A a posteriori subcell limiter to stabilize Discontinuous Galerkin numerical schemes.

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Summary of the talk

- Introduction
- (Quick) Description and reminder on DG
- Subcell resolution to be maintain whil limiting
- Numerical results DG-$\mathbb{P}_9$ (Sod, Lax, smooth vortex, Shu Osher, double Mach, forwrad facing step, shock vortex interaction, Rieman problems 1, 2, 3, 4, 5, 3D explosion)
- Conlusions and perspectives

Introduction

Context

Rvisiting the stabilization of DG method of high accuracy (space/time) for hydrodynamics system of equations in multiD on quad/hexa mesh.

Discontinuous Galerkin method

DG schemes satisfy a local cell entropy inequality for any polynomial degree N used for the approximation of the discrete solution $\Longrightarrow$ nonlinear stability in L_2 norm for arbitrary high order of accuracy.
By nature very robust and appropriate for the solution of nonlinear hyperbolic conservation laws

(Numerical) live is not so simple

Even the DG method needs some sort of nonlinear limiting to avoid the Gibbs phenomenon at discontinuities $\Longrightarrow$ stabilization/limiting must be designed
Vast literature exists mostly based on "troubled cell" detector on unlimited solution then the DG polynomial is "modified" (artificial viscosity, "slope/moments" limiters, WENO)

Introduction

Time discretization (explicit)

Usualy made by TVD Runge-Kutta under a very severe time step restriction scaling as $1/(2N+1)$. This has to be put in perspective that some subcell resolution exists.

MOOD for FV schemes

MOOD paradigm and DG limiting are somewhat close, but different by nature :
MOOD considers two time levels and recomute the solution with different schemes.
DG limiters detect on one time level $n+1$ and change the final solution by hand.

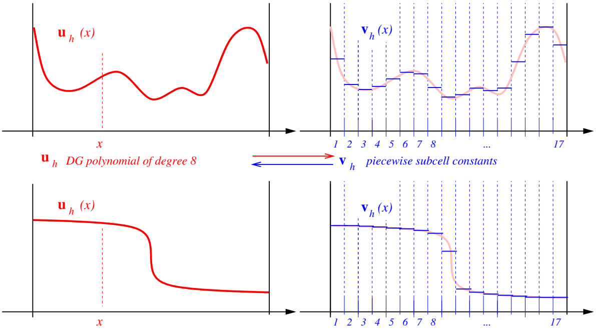
Our idea

1. Use MOOD detection procedure to replace the “trouble cell” indicator from DG limiter.
2. Use the idea of recomputing the numerical solution with a more dissipative scheme without loosing the subcell resolution property of DG $\Longrightarrow$ compute with (WENO, MUSCL, FV) on a subgrid in bad cells.

Subcell limiter for Discontinuous Galerkin (DG)

a posteriori MOOD type

Maintain subcell resolution of DG



$\mathbf{u}_h(\mathbf{x}, t)$ is $\mathbb{P}_N$ in main cell $T_i \Rightarrow$ $\mathbf{v}_h(\mathbf{x}, t)$ is $\mathbb{P}_0$ on fine subgrid of T_i, $\mathcal{S}_i = \bigcup_j S_{i,j}$.

Operators reconstruction $\mathcal{R}$ and projection $\mathcal{P}$ satisfy $\mathcal{R} \circ \mathcal{P} = \mathcal{I}$. Need to detect problematic cells on subscale level.

MOOD detection and decrementing using ADER-WENO-FV scheme on subcells

a posteriori detect if the unlimited DG candidate solution $\mathbf{u}_i^*(\mathbf{x})$ is problematic, if so recompute this cell on subcell level (after $\mathcal{P}$) with a high-order accurate ADER-WENO-FV scheme to get $\mathbf{v}_j^{n+1}$, j subcells, and retrieve back $\mathbf{u}_i^{n+1}(\mathbf{x})$ after $\mathcal{R}$.

MOOD For Discontinuous Galerkin (DG) schemes

One step ADER DG schemes Dumbser, Balsara, Toro, Munz. JCP, 227, 2008.

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{Q}) = 0, \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \quad t \in \mathbb{R}_0^+,$$

$\mathbf{Q}$ is represented within each cell T_i by piecewise polynomials of maximum degree $N \geq 0$, $\mathbf{u}_h(\mathbf{x}, t^n)$ the discrete “representation”

$$\mathbf{u}_h(\mathbf{x}, t^n) = \sum_l \Phi_l(\mathbf{x})\hat{\mathbf{u}}_l^n, \quad \mathbf{x} \in T_i,$$

