All-regime Lagrangian-Remap numerical schemes for the gas dynamics equations. Applications to the large friction and low Mach regimes

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Joint works with M. Girardin and S. Kokh

Outline

Outline

Introduction

Motivation : numerical study of two-phase flows in nuclear reactors

We consider the following model

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ &\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0 \\ &\partial_t (\rho E) + \nabla \cdot [(\rho E + p)\mathbf{u}] = 0 \end{aligned}$$

where ρ, $\mathbf{u} = (u, v)^t$, E denote respectively the density, the velocity vector and the total energy of the fluid.

Let $e = E - \frac{|\mathbf{u}|^2}{2}$ be the specific and $\tau = 1/\rho$ the covolume

Introduction

We are especially interested in the design of numerical schemes when the model depends on a parameter $\epsilon > 0$ in the following three flow regimes

Classical regime : $\epsilon = O(1)$
Low ϵ regime : $\epsilon << 1$
Limit regime : $\epsilon \to 0$

Our objective is to propose a numerical scheme that is

- all-regime : uniform stability and uniform consistency w.r.t. ϵ
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

These requirements will be specified later on...

Outline

Large friction regime

We consider the following model with friction and gravity

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \rho(\mathbf{g} - \alpha \mathbf{u}) \\
&\partial_t (\rho E) + \nabla \cdot [(\rho E + p)\mathbf{u}] = \rho \mathbf{u}.(\mathbf{g} - \alpha \mathbf{u})
\end{aligned}
$$

where $\mathbf{g}$, α denote the gravity field and the friction coefficient.

The large friction regime is obtained by replacing α with $\frac{\alpha}{\epsilon}$

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \rho(\mathbf{g} - \frac{\alpha}{\epsilon} \mathbf{u}) \\
&\partial_t (\rho E) + \nabla \cdot [(\rho E + p)\mathbf{u}] = \rho \mathbf{u}.(\mathbf{g} - \frac{\alpha}{\epsilon} \mathbf{u})
\end{aligned}
$$

with $\epsilon << 1$

Large friction regime

Setting

$$\mathbf{u} = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + O(\epsilon^2)$$

in

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \rho (\mathbf{g} - \frac{\alpha}{\epsilon} \mathbf{u}) \\
&\partial_t (\rho E) + \nabla \cdot [(\rho E + p) \mathbf{u}] = \rho \mathbf{u} . (\mathbf{g} - \frac{\alpha}{\epsilon} \mathbf{u})
\end{aligned}$$

the behaviour of the solutions is given by

$$\begin{aligned}
&\mathbf{u}_0 = 0 \\
&\partial_t \rho + \epsilon \nabla \cdot (\rho \mathbf{u}_1) = O(\epsilon^2) \\
&\nabla p = \rho (\mathbf{g} - \alpha \mathbf{u}_1) \\
&\partial_t (\rho E) + \epsilon \nabla \cdot [(\rho E + p) \mathbf{u}_1] = \epsilon \rho \mathbf{u}_1 . (\mathbf{g} - \alpha \mathbf{u}_1) + O(\epsilon^2)
\end{aligned}$$

Large friction regime

Note that replacing t with $\frac{t}{\epsilon}$ in

$$\begin{aligned}
&\mathbf{u}_0 = 0 \\
&\partial_t \rho + \epsilon \nabla \cdot (\rho \mathbf{u}_1) = O(\epsilon^2) \\
&\nabla p = \rho(\mathbf{g} - \alpha \mathbf{u}_1) \\
&\partial_t (\rho E) + \epsilon \nabla \cdot [(\rho E + p)\mathbf{u}_1] = \epsilon \rho \mathbf{u}_1.(\mathbf{g} - \alpha \mathbf{u}_1) + O(\epsilon^2)
\end{aligned}$$

the long time behaviour is given by

$$\begin{aligned}
&\mathbf{u}_0 = 0 \\
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}_1) = O(\epsilon) \\
&\nabla p = \rho(\mathbf{g} - \alpha \mathbf{u}_1) \\
&\partial_t (\rho e) + \nabla \cdot [(\rho e + p)\mathbf{u}_1] = \rho \mathbf{u}_1.(\mathbf{g} - \alpha \mathbf{u}_1) + O(\epsilon)
\end{aligned}$$

see Hsiao-Liu, Nishihara, Junca-Rascle, Lin-Coulombel, Coulombel-Goudon, Marcati-Milani... for rigorous proofs

