SHARK-FV 2015 Conference

SHARING HIGHER-ORDER ADVANCED RESEARCH KNOW-HOW on FINITE VOLUME

Ofir, Portugal

May 18 - 22, 2015

High-order finite volume scheme for the incompressible Navier-Stokes equations on staggered unstructured meshes

May 18-22, 2015, Ofir, Portugal

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Financed by FEDER Funds through COMPETE and by FCT within the Projects PEst-C/MAT/UI0013/2014, PTDC/MAT/121185/2010 and FCT-ANR/MAT-NAN/0122/2012

FCT Fundação para a Ciência e a Tecnologia MINISTÉRIO DA EDUCAÇÃO E CIÊNCIA

Escola de Ciências Universidade do Minho

COMPETE

QREN QUADRO DE REFERÊNCIA ESTRATÉGICO NACIONAL

UNIÃO EUROPEIA Fundo Europeu de Desenvolvimento Regional

In this presentation...

1. How to deal with div-grad dualities using high-order (finite volume) polynomial reconstructions free of oscilations and instabilities?

2. How to extend the method for non-linear problems?

3. How to preserve the quality of the fluxes approximations on smooth curved boundaries?

Outline

1. Incompressible Stokes Equations
   - Formulation and Discretization
   - Polynomial Reconstructions
   - Numerical Fluxes
   - Residual Operators
   - Numerical Results
2. Incompressible Navier-Stokes Equations
   - Formulation and Discretization
   - Polynomial Reconstructions
   - Numerical Fluxes
   - Residual Operators and Fixed-point Algorithm
   - Numerical Results
3. Dirichlet Condition to Curved Boundaries
   - Numerical Results
4. Conclusions and Current Studies

Incompressible Stokes Problem

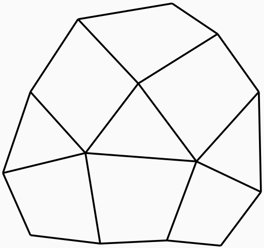
- Polygonal domain Ω with boundary $\partial\Omega$, steady-state and incompressible Stokes problem

$$
\begin{aligned}
&\nabla \cdot \left(-\nu \nabla U + \frac{1}{\rho} P I \right) = g, && \text{in } \Omega \\
&\nabla \cdot U = 0, && \text{in } \Omega \\
&U = U_{\mathrm{D}}, && \text{on } \partial\Omega
\end{aligned}
$$

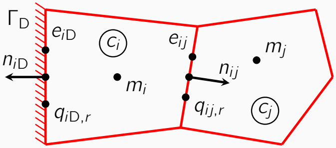
☞ The main difficulty is to handle the div-grad duality!

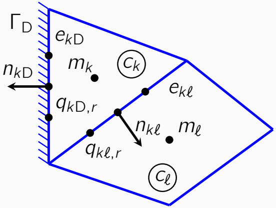
- Colocatted disctretization: all unknowns defined in one mesh; stabilization procedure required (e.g. Rhie-Chow interpolation)
- Staggered discretization: (primal) mesh and diamond mesh; stable and robust

Primal/Diamond Staggered Meshes



Primal Mesh





- Primal/diamond mean values (unkowns)

$$P_i \approx \frac{1}{|c_i|} \int_{c_i} P \,\mathrm{d}x, \quad U_{1,k} \approx \frac{1}{|c_k|} \int_{c_k} U_1 \,\mathrm{d}x, \quad U_{2,k} \approx \frac{1}{|c_k|} \int_{c_k} U_2 \,\mathrm{d}x$$

Finite Volume Discretization

- Momentum equation in diamond cells

$$\int_{c_k} \nabla \cdot \left(-\nu \nabla U + \frac{1}{\rho} P I_\mathrm{d} \right) \mathrm{d}X = \int_{c_k} g \mathrm{d}X \quad \Rightarrow$$

$$\frac{1}{|c_k|} \sum_{\ell \in \nu(k)} |e_{k\ell}| \left[\sum_{r=1}^{R} \zeta_r \left(\mathbb{F}^U_{\beta,k\ell,r} + \mathbb{F}^P_{\beta,k\ell,r} \right) \right] - g_{\beta,k} = \mathcal{O}(h_k^{2R}),\ k \in \mathcal{C}_\mathcal{D}$$

- Divergence-free equation in primal cells

$$\int_{c_i} \nabla \cdot U \mathrm{d}X = 0 \quad \Rightarrow \quad \frac{1}{|c_i|} \sum_{j \in \nu(i)} |e_{ij}| \left[\sum_{r=1}^{R} \zeta_r \mathbb{F}^\nabla_{ij,r} \right] = \mathcal{O}(h_i^{2R}),\ i \in \mathcal{C}_\mathcal{M}$$

- Physical fluxes

$$\mathbb{F}^U_{\beta,k\ell,r} = -\nu \nabla U_\beta(q_{k\ell,r}) \cdot n_{k\ell}, \qquad \mathbb{F}^P_{\beta,k\ell,r} = \frac{1}{\rho} P(q_{k\ell,r}) n_{\beta,k\ell}$$

$$\mathbb{F}^\nabla_{ij,r} = U(q_{ij,r}) \cdot n_{ij}$$

- Mean source term (Gaussian quadrature): $g_{\beta,k} \approx \int_{c_k} g_\beta \mathrm{d}X$

Polynomial Reconstructions for Primal Cells

- Conservative recontruction for primal cell c_i, $i \in \mathcal{C}_\mathcal{P}$

$$\frac{1}{|c_i|}\int_{c_i} \boldsymbol{P}_i(x)\,\mathrm{d}x = P_i, \qquad M_i^\alpha = \frac{1}{|c_i|}\int_{c_i} (x-m_i)^\alpha\,\mathrm{d}x$$

$$\boldsymbol{P}_i(x) = P_i + \sum_{1\le|\alpha|\le d} \mathcal{R}_i^\alpha \left[(x-m_i)^\alpha - M_i^\alpha\right]$$

- Coefficients $\mathcal{R}_i = (\mathcal{R}_i^\alpha)_{1\le|\alpha|\le d}$
- Minimization functional

$$E_i(\mathcal{R}_i) = \sum_{q\in S_{d,i}} \left[\frac{1}{|c_q|}\int_{c_q} \boldsymbol{P}_i(x)\,\mathrm{d}x - P_q\right]^2$$

☞ $\widehat{\mathcal{R}}_i$ minimizes $E_i(\mathcal{R}_i)$ in the least squares sense

