = \Delta x / L; \quad \varepsilon_S = 0.5$$

Convergence assessment: L^1- error: $\frac{1}{I}\sum_{i=1}^{I}\left|\phi_i^N - \phi_i^{ex}\right|$, L^∞- error: $\max_i \left|\phi_i^N - \phi_i^{ex}\right|$

- Numerical results

❑ LAKE AT REST

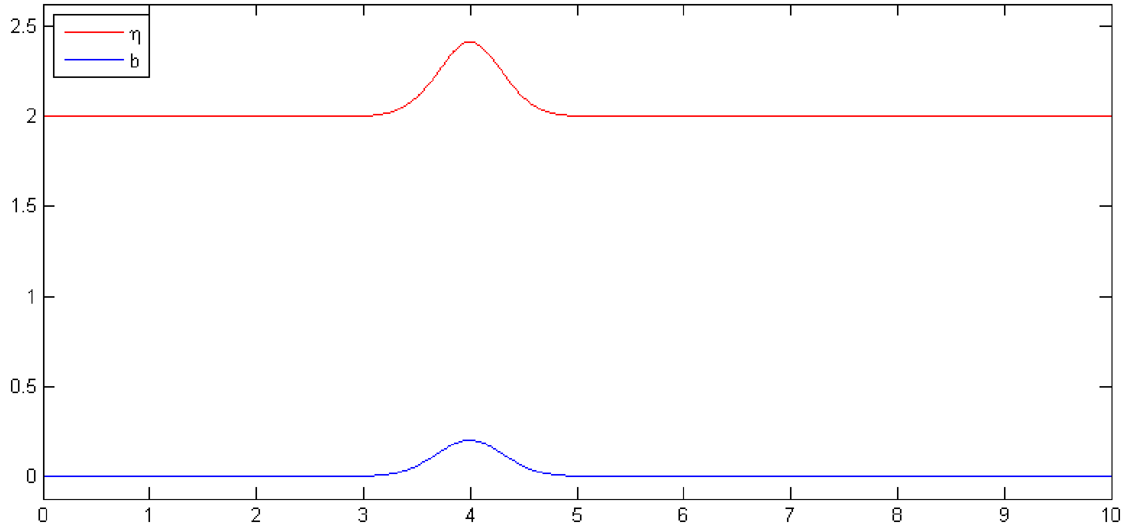


Meshes: $50, 100, 200$ cells; $T = 1 \rightarrow$ up to 2215 time steps.

L^1- error $< 10^{-15}$, L^∞- error $< 10^{-14}$ $\rightarrow$ C-property satisfied

- Numerical results

- SMOOTH SOLUTION (steady-state supercritical flow)



$\eta(0) = 2$

$q(x,t) = 13.29$

Meshes: 100, …,1600 cells; $T = 10$; $b(x) = 0.2\exp(-5(x-4)^2)$.

MUSCL detector only on h; MOOD detector.

- Numerical results

SMOOTH SOLUTION (steady-state supercritical flow)

Nb of Cells	η err_1		η err_∞		u err_1		u err_∞	
100	2.44e-03	—	4.76e-02	—	5.40e-03	—	7.91e-02	—
200	5.20e-04	2.2	2.05e-02	1.2	1.12e-03	2.3	3.38e-02	1.2
400	1.12e-04	2.2	5.77e-03	1.8	2.38e-04	2.2	9.52e-03	1.8
800	2.30e-05	2.3	1.47e-03	2.0	4.89e-05	2.3	2.43e-03	2.0
1600	5.43e-06	2.1	3.73e-04	2.0	1.16e-05	2.1	6.15e-04	2.0