Local space-time predictor $\mathbf{q}_h$

$\mathbf{u}_h(\mathbf{x}, t^n)$ is evolved in time according to a local weak formulation of the governing PDE in space-time (reference coord. system (ξ, τ)).
Skipping the math-nipulations, one gets the iterative scheme

$$\mathbf{q}_h = \mathbf{q}_h(\boldsymbol{\xi}, \tau) = \sum_l \theta_l(\boldsymbol{\xi}, \tau)\hat{\mathbf{q}}_l := \theta_l \hat{\mathbf{q}}_l, \qquad \mathbf{F}_h^* = \mathbf{F}_h^*(\boldsymbol{\xi}, \tau) = \sum_l \theta_l(\boldsymbol{\xi}, \tau)\hat{\mathbf{F}}_l^* := \theta_l \hat{\mathbf{F}}_l^*,$$

$$\left([\theta_k, \theta_l]^1 - \left\langle \frac{\partial}{\partial \tau}\theta_k, \theta_l \right\rangle \right) \hat{\mathbf{q}}_l^{r+1} = [\theta_k, \Phi_l]^0 \, \hat{\mathbf{u}}_l^n - \langle \theta_k, \nabla_\xi \theta_l \rangle \cdot \mathbf{F}^*(\hat{\mathbf{q}}_l^r),$$

MOOD For Discontinuous Galerkin (DG) schemes

Arbitrary high order accurate one-step DG (ADER-DG) scheme

Fully discrete one-step ADER-DG scheme : multiplication of the PDE by a test function Φ_k, integration over the space-time control volume $T_i \times [t^n; t^{n+1}]$, flux divergence term integrated by parts to get the weak formulation

$$\int_{t^n}^{t^{n+1}} \int_{T_i} \Phi_k \frac{\partial \mathbf{u}_h}{\partial t} d\mathbf{x}dt + \int_{t^n}^{t^{n+1}} \int_{\partial T_i} \Phi_k \, \mathbf{F}\left(\mathbf{u}_h\right) \cdot \mathbf{n} \, dSdt - \int_{t^n}^{t^{n+1}} \int_{T_i} \nabla \Phi_k \cdot \mathbf{F}\left(\mathbf{u}_h\right) d\mathbf{x}dt = 0,$$

Inserting $\mathbf{q}_h$ yields the arbitrary high order accurate one-step DG (ADER-DG) scheme :

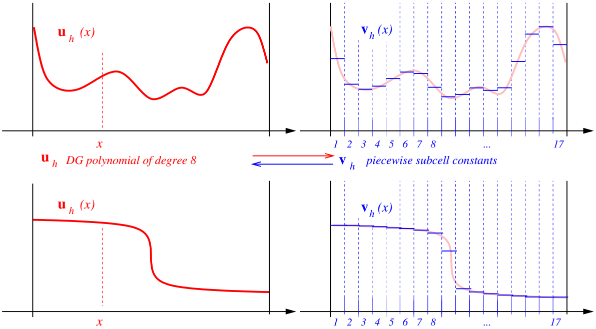
$$\left(\int_{T_i} \Phi_k \Phi_l d\mathbf{x} \right) \left(\hat{\mathbf{u}}_l^{n+1} - \hat{\mathbf{u}}_l^n \right) + \int_{t^n}^{t^{n+1}} \int_{\partial T_i} \Phi_k \, \mathcal{G}\left(\mathbf{q}_h^-, \mathbf{q}_h^+ \right) \cdot \mathbf{n} \, dSdt - \int_{t^n}^{t^{n+1}} \int_{T_i} \nabla \Phi_k \cdot \mathbf{F}\left(\mathbf{q}_h\right) d\mathbf{x}dt = 0.$$

Are we there yet ? Not yet ! We need some limiter

Classical procedures : artificial viscosity, WENO limiting, slope/moment reduction, etc.
Common design principle : Spurious numerical oscillations can be detected and corrected by looking at discrete solution at t^n usually without using the PDE.

MOOD For Discontinuous Galerkin (DG) FE schemes

Maintain subcell resolution of DG



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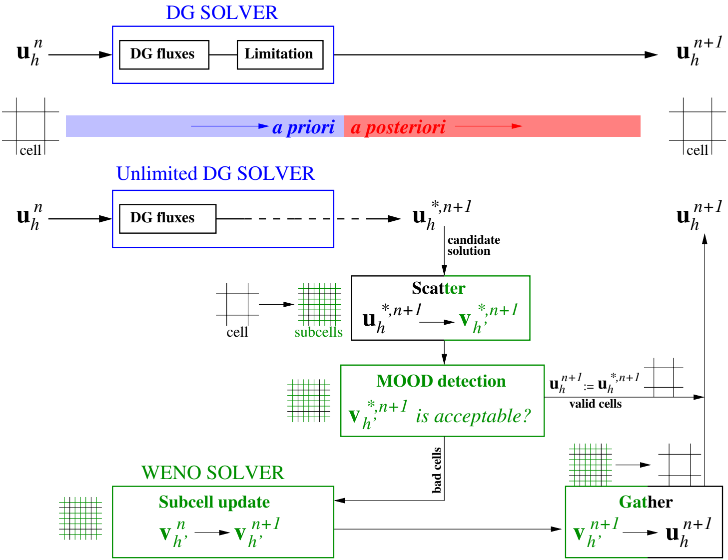
MOOD For Discontinuous Galerkin (DG) FE schemes