- $\widehat{\boldsymbol{P}}_i(x)$ has the coefficients $\widehat{\mathcal{R}}_i$

Polynomial Reconstructions for Diamond Cells

- Conservative recontruction for diamond cell c_k, $k \in \mathcal{C}_\mathcal{D}$

$$\frac{1}{|c_k|}\int_{c_k} \boldsymbol{U}_{\beta,k}(x)\,\mathrm{d}x = U_{\beta,k}, \qquad M_k^\alpha = \frac{1}{|c_k|}\int_{c_k} (x - m_k)^\alpha\,\mathrm{d}x$$

$$\boldsymbol{U}_{\beta,k}(x) = U_{\beta,k} + \sum_{1\leq|\alpha|\leq d} \mathcal{R}_{\beta,k}^\alpha \left[(x - m_k)^\alpha - M_k^\alpha\right]$$

- Coefficients $\mathcal{R}_{\beta,k} = (\mathcal{R}_{\beta,k}^\alpha)_{1\leq|\alpha|\leq d}$
- Minimization functional

$$E_{\beta,k}(\mathcal{R}_{\beta,k}) = \sum_{q\in S_{d,k}} \left[\frac{1}{|c_q|}\int_{c_q} \boldsymbol{U}_{\beta,k}(x)\,\mathrm{d}x - U_{\beta,q}\right]^2$$

☞ $\widehat{\mathcal{R}}_{\beta,k}$ minimizes $E_{\beta,k}(\mathcal{R}_{\beta,k})$ in the least squares sense

- $\widehat{\boldsymbol{U}}_{\beta,k}(x)$ has the coefficients $\widehat{\mathcal{R}}_{\beta,k}$

Polynomial Reconstructions for Diamond Edges

- Non-conservative recontruction for diamond edge $e_{k\ell}$, $k \in \mathcal{C}_{\mathcal{D}}$, $\ell \in \nu(k)$

$$\boldsymbol{U}_{\beta,k\ell}(x) = \sum_{0\leq|\alpha|\leq d} \mathcal{R}^{\alpha}_{\beta,k\ell}(x - m_{k\ell})^{\alpha}$$

- Coefficients $\mathcal{R}_{\beta,k\ell} = (\mathcal{R}^{\alpha}_{\beta,k\ell})_{0\leq|\alpha|\leq d}$
- Minimization functional

$$E_{\beta,k\ell}(\mathcal{R}_{\beta,k\ell}) = \sum_{q\in S_{d,k\ell}} \omega_{\beta,k\ell,q} \left[\frac{1}{|c_q|}\int_{c_q} \boldsymbol{U}_{\beta,k\ell}(x)\,\mathrm{d}x - U_{\beta,q}\right]^2$$

☞ $\widetilde{\mathcal{R}}_{\beta,k\ell}$ minimizes $E_{\beta,k\ell}(\mathcal{R}_{\beta,k\ell})$ in the least squares sense

- $\widetilde{\boldsymbol{U}}_{\beta,k\ell}(x)$ has the coefficients $\widetilde{\mathcal{R}}_{\beta,k\ell}$

Polynomial Reconstructions for Diamond Edges

- Conservative recontruction for diamond edge $e_{k\mathrm{D}}$, $k \in \mathcal{C}_{\mathcal{D}}$

$$\frac{1}{|e_{k\mathrm{D}}|}\int_{e_{k\mathrm{D}}} \boldsymbol{U}_{\beta,k\mathrm{D}}(x)\,\mathrm{d}s = \overline{U}_{\beta,k\mathrm{D}}, \qquad M^{\alpha}_{k\mathrm{D}} = \frac{1}{|e_{k\mathrm{D}}|}\int_{e_{k\mathrm{D}}} (x - m_{k\mathrm{D}})^{\alpha}\,\mathrm{d}x$$

$$\boldsymbol{U}_{\beta,k\mathrm{D}}(x) = \overline{U}_{\beta,k\mathrm{D}} + \sum_{1\le|\alpha|\le d} \mathcal{R}^{\alpha}_{\beta,k\mathrm{D}}\left[(x - m_{k\mathrm{D}})^{\alpha} - M^{\alpha}_{k\mathrm{D}}\right]$$

- Coefficients $\mathcal{R}_{\beta,k} = (\mathcal{R}^{\alpha}_{\beta,k})_{1\le|\alpha|\le d}$
- Weighted minimization functional, $\omega_{\beta,k\mathrm{D}} = (\omega_{\beta,k\mathrm{D},q})_{q=1,\dots,\#S_{d,k\mathrm{D}}}$

$$E_{\beta,k\mathrm{D}}(\mathcal{R}_{\beta,k\mathrm{D}}) = \sum_{q\in S_{d,k\mathrm{D}}} \omega_{\beta,k\mathrm{D},q}\left[\frac{1}{|c_q|}\int_{c_q} \boldsymbol{U}_{\beta,k\mathrm{D}}(x)\,\mathrm{d}x - U_{\beta,q}\right]^2$$

☞ $\widehat{\mathcal{R}}_{\beta,k\ell}$ minimizes $E_{\beta,k\mathrm{D}}(\mathcal{R}_{\beta,k\mathrm{D}})$ in the least squares sense

- $\widehat{\boldsymbol{U}}_{\beta,k\ell}(x)$ has the coefficients $\widehat{\mathcal{R}}_{\beta,k\ell}$

Numerical Fluxes

	c_k	$e_{k\ell}$	$e_{k\mathrm{D}}$	c_i
U	$\widehat{\boldsymbol{U}}_{1,k}$, $\widehat{\boldsymbol{U}}_{2,k}$	$\widetilde{\boldsymbol{U}}_{2,k\ell}$, $\widetilde{\boldsymbol{U}}_{2,k\ell}$	$\widehat{\boldsymbol{U}}_{k\mathrm{D}}$, $\widehat{\boldsymbol{U}}_{2,k\mathrm{D}}$	—
P	—	—	—	$\widehat{\boldsymbol{P}}_i$

- Inner diamond edge $e_{k\ell}$, with $i = \Pi_{\mathcal{M}}(k,\ell)$

$$\mathcal{F}^{U}_{\beta,k\ell,r} = -\nu\nabla\widetilde{\boldsymbol{U}}_{\beta,k\ell}(q_{k\ell,r}) \cdot n_{k\ell}, \quad \mathcal{F}^{P}_{\beta,k\ell,r} = \frac{1}{\rho}\widehat{\boldsymbol{P}}_i(q_{k\ell,r})n_{\beta,k\ell}$$