MUSCL

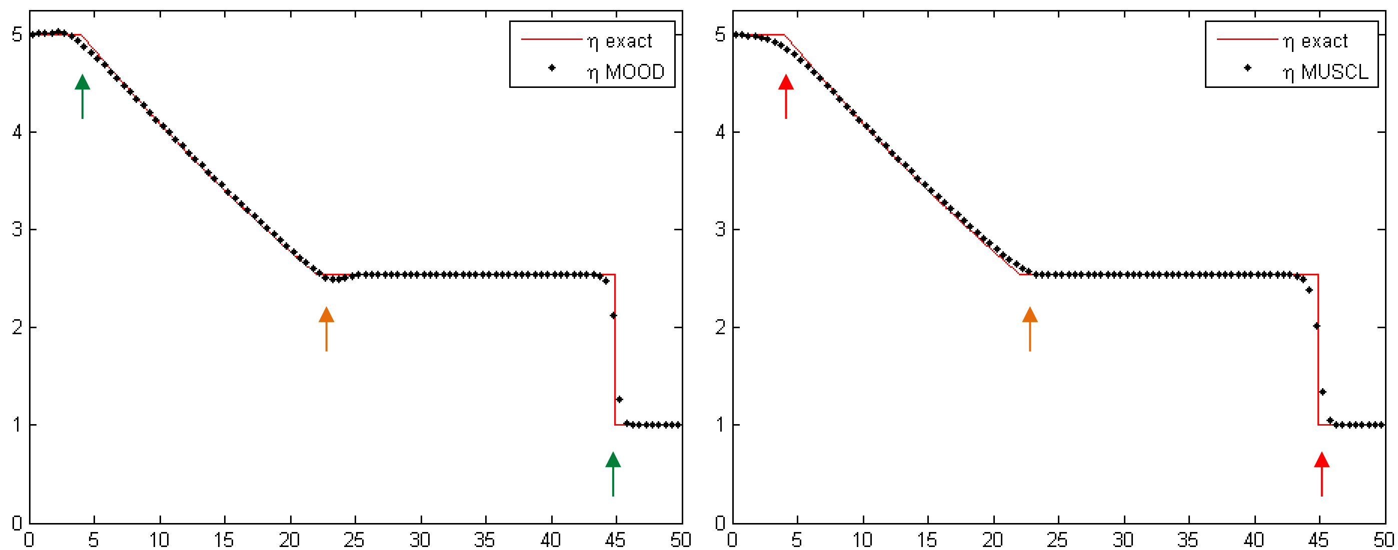
Nb of Cells	η err_1		η err_∞		u err_1		u err_∞	
100	1.91e-03	—	3.15e-02	—	5.06e-03	—	6.88e-02	—
200	5.23e-05	5.2	8.19e-04	5.3	1.24e-04	5.4	1.61e-03	5.4
400	8.49e-06	2.6	1.29e-04	2.7	2.13e-05	2.5	2.73e-04	2.6
800	1.45e-06	2.5	2.21e-05	2.5	3.81e-06	2.5	5.12e-05	2.4
1600	2.86e-07	2.3	4.26e-06	2.4	7.84e-07	2.3	1.07e-05	2.3

MOOD

- Numerical results

❑ DAM BREAK ON A WET BED

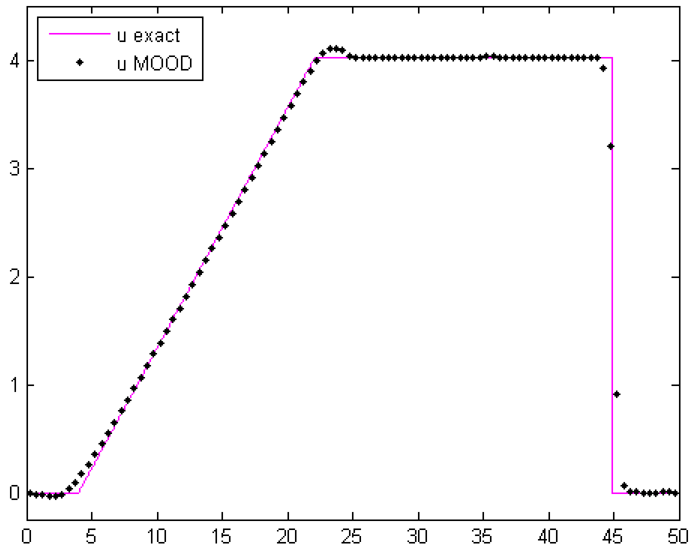
Mesh: 100 cells; $T = 3$; $\eta(x,0) = \begin{cases} 5, & 0 \le x \le 25, \\ 1, & 25 \le x \le 50. \end{cases}$