MOOD For Discontinuous Galerkin (DG) FE schemes

Simplest *a posteriori* PAD and NAD detection for hydrodynamics equations

- PAD : candidate solution $\mathbf{u}_h^*(\mathbf{x}, t^{n+1})$ is physically valid in cell T_i if $\rho_h^*(\mathbf{x}, t^{n+1}) > 0,\ p_h^*(\mathbf{x}, t^{n+1}),\ \forall \mathbf{x} \in T_i$
- NAD : relaxed DMP.
$$\min_{\mathbf{y}\in\mathcal{V}_i} \left(\mathbf{u}_h(\mathbf{y}, t^n)\right) - \delta \le \mathbf{u}_h^*(\mathbf{x}, t^{n+1}) \le \max_{\mathbf{y}\in\mathcal{V}_i} \left(\mathbf{u}_h(\mathbf{y}, t^n)\right) + \delta, \qquad \forall \mathbf{x} \in T_i,$$
we use the "discrete subcell" version of previous equation
$$\min_{\mathbf{y}\in\mathcal{V}_i} \left(\mathbf{v}_h(\mathbf{y}, t^n)\right) - \delta \le \mathbf{v}_h^*(\mathbf{x}, t^{n+1}) \le \max_{\mathbf{y}\in\mathcal{V}_i} \left(\mathbf{v}_h(\mathbf{y}, t^n)\right) + \delta, \qquad \forall \mathbf{x} \in T_i,$$

Summary

Detection criteria : PAD and NAD on subcell scale
Decrementing : none really ! Only DG solver on cells replaced by WENO solver on subcells
Cascade of schemes : DG9-cell→WENO3-subcell
Parachute scheme : WENO3

Numerical code

3D MPI parallel structured DG code ADER-DG-$\mathbb{P}_N$ ($N = 5$ or 9) + *a posteriori* SubCell Limiter (SCL). ADER-WENO3 scheme is acting at the subcell level ($\mathbb{P}_2$ reconstructions).
The full scheme is DG-$\mathbb{P}_N$+WENO3 SCL.

MOOD For Discontinuous Galerkin (DG) FE schemes

Subgrid, time step

Subgrid

The subgrid is chosen as to maintain the same Δt_{DG} this implies that $N_s = 2N + 1$. Indeed DG constrain is :

$$\Delta t \leq \frac{1}{d} \frac{1}{(2N+1)} \frac{h}{|\lambda_{\max}|}, \tag{1}$$

FV method on the subgrid must satisfy

$$\Delta t \leq \frac{1}{d} \frac{1}{N_s} \frac{h}{|\lambda_{\max}|}. \tag{2}$$

h/N_s is the size of the subcell.
Note that $2N + 1$ is greater than the minimal number of subcell needed to represent the whole DG information.

MOOD For Discontinuous Galerkin (DG) FE schemes

Code, testing campaign

Numerical code on hydrodynamics

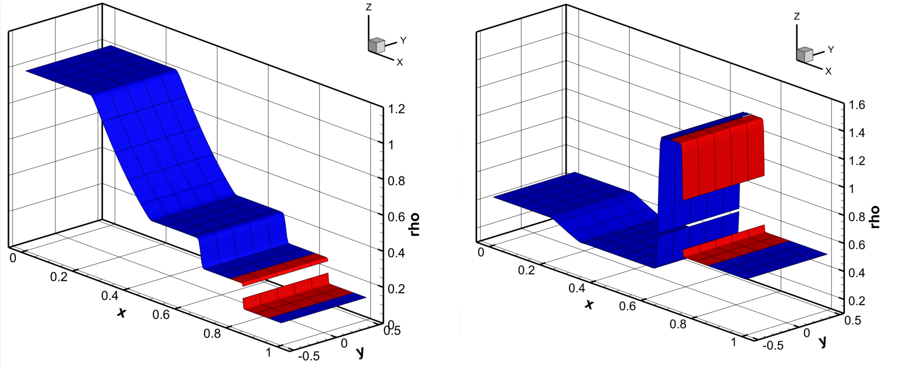
3D MPI parallel structured DG code ADER-DG-$\mathbb{P}_N$ ($N = 5$ or 9) + *a posteriori* SubCell Limiter (SCL). ADER-WENO3 scheme is acting at the subcell level ($\mathbb{P}_2$ reconstructions).
The full scheme is DG-$\mathbb{P}_N$+WENO3 SCL.
Formally 6th or 10th order accurate scheme on smooth solution. On non smooth solution the limiting acts on subcell features

Test methodology

- Try 1D (with 2D code) : Sod, Lax shock tubes
- Verify effective high accuracy on smooth vortex (note that the limiter has the choice to act is it feels the need)
- Capture small scales with Shu-Osher test and limit when these become shocks
- Classical physical tests (Double Mach, FFstep, shock vortex)
- Classical tests to write articles (RP : 1, 2, 3, 4, 5)
- 3D performance for the ego (3D, $10^1 0$ dof, 8000 CPUs, etc.)