- Boundary diamond edge $e_{k\mathrm{D}}$ with $i = \Pi_{\mathcal{M}}(k,\mathrm{D})$

$$\mathcal{F}^{U}_{\beta,k\mathrm{D},r} = -\nu\nabla\widehat{\boldsymbol{U}}_{\beta,k\mathrm{D}}(q_{k\mathrm{D},r}) \cdot n_{k\mathrm{D}}, \quad \mathcal{F}^{P}_{\beta,k\mathrm{D},r} = \frac{1}{\rho}\widehat{\boldsymbol{P}}_i(q_{k\mathrm{D},r})n_{\beta,k\mathrm{D}}$$

- Inner/boundary primal edge e_{ij} with $k = \Pi_{\mathcal{D}}(i,j)$

$$\mathcal{F}^{\nabla}_{ij,r} = \widehat{\boldsymbol{U}}_{1,k}(q_{ij,r})n_{1,ij} + \widehat{\boldsymbol{U}}_{2,k}(q_{ij,r})n_{2,ij}$$

$$\mathcal{F}^{\nabla}_{i\mathrm{D},r} = \widehat{\boldsymbol{U}}_{1,k\mathrm{D}}(q_{i\mathrm{D},r})n_{1,i\mathrm{D}} + \widehat{\boldsymbol{U}}_{2,k\mathrm{D}}(q_{i\mathrm{D},r})n_{2,i\mathrm{D}}$$

Residual Operators

- Residual for each primal/diamond cell, $\Phi = (\mathbb{U}_1, \mathbb{U}_2, \mathbb{P}) \in \mathbb{R}^{2K+I}$

$$\mathcal{G}_k^{\beta}(\Phi) = \sum_{\ell \in \nu(k)} \frac{|e_{k\ell}|}{|c_k|} \left[\sum_{r=1}^{R} \zeta_r \left(\mathcal{F}^{U}_{\beta,k\ell,r} + \mathcal{F}^{P}_{\beta,k\ell,r} \right) \right] - f_{\beta,k}, \quad k \in \mathcal{C}_{\mathcal{D}}$$

$$\mathcal{G}_i^{\nabla}(\Phi) = \sum_{j \in \nu(i)} \frac{|e_{ij}|}{|c_i|} \sum_{r=1}^{R} \zeta_r \mathcal{F}^{\nabla}_{ij,r}, \quad i \in \mathcal{C}_{\mathcal{M}}$$

- Global residual operators

$$\mathcal{G}^{\beta}(\Phi) = \left(\mathcal{G}_k^{\beta}(\Phi) \right)_{k=I+1,\dots,I+K}, \qquad \mathcal{G}^{\nabla}(\Phi) = \left(\mathcal{G}_i^{\nabla}(\Phi) \right)_{i=1,\dots,I}$$

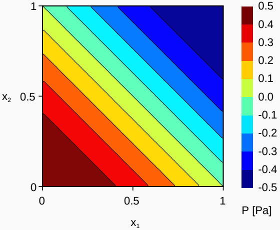
- Affine operator from $\mathbb{R}^{2K+I}$ into $\mathbb{R}^{2K+I}$

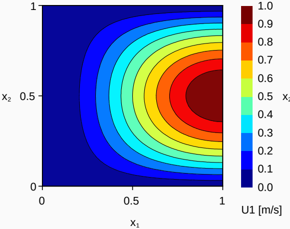
$$\mathcal{H}(\Phi) = \left(\mathcal{G}^{1}(\Phi), \mathcal{G}^{2}(\Phi), \mathcal{G}^{\nabla}(\Phi) \right)^{T},$$

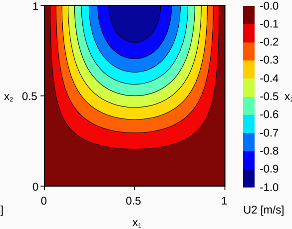
- Find the solution $\Phi^{\star} = (\mathbb{U}_1^{\star}, \mathbb{U}_2^{\star}, \mathbb{P}^{\star})^{T} \in \mathbb{R}^{2K+I}$ solving $\mathcal{H}(\Phi) = 0$

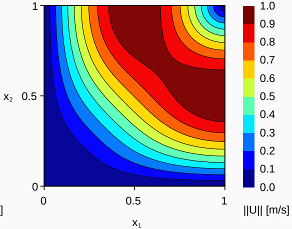
Numerical Results

- $\Omega = [0,1]^2,\ \rho = \nu = 1$
- $U_1(x) = \dfrac{1}{2}\,(1 - \cos(\pi x_1))\,\sin(\pi x_2)$
- $U_2(x) = \dfrac{1}{2}\,\sin(\pi x_1)\,(\cos(\pi x_2) - 1)$
- $P(x) = \dfrac{1}{2}\cos\left(\dfrac{\pi}{2}(x_1 + x_2)\right)$

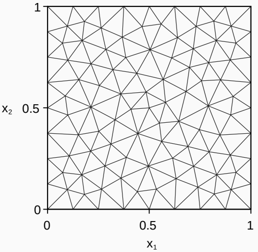


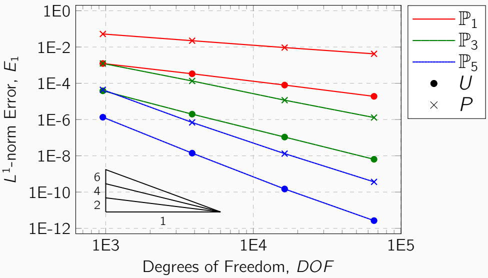


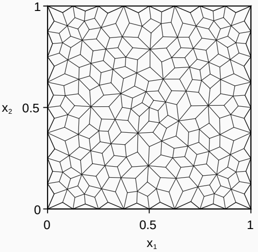


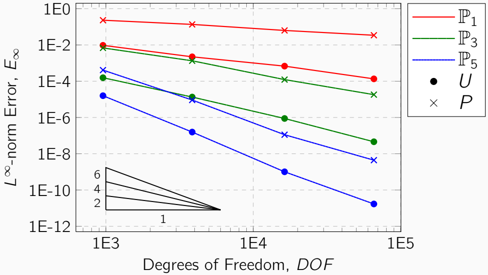


Error Analysis (Delaunay Mesh)