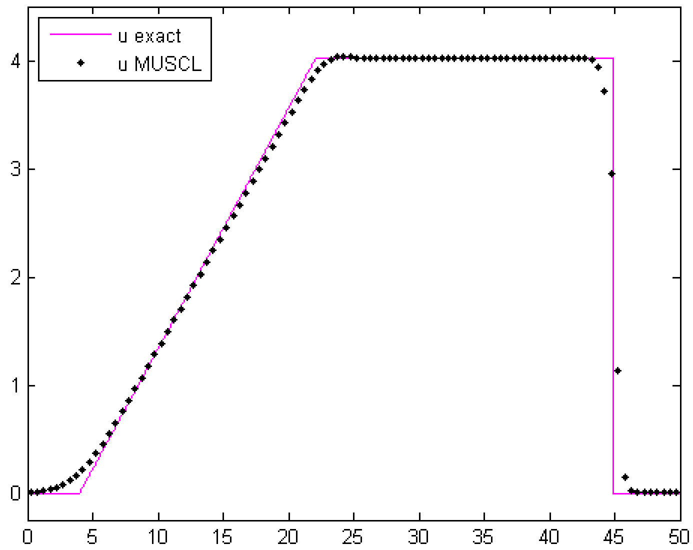
MOOD MUSCL

- Numerical results

DAM BREAK ON A WET BED



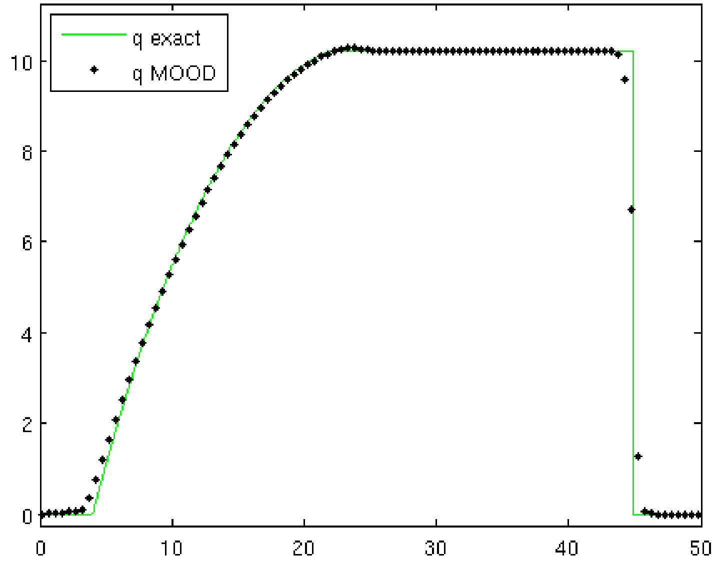
MOOD



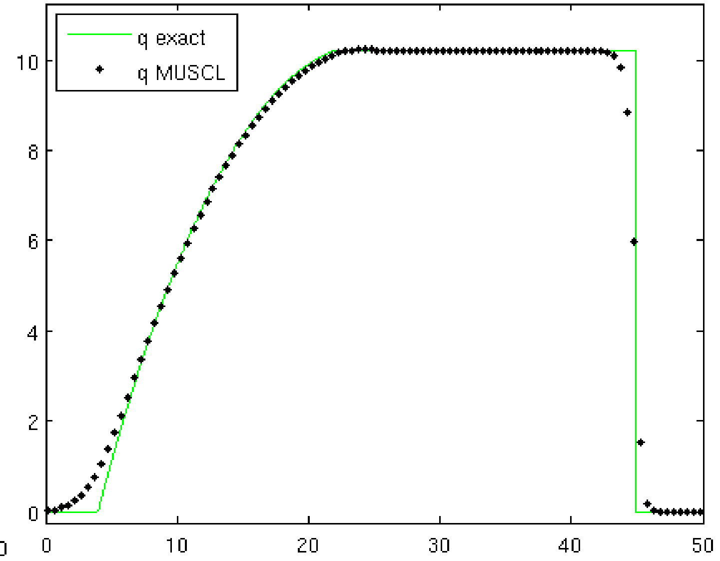
MUSCL

- Numerical results

DAM BREAK ON A WET BED



MOOD



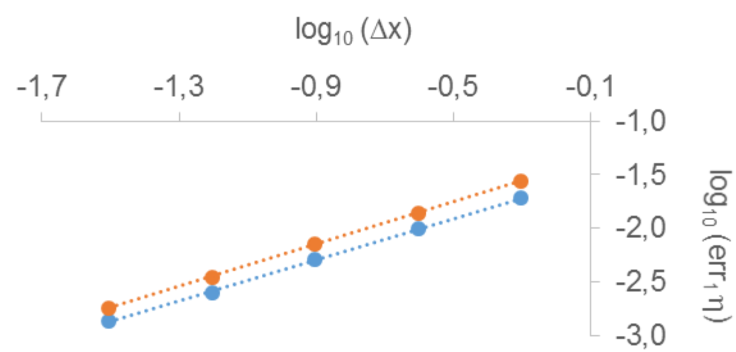
MUSCL

- Numerical results

DAM BREAK ON A WET BED

Overall convergence:

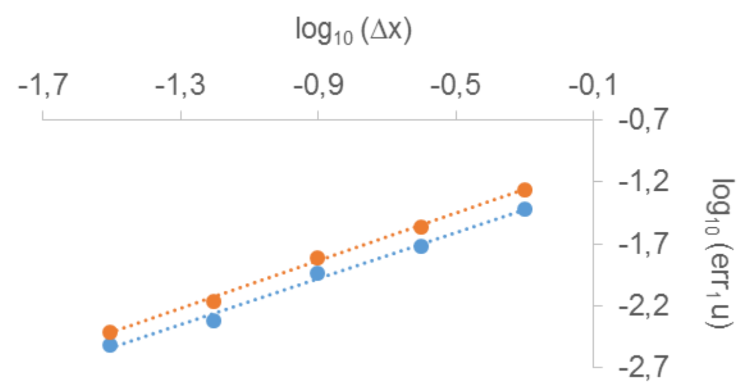
$$\mathrm{err}_1 = C\,(\Delta x)^{\beta}$$



MOOD:

$$C_\eta = 0.037$$

$$C_u = 0.072$$



MUSCL:

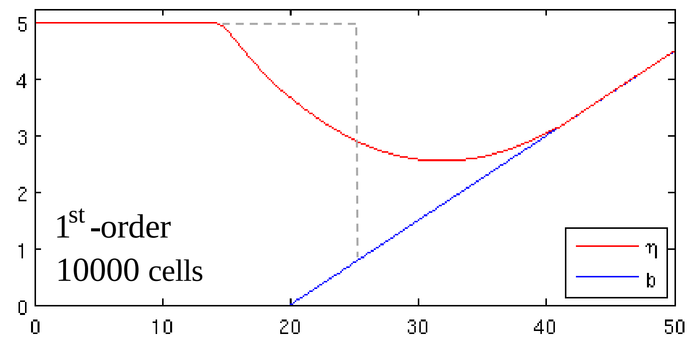
$$C_\eta = 0.055$$

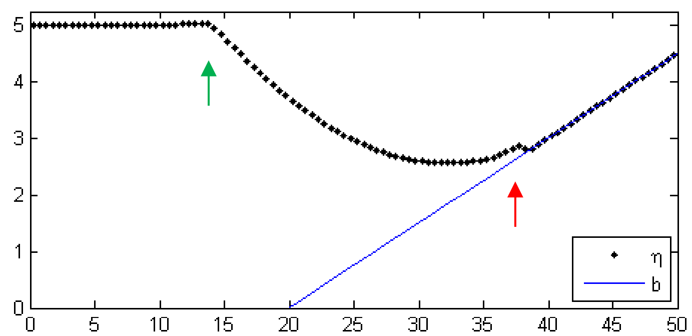
$$C_u = 0.107$$

- Numerical results

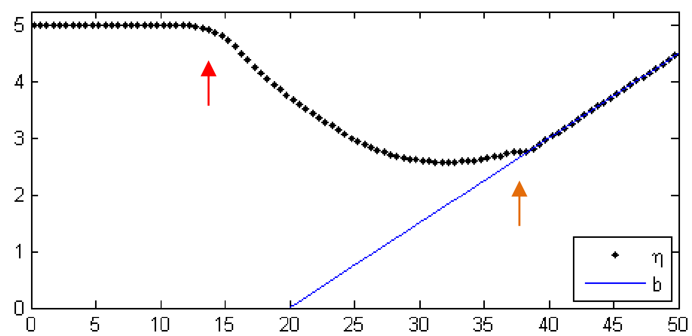
❑ DRY/WET SIMULATION (smooth bathymetry)

Mesh: 100 cells; $T = 1.5$.