MOOD For Discontinuous Galerkin (DG) FE schemes

Sod and Lax shock tube



Sod shock tube problem (left) at $t_{\text{final}} = 0.2$ and Lax problem (right) at $t_{\text{final}} = 0.14$. Coarse mesh of only 20×5 cells on the main grid. ADER-DG-$\mathbb{P}_9$ with WENO3 subcell limiter. The density variable is displayed. Troubled cells are shown in red, while blue cells (unlimited ADER-DG-$\mathbb{P}_9$ on the main grid).

MOOD For Discontinuous Galerkin (DG) FE schemes

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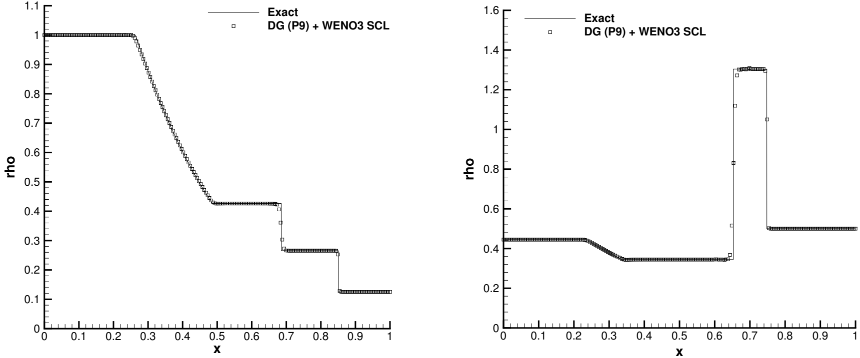
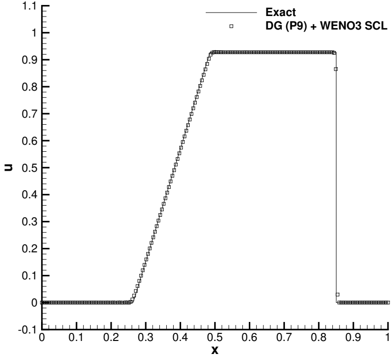
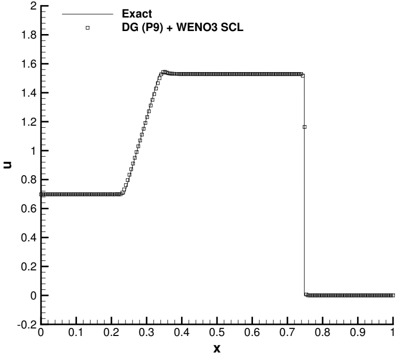


Sod shock tube problem (left) at $t_{\text{final}} = 0.2$ and Lax problem (right) at $t_{\text{final}} = 0.14$. ADER-DG-$\mathbb{P}_9$ with WENO3 subcell limiter Coarse mesh of 20×5 cells .
1D cut on 200 equidistant sample points through the numerical solution (symbols) *vs* exact solution for density.

MOOD For Discontinuous Galerkin (DG) FE schemes

Sod and Lax shock tube





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MOOD For Discontinuous Galerkin (DG) FE schemes

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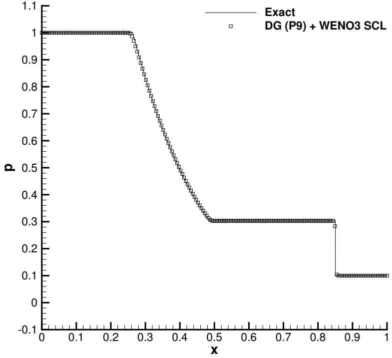

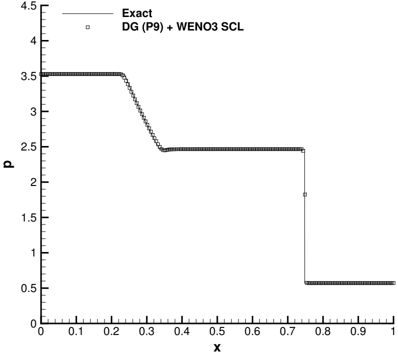

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1D cut on 200 equidistant sample points through the numerical solution (symbols) *vs* exact solution for pressure.