Low Mach regime

Introducing the characteristic and non-dimensional quantities :

$$x = \frac{x}{L}, \quad t = \frac{t}{T}, \quad \rho = \frac{\rho}{\rho_0}, \quad u = \frac{u}{u_0},$$

$$v = \frac{v}{v_0}, \quad e = \frac{e}{e_0}, \quad p = \frac{p}{p_0}, \quad c = \frac{c}{c_0}$$

with $u_0 = v_0 = \frac{L}{T}$, $e_0 = p_0\rho_0$ and $p_0 = \rho_0 c_0^2$, the non-dimensional system is

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M^2} \nabla p = 0 \\
&\partial_t (\rho e) + \nabla \cdot [(\rho e + p)\mathbf{u}] + \frac{M^2}{2} \big(\partial_t (\rho \mathbf{u.u}) + \nabla \cdot (\rho \mathbf{u.uu})\big) = 0
\end{aligned}$$

where $M = \frac{u_0}{c_0}$ denotes the Mach number and plays the role of ϵ

Low Mach regime

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$
$$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M^2} \nabla p = 0$$
$$\partial_t (\rho e) + \nabla \cdot [(\rho e + p)\mathbf{u}] + \frac{M^2}{2} \big(\partial_t (\rho \mathbf{u}.\mathbf{u}) + \nabla \cdot (\rho \mathbf{u}.\mathbf{u}\mathbf{u}) \big) = 0$$

Remark 1. The flow is said to be in the low Mach regime if $M << 1$ and $\nabla p = O(M^2)$

Remark 2. Using asymptotic expansions of $\rho, \mathbf{u}, p, c$ in powers of M in the governing equations of $\rho, \mathbf{u}, p$, together with boundary conditions on a given domain $\mathcal{D}$ (global argument), we get

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \mathbf{u}_0) = 0$$
$$\partial_t \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + \frac{1}{\rho_0} \nabla p_2 = 0$$
$$\nabla \cdot \mathbf{u}_0 = 0$$

Numerical issue in the Low Mach regime

Accurate time-explicit computations of solutions generally require

- a mesh size $h = o(M)$
- a time step $\Delta t = O(hM)$

which is out of reach in practice

More details can be found in the large body of literature on this subject : A. Majda, E. Turkel, H. Guillard, C. Viozat, B. Thornber, S. Dellacherie, P. Omnes, P-A. Raviart, F. Rieper, Y. Penel, P. Degond, S. Jin, J.-G. Liu, P. Colella, K. Pao, E. Turkel, R. Klein, J-P Vila, M.G., B. Després, M. Ndjinga, J. Jung, M. Sun, ...

General cure : change the treatment of acoustic waves in the low Mach regime by centering the pressure gradient

Numerical issue in the large friction regime

Accurate time-explicit computations of solutions generally require

- a mesh size $h = o(\epsilon)$
- a time step $\Delta t = O(\epsilon)$

which is out of reach in practice

More details can be found in the large body of literature on this subject : L. Hsiao, T.-P. Liu, S. Jin, L. Pareschi, L. Gosse, G. Toscani, F. Bouchut, H. Ounaissa, B. Perthame, C. C., F. Coquel, E. Godlewski, P.-A. Raviart, N. Seguin, C. Berthon, P.-G. LeFloch, R. Turpault, F. Filbet, A. Rambaud, M. Girardin, S. Kokh, C. Cancès, H. Mathis, N. Seguin, S. Cordier, B. Després, E. Franck, C. Buet, ...