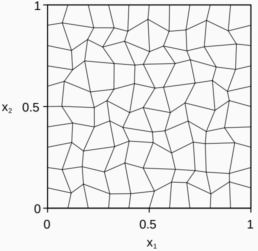


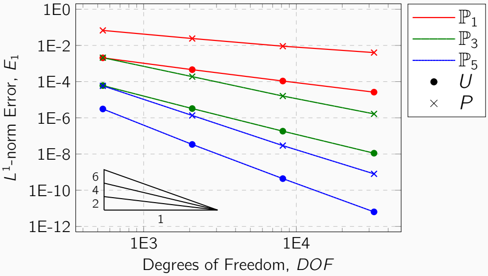


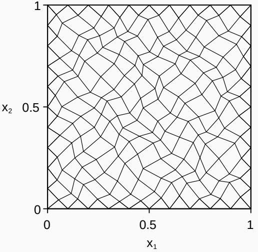


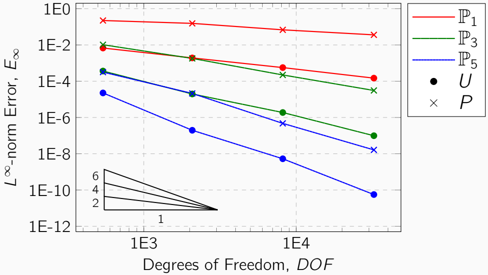


Error Analysis (Deformed Mesh)

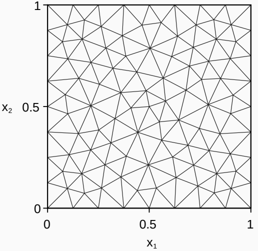


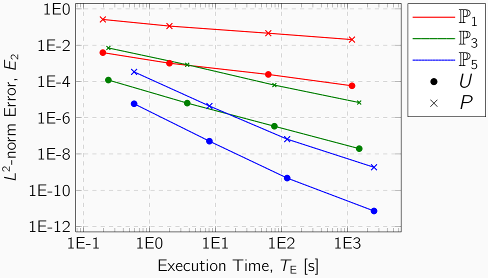


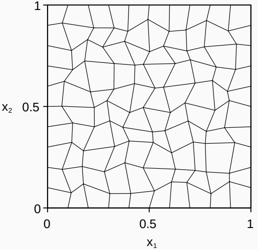


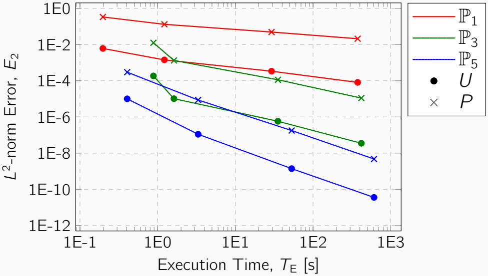


Time Consumption Analysis









Incompressible Navier-Stokes Problem

- Polygonal domain Ω with boundary $\partial\Omega$, steady-state and incompressible Navier-Stokes problem

$$\begin{aligned} &\nabla \cdot \left(U \otimes U - \nu \nabla U + \frac{1}{\rho} P I \right) = g, && \text{in } \Omega \\ &\nabla \cdot U = 0, && \text{in } \Omega \\ &U = U_{\mathrm{D}}, && \text{on } \partial\Omega \end{aligned}$$

- Linear formulation with $\Lambda = (\Lambda_1, \Lambda_2)$ equivalent when $\Lambda \to U$

$$\begin{aligned} &\nabla \cdot \left(U \otimes \Lambda - \nu \nabla U + \frac{1}{\rho} P I \right) = g, && \text{in } \Omega \\ &\nabla \cdot U = 0, && \text{in } \Omega \\ &U = U_{\mathrm{D}}, && \text{on } \partial\Omega \end{aligned}$$

Finite Volume Discretization

- Momentum equation in diamond cells

$$\frac{1}{|c_k|}\sum_{\ell\in\nu(k)}|e_{k\ell}|\left[\sum_{r=1}^{R}\zeta_r\left(\mathbb{F}^{\otimes}_{\beta,k\ell,r}+\mathbb{F}^{U}_{\beta,k\ell,r}+\mathbb{F}^{P}_{\beta,k\ell,r}\right)\right]-g_{\beta,k}=\mathcal{O}(h_k^{2R}),\ k\in\mathcal{C}_{\mathcal{D}}$$

- Divergence-free equation in primal cells

$$\frac{1}{|c_i|}\sum_{j\in\nu(i)}|e_{ij}|\left[\sum_{r=1}^{R}\zeta_r\mathbb{F}^{\nabla}_{ij,r}\right]=\mathcal{O}(h_i^{2R}),\ i\in\mathcal{C}_{\mathcal{M}}$$

- Physical fluxes

$$\mathbb{F}^{U}_{\beta,k\ell,r}=-\nu\nabla U_\beta(q_{k\ell,r})\cdot n_{k\ell},\qquad \mathbb{F}^{P}_{\beta,k\ell,r}=\frac{1}{\rho}P(q_{k\ell,r})n_{\beta,k\ell}$$

$$\mathbb{F}^{\otimes}_{\beta,k\ell,r}=U_\beta(q_{k\ell,r})\left(\Lambda(q_{k\ell,r})\cdot n_{\beta,k\ell}\right),\quad \mathbb{F}^{\nabla}_{ij,r}=U(q_{ij,r})\cdot n_{ij}$$

- Mean source term (Gaussian quadrature): $g_{\beta,k}\approx\int_{c_k}g_\beta\mathrm{d}X$

Polynomial Reconstructions

	c_k	$e_{k\ell}$	$e_{k\mathrm{D}}$	c_i
U	$\widehat{\boldsymbol{U}}_{1,k}$, $\widehat{\boldsymbol{U}}_{2,k}$	$\widetilde{\boldsymbol{U}}_{2,k\ell}$, $\widetilde{\boldsymbol{U}}_{2,k\ell}$	$\widehat{\boldsymbol{U}}_{k\mathrm{D}}$, $\widehat{\boldsymbol{U}}_{2,k\mathrm{D}}$	—
Λ	$\widehat{\boldsymbol{\Lambda}}_{1,k}$, $\widehat{\boldsymbol{\Lambda}}_{2,k}$	$\widetilde{\boldsymbol{\Lambda}}_{1,k\ell}$, $\widetilde{\boldsymbol{\Lambda}}_{2,k\ell}$	$\widetilde{\boldsymbol{\Lambda}}_{1,k\mathrm{D}}$, $\widetilde{\boldsymbol{\Lambda}}_{2,k\mathrm{D}}$	—
P	—	—	—	$\widehat{\boldsymbol{P}}_i$