MOOD For Discontinuous Galerkin (DG) FE schemes

Smooth vortex

	N_x	L^1 error	L^2 error	L^∞ error	L^1 order	L^2 order	L^∞ order	Theor.
DG-$\mathbb{P}_2$	25	9.33E-03	2.07E-03	2.02E-03	—	—	—	3
	50	6.70E-04	1.58E-04	1.66E-04	3.80	3.71	3.60	
	75	1.67E-04	4.07E-05	4.45E-05	3.43	3.35	3.25	
	100	6.74E-05	1.64E-05	1.82E-05	3.15	3.15	3.10	
DG-$\mathbb{P}_3$	25	5.77E-04	9.42E-05	7.84E-05	—	—	—	4
	50	2.75E-05	4.52E-06	4.09E-06	4.39	4.38	4.26	
	75	4.36E-06	7.89E-07	7.55E-07	4.55	4.30	4.17	
	100	1.21E-06	2.37E-07	2.38E-07	4.46	4.17	4.01	
DG-$\mathbb{P}_4$	20	1.54E-04	2.18E-05	2.20E-05	—	—	—	5
	30	1.79E-05	2.46E-06	2.13E-06	5.32	5.37	5.75	
	40	3.79E-06	5.35E-07	5.18E-07	5.39	5.31	4.92	
	50	1.11E-06	1.61E-07	1.46E-07	5.50	5.39	5.69	
DG-$\mathbb{P}_5$	10	9.72E-04	1.59E-04	2.00E-04	—	—	—	6
	20	1.56E-05	2.13E-06	2.14E-06	5.96	6.22	6.55	
	30	1.14E-06	1.64E-07	1.91E-07	6.45	6.33	5.96	
	40	2.17E-07	2.97E-08	3.59E-08	5.77	5.93	5.82	
DG-$\mathbb{P}_6$	5	2.24E-02	4.15E-03	3.11E-03	—	—	—	7
	10	1.76E-04	2.75E-05	2.86E-05	6.99	7.24	6.76	
	20	1.67E-06	2.28E-07	2.26E-07	6.72	6.91	6.98	
	25	3.60E-07	4.96E-08	6.27E-08	6.86	6.84	5.74	

MOOD For Discontinuous Galerkin (DG) FE schemes

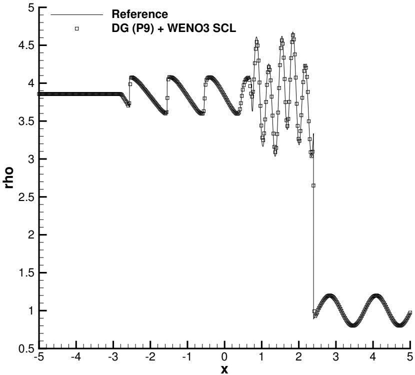
Smooth vortex

	N_x	L^1 error	L^2 error	L^∞ error	L^1 order	L^2 order	L^∞ order	Theor.
DG-$\mathbb{P}_7$	5	5.50E-03	1.22E-03	1.46E-03	—	—	—	8
	10	4.63E-05	6.26E-06	6.95E-06	6.89	7.61	7.71	
	15	1.62E-06	2.20E-07	2.29E-07	8.28	8.26	8.42	
	20	2.05E-07	2.80E-08	2.28E-08	7.18	7.17	8.01	
DG-$\mathbb{P}_8$	4	9.11E-03	1.80E-03	3.44E-03	—	—	—	9
	8	4.97E-05	7.51E-06	6.93E-06	7.52	7.90	8.96	
	10	7.50E-06	1.05E-06	1.18E-06	8.47	8.81	7.95	
	15	2.40E-07	3.34E-08	3.09E-08	8.49	8.51	8.98	
DG-$\mathbb{P}_9$	4	3.95E-03	7.89E-04	1.42E-03	—	—	—	10
	8	1.01E-05	1.44E-06	1.52E-06	8.61	9.09	9.87	
	10	1.44E-06	2.00E-07	2.27E-07	8.74	8.85	8.51	
	12	2.67E-07	3.70E-08	3.77E-08	9.26	9.25	9.85	

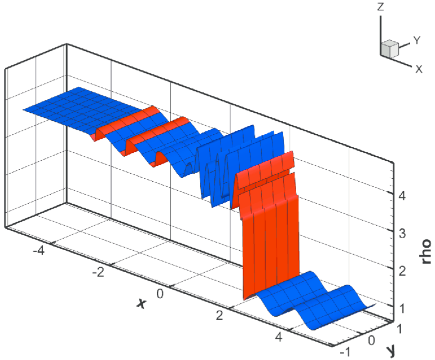
MOOD For Discontinuous Galerkin (DG) FE schemes

Shu-Osher oscillatory test

ADER-DG-$\mathbb{P}_9$ with ADER-WENO3 subcell limiter on a 40×5 mesh



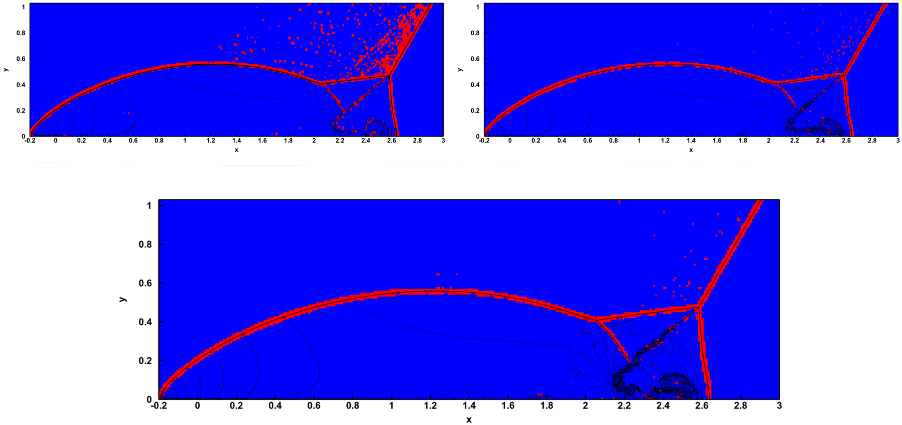
1D cut (symbols) *vs* reference solution. (ultra-fine ADER-WENO solution straight line). $\mathbb{P}_9$ polynomial represented by 10 sample points per cell



Red-troubled cells updated with ADER-WENO3 on the subgrid.
Blue-unlimited ADER-DG-$\mathbb{P}_9$ updated cells.