MOOD

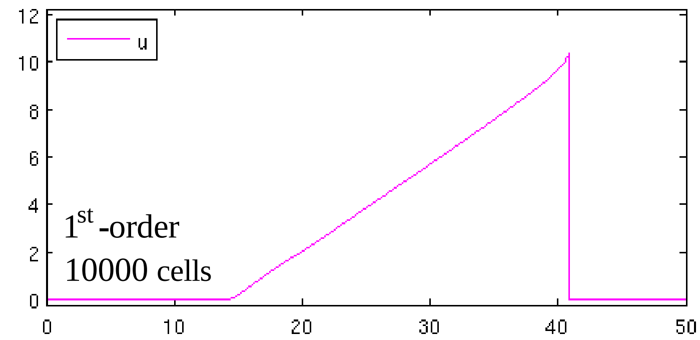


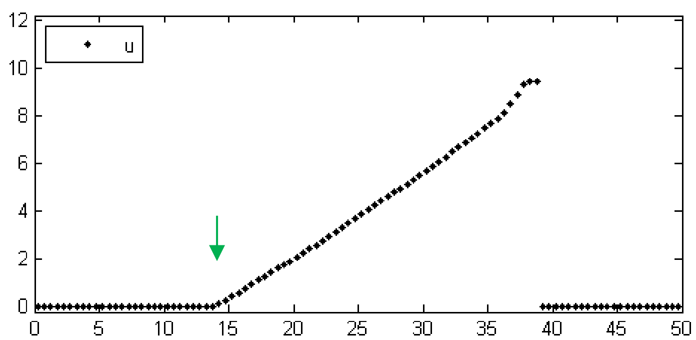
MUSCL

- Numerical results

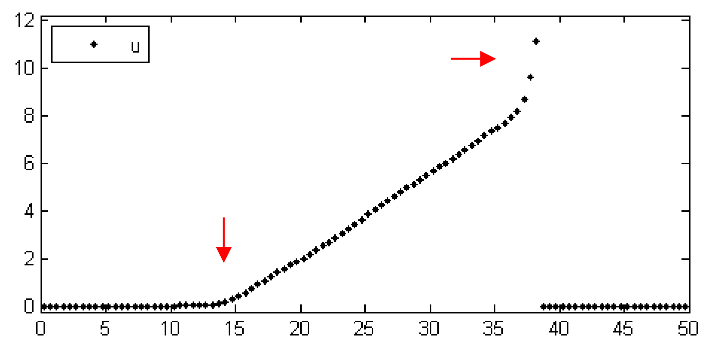
DRY/WET SIMULATION (smooth bathymetry)

Mesh: 100 cells; $T = 1.5$.





MOOD



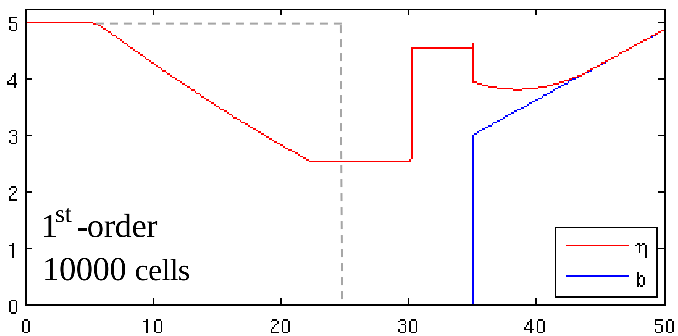
MUSCL

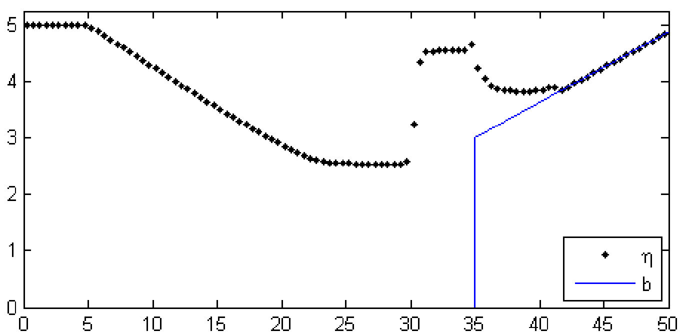
- Numerical results

- DRY/WET SIMULATION (discontinuous bathymetry)