- Centred/upwind approximation for inner diamond edge $e_{k\ell}$

$$\Lambda_{k\ell,r} = \widetilde{\boldsymbol{\Lambda}}_{k\ell}(q_{k\ell,r}),$$

$$\Lambda_{k\ell,r} = \Upsilon^+\left(\widetilde{\boldsymbol{\Lambda}}_{k\ell}(q_{k\ell,r}) \cdot n_{k\ell}\right)\widehat{\boldsymbol{\Lambda}}_k(q_{k\ell,r}) + \Upsilon^-\left(\widetilde{\boldsymbol{\Lambda}}_{k\ell}(q_{k\ell,r}) \cdot n_{k\ell}\right)\widehat{\boldsymbol{\Lambda}}_\ell(q_{k\ell,r})$$

- Centred/upwind approximation for boundary diamond edge $e_{k\mathrm{D}}$

$$\Lambda_{k\mathrm{D},r} = \widetilde{\boldsymbol{\Lambda}}_{k\mathrm{D}}(q_{k\mathrm{D},r}),$$

$$\Lambda_{k\mathrm{D},r} = \Upsilon^+\left(\widetilde{\boldsymbol{\Lambda}}_{k\mathrm{D}}(q_{k\mathrm{D},r}) \cdot n_{k\mathrm{D}}\right)\widehat{\boldsymbol{\Lambda}}_k(q_{k\mathrm{D},r}) + \Upsilon^-\left(\widetilde{\boldsymbol{\Lambda}}_{k\mathrm{D}}(q_{k\mathrm{D},r}) \cdot n_{k\mathrm{D}}\right)\widetilde{\boldsymbol{\Lambda}}_{k\mathrm{D}}(q_{k\mathrm{D},r})$$

Numerical Fluxes

- Inner diamond edge $e_{k\ell}$, with $i = \Pi_{\mathcal{M}}(k, \ell)$

$$\mathcal{F}^{\otimes}_{\beta,k\ell,r} = [\Lambda_{k\ell,r} \cdot n_{k\ell}]^{+}\, \widehat{\boldsymbol{U}}_{\beta,k}(q_{k\ell,r}) + [\Lambda_{k\ell,r} \cdot n_{k\ell}]^{-}\, \widehat{\boldsymbol{U}}_{\beta,\ell}(q_{k\ell,r})$$

$$\mathcal{F}^{U}_{\beta,k\ell,r} = -\nu \nabla \widetilde{\boldsymbol{U}}_{\beta,k\ell}(q_{k\ell,r}) \cdot n_{k\ell}, \quad \mathcal{F}^{P}_{\beta,k\ell,r} = \frac{1}{\rho} \widehat{\boldsymbol{P}}_{i}(q_{k\ell,r}) n_{\beta,k\ell}$$

- Boundary diamond edge $e_{k\mathrm{D}}$ with $i = \Pi_{\mathcal{M}}(k, \mathrm{D})$

$$\mathcal{F}^{\otimes}_{\beta,k\mathrm{D},r} = [\Lambda_{k\mathrm{D},r} \cdot n_{k\mathrm{D}}]^{-}\, \widehat{\boldsymbol{U}}_{\beta,k}(q_{k\mathrm{D},r}) + [\Lambda_{k\mathrm{D},r} \cdot n_{k\mathrm{D}}]^{-}\, U_{\beta,\mathrm{D}}(q_{k\mathrm{D},r})$$

$$\mathcal{F}^{U}_{\beta,k\mathrm{D},r} = -\nu \nabla \widehat{\boldsymbol{U}}_{\beta,k\mathrm{D}(q_{k\mathrm{D},r})} \cdot n_{k\mathrm{D}}, \quad \mathcal{F}^{P}_{\beta,k\mathrm{D},r} = \frac{1}{\rho} \widehat{\boldsymbol{P}}_{i}(q_{k\mathrm{D},r}) n_{\beta,k\mathrm{D}}$$

- Inner/boundary primal edge $e_{ij}/e_{i\mathrm{D}}$ with $k = \Pi_{\mathcal{D}}(i, j)$

$$\mathcal{F}^{\nabla}_{ij,r} = \widehat{\boldsymbol{U}}_{1,k}(q_{ij,r}) n_{1,ij} + \widehat{\boldsymbol{U}}_{2,k}(q_{ij,r}) n_{2,ij},$$

$$\mathcal{F}^{\nabla}_{i\mathrm{D},r} = \widehat{\boldsymbol{U}}_{1,k\mathrm{D}}(q_{i\mathrm{D},r}) n_{1,i\mathrm{D}} + \widehat{\boldsymbol{U}}_{2,k\mathrm{D}}(q_{i\mathrm{D},r}) n_{2,i\mathrm{D}}$$

Residual Operators and Fixed-point Algorithm

- Residual for each primal/diamond cell, $\Phi = \left(\Phi^U, \Phi^P\right) \in \mathbb{R}^{2(K+I)}$, $\Psi = \left(\Phi^\Lambda\right) \in \mathbb{R}^{2K}$

$$\mathcal{G}^{\mathrm{NS}}_{\beta,k}(\Phi;\Psi) = \sum_{\ell\in\nu(k)} \frac{|e_{k\ell}|}{|c_k|}\left[\sum_{r=1}^{R} \zeta_r \left(\mathcal{F}^{\otimes}_{\beta,k\ell,r} + \mathcal{F}^{U}_{\beta,k\ell,r} + \mathcal{F}^{P}_{\beta,k\ell,r}\right)\right] - f_{\beta,k}$$

$$\mathcal{G}^{\nabla}_{i}(\Phi) = \sum_{j\in\nu(i)} \frac{|e_{ij}|}{|c_i|}\left[\sum_{r=1}^{R} \zeta_r \mathcal{F}^{\nabla}_{ij,r}\right],$$