MOOD For Discontinuous Galerkin (DG) FE schemes

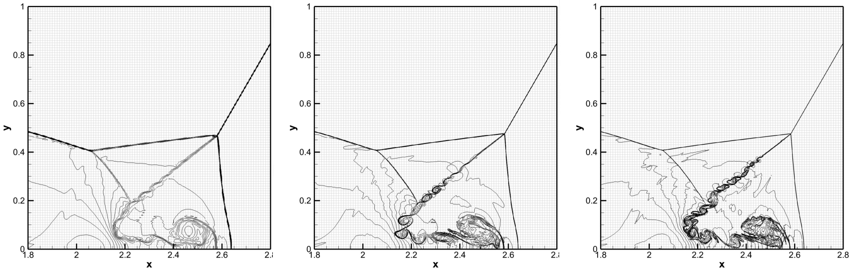
Double Mach reflection : ADER-DG-$\mathbb{P}_N$+ADER-WENO3 subcell limiter 350 $\times$ 100



Bad cells (red) for ADER-DG-$\mathbb{P}_2$, ADER-DG-$\mathbb{P}_5$
ADER-DG-$\mathbb{P}_9$

MOOD For Discontinuous Galerkin (DG) FE schemes

Double Mach reflection : ADER-DG-$\mathbb{P}_N$+ADER-WENO3 subcell limiter 350 $\times$ 100



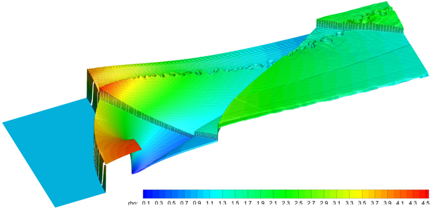
Density and main mesh for ADER-DG-$\mathbb{P}_2$, ADER-DG-$\mathbb{P}_5$, ADER-DG-$\mathbb{P}_9$.

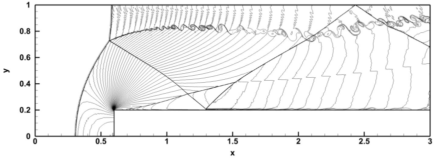
Because few cells are problematic in the vortex region then the unlimited (most accurate) scheme is used here. This is valid as the flow is exempt from shocks or steep gradients there.

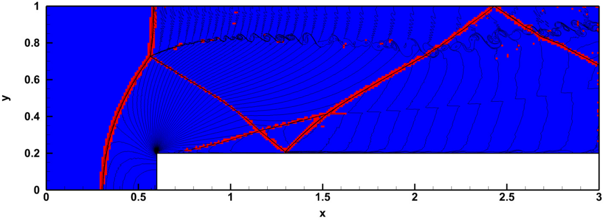
MOOD For Discontinuous Galerkin (DG) FE schemes

Forward facing step : ADER-DG-$\mathbb{P}_5$+ADER-WENO3 subcell limiter 300 $\times$ 100

Mach 3 wind tunnel with a step. Initialization $\rho = \gamma$, $p = 1$, velocity $u = 3$, $v = 0$ and $\gamma = 1.4$.





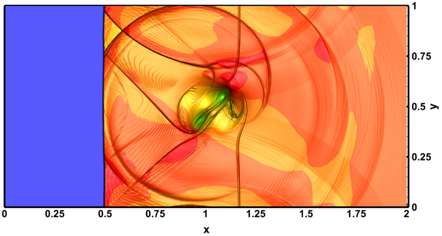


MOOD For Discontinuous Galerkin (DG) FE schemes

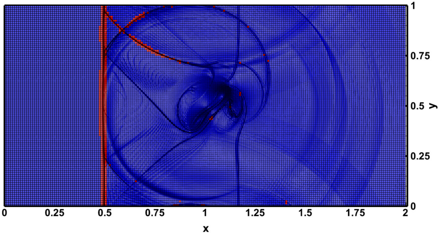
Shock vortex interaction : ADER-DG-$\mathbb{P}_5$+ADER-WENO3 subcell limiter 200×100

Initialization from Rault, Chiavassa, Donat, J. Sci. Comput. 19 (2003).

Density



Bad cells (red)



Ability to capture at the same time shock waves and smooth vortex features that produce small amplitude acoustic waves.