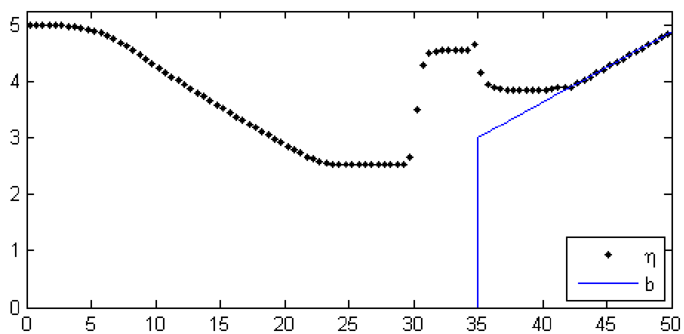
Mesh: 100 cells; $T = 2.75$.

MOOD involves h and u!





MOOD

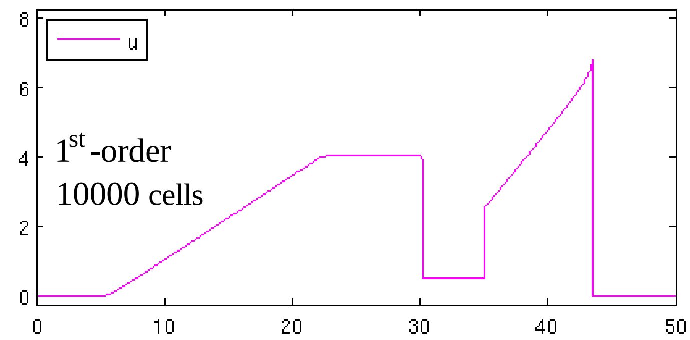


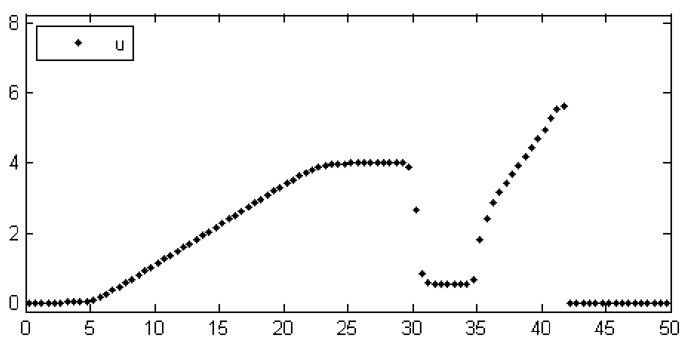
MUSCL

- Numerical results

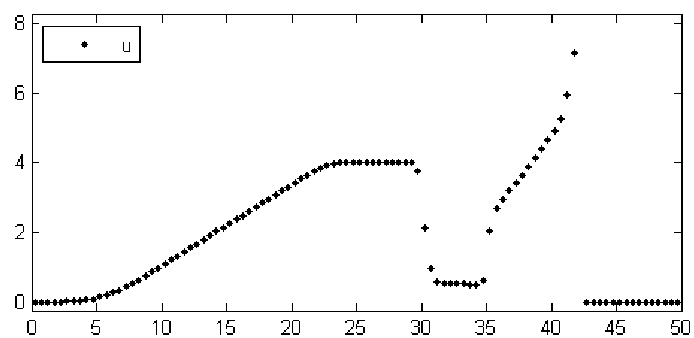
DRY/WET SIMULATION (discontinuous bathymetry)

Mesh: 100 cells; $T = 2.75$.





MOOD



MUSCL

- Conclusions and future work

- We performed comparisons between the 2nd-order MOOD and the MUSCL methods for the non-linear shallow water system with varying bathymetry.

- Both methods comply with the C-property, but the MOOD method is less diffusive than the MUSCL method (shock, rarefaction propagation), and is more accurate when dealing with smooth solutions.

- Dry/wet interfaces with smooth/discontinuous bathymetry are reasonably well handled by MOOD technique (better velocity / kinetic energy estimates).

- Still, new detectors and schemes have to be proposed to improve the treatment of wet/dry interfaces (location, height and velocity profiles).