- Global residual operators

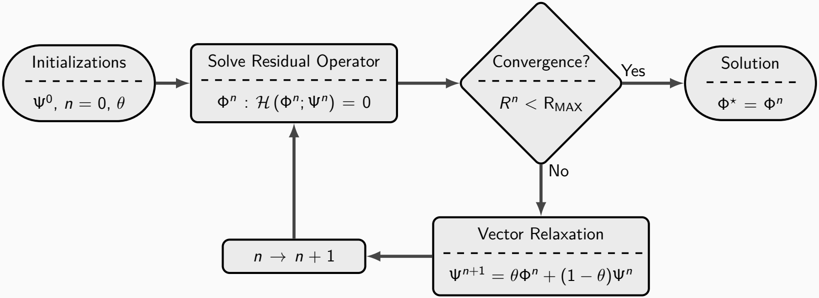
$$\mathcal{G}^{\mathrm{NS}}(\Phi;\Psi) = \left(\mathcal{G}^{\mathrm{NS}}_{k}(\Phi;\Psi)\right)_{k=I+1,\dots,I+K}, \qquad \mathcal{G}^{\nabla}(\Phi) = \left(\mathcal{G}^{\nabla}_{i}(\Phi)\right)_{i=1,\dots,I}$$

- Affine operator from $\mathbb{R}^{2(K+I)}$ into $\mathbb{R}^{2(K+I)}$

$$\mathcal{H}(\Phi;\Psi) = \left(\mathcal{G}^{\mathrm{NS}}(\Phi;\Psi), \mathcal{G}^{\nabla}(\Phi)\right)^{\boldsymbol{T}}.$$

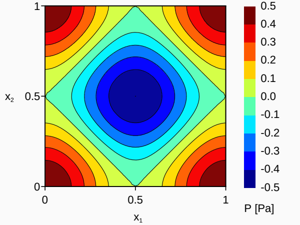
Residual Operators and Fixed-point Algorithm

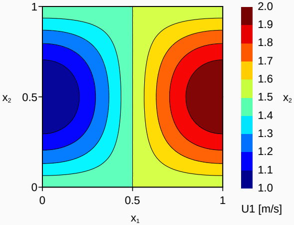
- Fixed-point algorithm

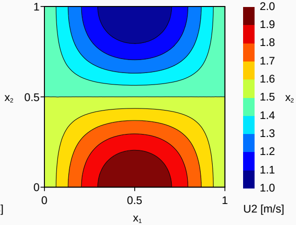


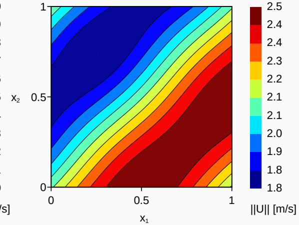
- R^n – residual
- R_{MAX} – tolerance
- θ – relaxation coefficient

- $\Omega = [0,1]^2,\ \rho = 1,\ \nu = 10^{-5}$
- $U_1(x) = \frac{1}{2}\left(3 - \cos(\pi x_1)\sin(\pi x_2)\right)$
- $U_2(x) = \frac{1}{2}\left(3 - \sin(\pi x_1)\cos(\pi x_2)\right)$
- $P(x) = \frac{1}{4}\left(\cos(2\pi x_1) + \cos(2\pi x_2)\right)$









Error Analysis

Centred

		$\mathbb{P}_1$		$\mathbb{P}_3$		$\mathbb{P}_5$	
	DOF	E_1	O_1	E_1	O_1	E_1	O_1
U_1	168	1.06E−02	—	2.22E−03	—	4.61E−04	—
	585	3.03E−03	2.01	1.40E−04	4.44	1.28E−05	5.75
	2373	7.27E−04	2.04	8.81E−06	3.95	1.89E−07	6.02
	9339	1.90E−04	1.96	5.58E−07	4.03	4.04E−09	5.61
U_2	168	9.86E−03	—	2.29E−03	—	4.74E−04	—
	585	2.85E−03	1.99	1.45E−04	4.42	1.31E−05	5.76
	2373	7.25E−04	1.95	9.02E−06	3.97	1.93E−07	6.02
	9339	1.88E−04	1.97	5.57E−07	4.07	4.05E−09	5.64
P	104	2.03E−02	—	4.24E−03	—	7.30E−04	—
	374	4.97E−03	2.20	2.63E−04	4.35	1.68E−05	5.89
	1550	1.09E−03	2.13	1.49E−05	4.04	1.71E−07	6.45
	6162	2.59E−04	2.08	9.11E−07	4.05	3.22E−09	5.76

Error Analysis

Centred

	DOF	$\mathbb{P}_1$ E_1	$\mathbb{P}_1$ O_1	$\mathbb{P}_3$ E_1	$\mathbb{P}_3$ O_1	$\mathbb{P}_5$ E_1	$\mathbb{P}_5$ O_1
U_1	168	1.06E−02	—	2.22E−03	—	4.61E−04	—
	585	3.03E−03	2.01	1.40E−04	4.44	1.28E−05	5.75
	2373	7.27E−04	2.04	8.81E−06	3.95	1.89E−07	6.02
	9339	1.90E−04	1.96	5.58E−07	4.03	4.04E−09	5.61
U_2	168	9.86E−03	—	2.29E−03	—	4.74E−04	—
	585	2.85E−03	1.99	1.45E−04	4.42	1.31E−05	5.76
	2373	7.25E−04	1.95	9.02E−06	3.97	1.93E−07	6.02
	9339	1.88E−04	1.97	5.57E−07	4.07	4.05E−09	5.64
P	104	2.03E−02	—	4.24E−03	—	7.30E−04	—
	374	4.97E−03	2.20	2.63E−04	4.35	1.68E−05	5.89
	1550	1.09E−03	2.13	1.49E−05	4.04	1.71E−07	6.45
	6162	2.59E−04	2.08	9.11E−07	4.05	3.22E−09	5.76

Upwind

	DOF	$\mathbb{P}_1$ E_1	$\mathbb{P}_1$ O_1	$\mathbb{P}_3$ E_1	$\mathbb{P}_3$ O_1	$\mathbb{P}_5$ E_1	$\mathbb{P}_5$ O_1
U_1	168	1.63E−02	—	2.70E−03	—	5.15E−04	—
	585	3.87E−03	2.31	1.16E−04	5.05	1.33E−05	5.86
	2373	9.71E−04	1.98	8.74E−06	3.69	2.09E−07	5.93
	9339	2.85E−04	1.79	5.30E−07	4.09	3.43E−09	6.00
U_2	168	1.55E−02	—	2.70E−03	—	5.15E−04	—
	585	3.85E−03	2.23	1.20E−04	5.00	1.35E−05	5.84
	2373	9.62E−04	1.98	8.87E−06	3.72	2.12E−07	5.93
	9339	2.85E−04	1.78	5.29E−07	4.12	3.45E−09	6.01
P	104	3.33E−02	—	4.59E−03	—	8.76E−04	—
	374	8.13E−03	2.20	2.20E−04	4.75	1.75E−05	6.12
	1550	1.96E−03	2.00	1.45E−05	3.83	1.61E−07	6.59
	6162	5.03E−04	1.97	8.79E−07	4.06	2.36E−09	6.12