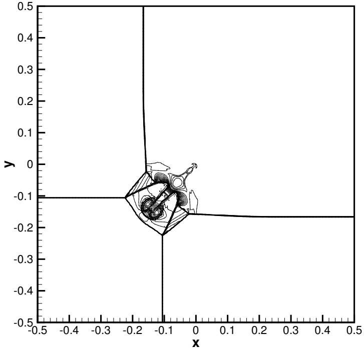
MOOD For Discontinuous Galerkin (DG) FE schemes

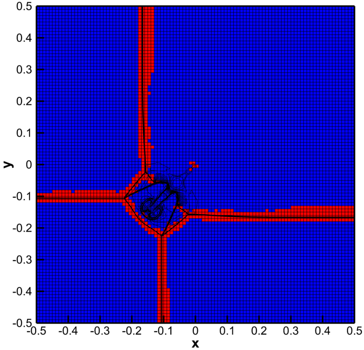
Riemann problems 1, 2, 3, 4, 5

#		ρ ($x \leq 0$)	u ($x \leq 0$)	v ($x \leq 0$)	p ($x \leq 0$)	ρ ($x > 0$)	u ($x > 0$)	v ($x > 0$)	p ($x > 0$)	t_{final}
RP1	$y > 0$	0.5323	1.206	0.0	0.3	1.5	0.0	0.0	1.5	0.25
	$y \leq 0$	0.138	1.206	1.206	0.029	0.5323	0.0	1.206	0.3	
RP2	$y > 0$	0.5065	0.8939	0.0	0.35	1.1	0.0	0.0	1.1	0.25
	$y \leq 0$	1.1	0.8939	0.8939	1.1	0.5065	0.0	0.8939	0.35	
RP3	$y > 0$	2.0	0.75	0.5	1.0	1.0	0.75	-0.5	1.0	0.30
	$y \leq 0$	1.0	-0.75	0.5	1.0	3.0	-0.75	-0.5	1.0	
RP4	$y > 0$	1.0	-0.6259	0.1	1.0	0.5197	0.1	0.1	0.4	0.25
	$y \leq 0$	0.8	0.1	0.1	1.0	1.0	0.1	-0.6259	1.0	
RP5	$y > 0$	1.0	0.7276	0.0	1.0	0.5313	0.0	0.0	0.4	0.25
	$y \leq 0$	0.8	0.0	0.0	1.0	1.0	0.0	0.7276	1.0	

MOOD For Discontinuous Galerkin (DG) FE schemes

Riemann problems 1





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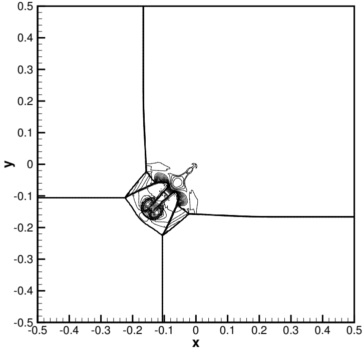

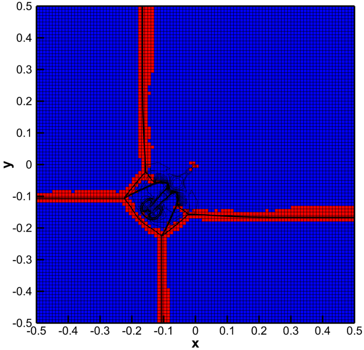

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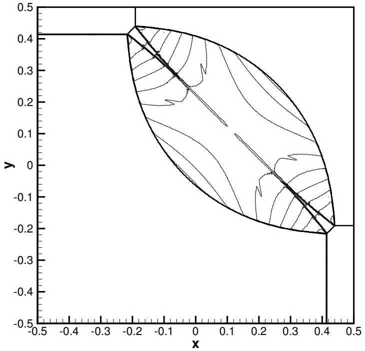
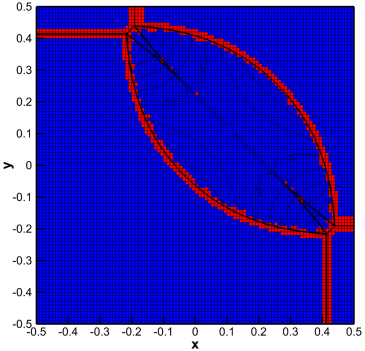
Riemann problems 2





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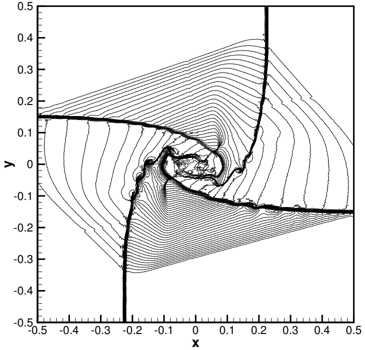
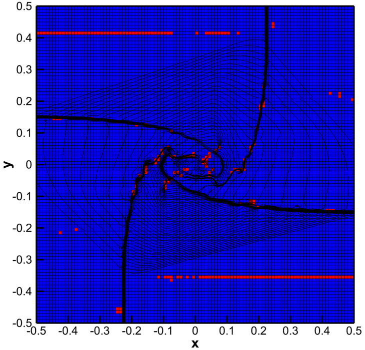
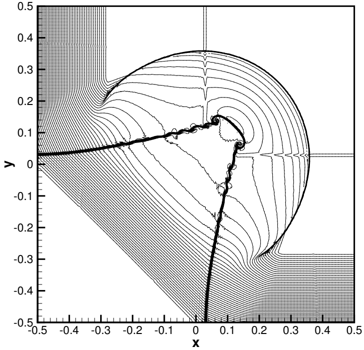
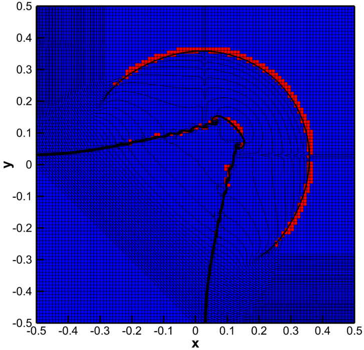