Error Analysis

Centred

	DOF	$\mathbb{P}_1$ E_∞	$\mathbb{P}_1$ O_∞	$\mathbb{P}_3$ E_∞	$\mathbb{P}_3$ O_∞	$\mathbb{P}_5$ E_∞	$\mathbb{P}_5$ O_∞
U_1	168	5.07E−02	—	6.98E−03	—	1.79E−03	—
	585	2.43E−02	1.18	5.97E−04	3.94	5.82E−05	5.49
	2373	5.12E−03	2.22	5.01E−05	3.54	1.28E−06	5.46
	9339	2.15E−03	1.27	4.53E−06	3.51	1.86E−08	6.17
U_2	168	6.92E−02	—	8.76E−03	—	1.74E−03	—
	585	1.20E−02	2.81	7.75E−04	3.89	5.03E−05	5.68
	2373	4.31E−03	1.46	5.31E−05	3.83	1.05E−06	5.52
	9339	2.21E−03	0.98	4.69E−06	3.54	1.63E−08	6.08
P	104	8.86E−02	—	1.33E−02	—	2.37E−03	—
	374	2.37E−02	2.06	1.01E−03	4.03	1.34E−04	4.49
	1550	7.71E−03	1.58	6.65E−05	3.83	2.12E−06	5.83
	6162	2.14E−03	1.85	5.10E−06	3.72	3.20E−08	6.08

Error Analysis

Centred

	DOF	$\mathbb{P}_1$ E_∞	$\mathbb{P}_1$ O_∞	$\mathbb{P}_3$ E_∞	$\mathbb{P}_3$ O_∞	$\mathbb{P}_5$ E_∞	$\mathbb{P}_5$ O_∞
U_1	168	5.07E−02	—	6.98E−03	—	1.79E−03	—
	585	2.43E−02	1.18	5.97E−04	3.94	5.82E−05	5.49
	2373	5.12E−03	2.22	5.01E−05	3.54	1.28E−06	5.46
	9339	2.15E−03	1.27	4.53E−06	3.51	1.86E−08	6.17
U_2	168	6.92E−02	—	8.76E−03	—	1.74E−03	—
	585	1.20E−02	2.81	7.75E−04	3.89	5.03E−05	5.68
	2373	4.31E−03	1.46	5.31E−05	3.83	1.05E−06	5.52
	9339	2.21E−03	0.98	4.69E−06	3.54	1.63E−08	6.08
P	104	8.86E−02	—	1.33E−02	—	2.37E−03	—
	374	2.37E−02	2.06	1.01E−03	4.03	1.34E−04	4.49
	1550	7.71E−03	1.58	6.65E−05	3.83	2.12E−06	5.83
	6162	2.14E−03	1.85	5.10E−06	3.72	3.20E−08	6.08

Upwind

	DOF	$\mathbb{P}_1$ E_∞	$\mathbb{P}_1$ O_∞	$\mathbb{P}_3$ E_∞	$\mathbb{P}_3$ O_∞	$\mathbb{P}_5$ E_∞	$\mathbb{P}_5$ O_∞
U_1	168	9.21E−02	—	8.55E−03	—	1.96E−03	—
	585	2.59E−02	2.04	4.97E−04	4.56	5.44E−05	5.74
	2373	5.71E−03	2.16	4.59E−05	3.40	1.18E−06	5.47
	9339	3.14E−03	0.82	4.17E−06	3.50	1.90E−08	6.03
U_2	168	4.76E−02	—	7.57E−03	—	2.08E−03	—
	585	3.35E−02	0.57	5.85E−04	4.11	4.10E−05	6.29
	2373	6.17E−03	2.42	4.23E−05	3.75	1.06E−06	5.22
	9339	3.35E−03	0.89	4.25E−06	3.35	1.85E−08	5.91
P	104	1.63E−01	—	1.55E−02	—	2.89E−03	—
	374	3.80E−02	2.28	9.02E−04	4.44	1.24E−04	4.93
	1550	1.26E−02	1.56	7.41E−05	3.52	2.02E−06	5.79
	6162	3.73E−03	1.77	5.24E−06	3.84	3.94E−08	5.70

Convergence Analysis

DOF			440		1544		6296		24840	
		θ	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}
$\mathbb{P}_1$	Centred	1	6776	31	15334	33	42236	35	94229	39
		0.8	7378	32	19004	42	25668	23	48726	22
		0.5	13243	57	31756	69	47933	40	90657	38

Convergence Analysis

DOF			440		1544		6296		24840	
		θ	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}
$\mathbb{P}_1$	Centred	1	6776	31	15334	33	42236	35	94229	39
		0.8	7378	32	19004	42	25668	23	48726	22
		0.5	13243	57	31756	69	47933	40	90657	38
	Upwind	1	—	nc	—	nc	—	nc	—	nc
		0.8	18609	84	52846	113	144663	123	182490	81
		0.5	30808	138	25635	55	99902	91	301373	130

Convergence Analysis

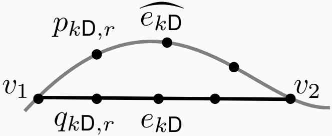
DOF			440		1544		6296		24840	
		θ	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}
$\mathbb{P}_1$	Centred	1	6776	31	15334	33	42236	35	94229	39
		0.8	7378	32	19004	42	25668	23	48726	22
		0.5	13243	57	31756	69	47933	40	90657	38
	Upwind	1	—	nc	—	nc	—	nc	—	nc
		0.8	18609	84	52846	113	144663	123	182490	81
		0.5	30808	138	25635	55	99902	91	301373	130
$\mathbb{P}_3$	Centred	1	24341	102	55196	109	74941	66	81781	38
		0.8	7533	31	14300	28	24621	21	48927	21
		0.5	12964	53	25855	49	48759	38	95547	38
	Upwind	1	—	nc	—	nc	—	nc	—	nc
		0.8	10548	45	26565	56	99173	93	142113	72
		0.5	11729	48	25151	47	65232	52	97303	39

Convergence Analysis

DOF			440		1544		6296		24840	
		θ	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}	N_{GMRES}	N_{FP}
$\mathbb{P}_1$	Centred	1	6776	31	15334	33	42236	35	94229	39
		0.8	7378	32	19004	42	25668	23	48726	22
		0.5	13243	57	31756	69	47933	40	90657	38
	Upwind	1	—	nc	—	nc	—	nc	—	nc
		0.8	18609	84	52846	113	144663	123	182490	81
		0.5	30808	138	25635	55	99902	91	301373	130
$\mathbb{P}_3$	Centred	1	24341	102	55196	109	74941	66	81781	38
		0.8	7533	31	14300	28	24621	21	48927	21
		0.5	12964	53	25855	49	48759	38	95547	38
	Upwind	1	—	nc	—	nc	—	nc	—	nc
		0.8	10548	45	26565	56	99173	93	142113	72
		0.5	11729	48	25151	47	65232	52	97303	39
$\mathbb{P}_5$	Centred	1	—	nc	—	nc	—	nc	—	nc
		0.8	11184	44	18741	32	39670	31	71900	30
		0.5	21337	84	33272	56	68526	52	112809	45
	Upwind	1	—	nc	—	nc	—	nc	—	nc
		0.8	9330	37	18449	32	29441	25	62532	25
		0.5	14825	57	31561	52	52433	39	109796	42

Dirichlet Condition on Curved Boundary

- Find $U^{\star}_{\beta,kD} \neq \overline{U}_{\beta,kD}$ wich best approximates $\widehat{U}_{\beta,kD}(x)$ at $q_{kD,r}$
- $\widehat{e_{k\mathrm{D}}}$ – curved edge on boundary
- $p_{k\mathrm{D},r}$ – Gauss point on $\widehat{e_{k\mathrm{D}}}$



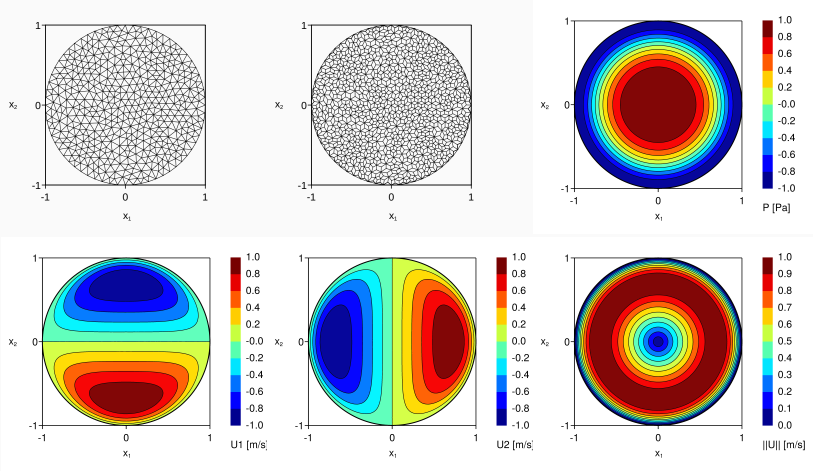
$$H(U^{\star}_{\beta,kD}) = \min H(U_{\beta,kD})$$

$$H(U_{\beta,kD}) = \sum_{r=1}^{R} \left(\widehat{U}_{\beta,kD}(p_{kD,r}) - U_{\beta,D}(p_{kD,r}) \right)^2$$

1. With $U^n_{\beta,kB}$ compute $\widehat{U}_{\beta,kD}(x)$, if $n = 0$ set $U^0_{\beta,kD} = \overline{U}_{\beta,kD}$
2. Evaluate the errors $\delta^n_r = U_{\beta,D}(p_{kD,r}) - \widehat{U}_{\beta,kD}(p_{kD,r})$,
3. Update the mean value with $U^{n+1}_{\beta,kD} = U^n_{\beta,kD} + \sum_{r=1}^{R} \xi_r \delta^n_r$
4. Stop if $|U^{n+1}_{\beta,kD} - U^n_{\beta,kD}| < \epsilon_B \overline{U}_{\beta,nD}$ and set $U^{\star}_{\beta,kD} = U^{n+1}_{\beta,kD}$
5. Else go to step (1)

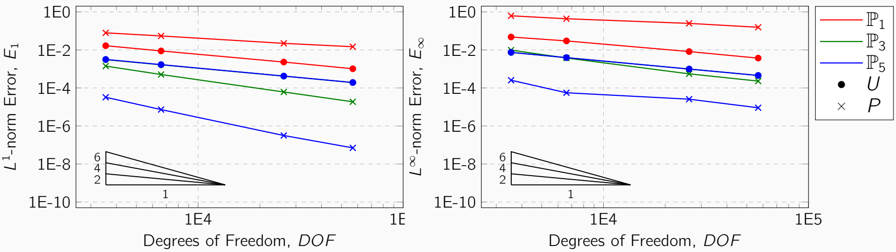
Numerical Results

- $\Omega = \{x : \tau^2 < 1\},\ \tau^2 = x_1^2 + x_2^2,\ a = 1/\left((1/(2\sqrt{2}))\exp(1/2)\right)$
- $\rho = \nu = 1,\ U(x) = a\exp(\tau^2)(1-\tau^2)\,[-y\ \ x]^T,\ P(x) = \cos(\pi\tau^2)$

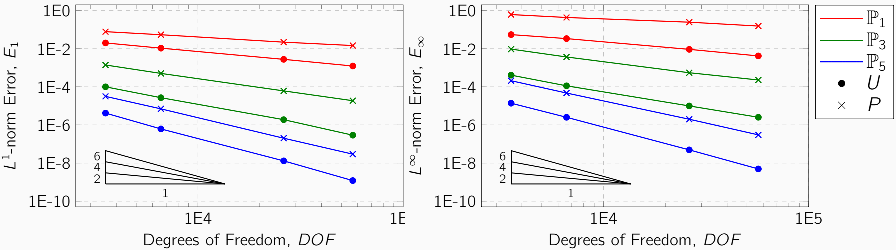


Error Analysis

No boundary correction



Boundary correction



Conclusions and Current Studies

- **Conclusions:**

☑ High-order scheme for the (Navier-)Stokes problem

☑ Robust and better accuracy *vs* time consumption performance

☑ Simple algorithm to preserve the accuracy on smooth curved boundaries

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- HO-FV scheme for the NS problem coupled with the energy equation
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