MOOD For Discontinuous Galerkin (DG) FE schemes

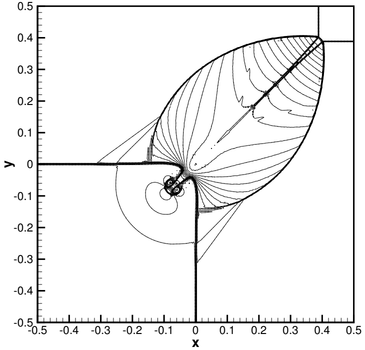
Riemann problems 4

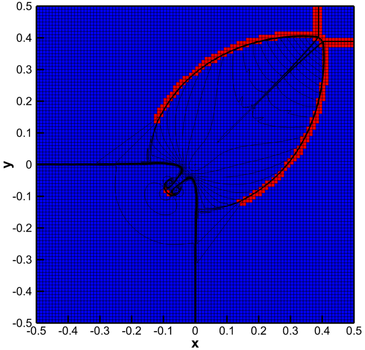




MOOD For Discontinuous Galerkin (DG) FE schemes

Riemann problems 5

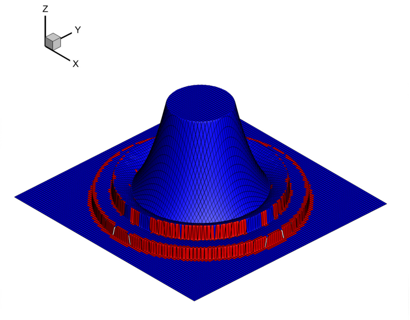


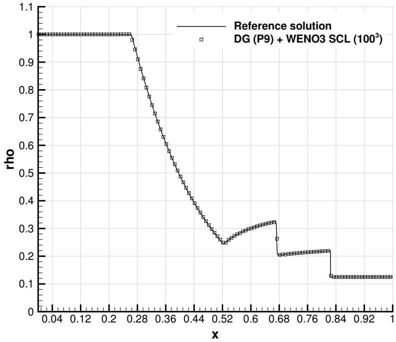


MOOD For Discontinuous Galerkin (DG) FE schemes

3D Sod problem : ADER-DG-$\mathbb{P}_9$+ADER-WENO3 subcell limiter 25^3 and 100^3

Mesh $100^3 \rightarrow N = 9$ and DOF$= (N+1)^4 = 10^{10}$, MPI run 8000 cores on SuperMUC (Munich)





X axis solution sampled on 125 equidistant points in order to represent the subcell resolution capabilities of the DG method on the coarse grid.

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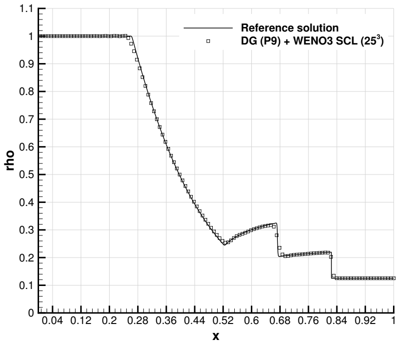

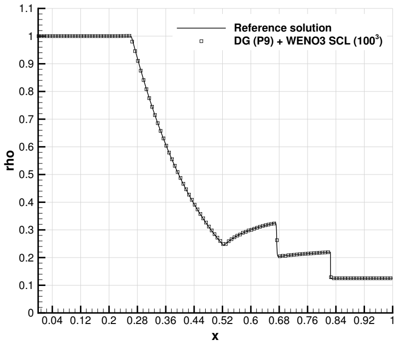

X axis solution sampled on 125 equidistant points in order to represent the subcell resolution capabilities of the DG method on the coarse grid. Even at the lowest resolution (12 elements for the x axis), the high degree of the DG polynomial performs well.
Note that the vertical gridlines correspond to the cell size of the main 25^3 grid

Conclusion

MOOD subcell limiter for DG schemes

- Subcell limiter for DG that preserves the subcell resolution.
- Test a candidate DG solution (at t^{n+1}) on subcell scale with NAD, PAD.
- For problematic cells, recompute with a subcell-based WENO/FV scheme (parachute)
- Retrieve back a DG polynomial on cells.
- Numerical tests on hydrodynamics 2D, 3D.

Perspectives

Future extensions

A lot...

Thank you for your attention

Acknowledgments

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