General cure : upwinding of the source terms at interfaces (USI)

Numerical strategies

Several approaches can be envisaged to compute accurate solutions when $\epsilon << 1$

- Use and discretize the limit model (the nature of which changes)
- Couple the original and limit models at moving interfaces
- Design Asymptotic-Preserving schemes (consistency with the limit model when $\epsilon \to 0$ and with the original model when $\epsilon \to 0$, no coupling)
- Consider all-regime stability and consistency properties (ϵ is kept constant in order to compute accurate solutions also in intermediate regimes)

A couple of definitions

Uniform stability

A scheme is said to be stable in the uniform sense if the CFL condition is uniform with respect to ϵ

This avoids stringent CFL restrictions $\Delta t = O(hM)$ or $\Delta t = O(\epsilon)$

Uniform consistency

A scheme is said to be consistent in the uniform sense if the truncation error is uniform with respect to ϵ

This avoids large numerical diffusion and mesh size restrictions $h = o(M)$ or $h = O(\epsilon)$

All-regime scheme

A scheme is said to be all-regime if it is able to compute accurate solutions with a mesh size h and a time step Δt independent of ϵ

Objectives

Our objective is to propose a numerical scheme that is

- all-regime : uniform stability and uniform consistency w.r.t. ϵ
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

How to do that...

Outline

How to reach these objectives

How to get the uniform stability ?

- implicit treatment of the fast phenomenon
- explicit treatment of the slow phenomenon (sake of accuracy)

→ Lagrange-Projection strategy Coquel-Nguyen-Postel-Tran

How to get the uniform consistency ?

- modify the numerical fluxes to reduce the numerical diffusion

→ Truncation errors in equivalent equations

How to deal with any (possibly strongly nonlinear) pressure law p ?

- overcome the non linearities, "linearization"

→ Relaxation strategy Suliciu, Jin-Xin, Bouchut, C.-Coquel, C.-Coulombel

How to deal with unstructured meshes in multi-D ?

- work on a fixed mesh (no need to deform unstructured meshes)

→ Operator splitting strategy and rotational invariance

Lagrange-Projection strategy

Let us first focus on the 1D system

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0 \\ \partial_t \varrho u + \partial_x (\varrho u^2 + p) = 0 \\ \partial_t (\varrho E) + \partial_x (\varrho E u + p u) = 0 \end{cases}$$

Using chain rule arguments, we also have

$$\begin{cases} \partial_t \varrho + u \partial_x \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + u \partial_x \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + u \partial_x \varrho E + \varrho E \partial_x u + \partial_x p u = 0 \end{cases}$$

so that splitting the transport part leads to

$$\begin{cases} \partial_t \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + \varrho E \partial_x u + \partial_x p u = 0 \end{cases} \qquad \begin{cases} \partial_t \varrho + u \partial_x \varrho = 0 \\ \partial_t \varrho u + u \partial_x \varrho u = 0 \\ \partial_t \varrho E + u \partial_x \varrho E = 0 \end{cases}$$

Lagrangian-step — Transport-step

Lagrange-Projection strategy

The Lagrangian-step

$$\begin{cases} \partial_t \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + \varrho E \partial_x u + \partial_x pu = 0 \end{cases} \quad \text{also writes} \quad \begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m pu = 0 \end{cases}$$

with $\tau = 1/\varrho$ and $\tau \partial_x = \partial_m$.

- Eigenvalues are given by $-\rho c, \ 0, \ \rho c$
- Usual CFL conditions for time-explicit schemes write

$$\frac{\Delta t}{h} \max(\rho c) \leq \frac{1}{2}$$

The idea is to propose a time-implicit scheme to avoid this time-step restriction $(\Delta t = O(hM)$ in the low Mach regime)

Lagrange-Projection strategy

The Transport-step is

$$\begin{cases} \partial_t \varrho + u \partial_x \varrho = 0 \\ \partial_t \varrho u + u \partial_x \varrho u = 0 \\ \partial_t \varrho E + u \partial_x \varrho E = 0 \end{cases} \quad \text{also writes} \quad \begin{cases} \partial_t \varrho + \partial_x \varrho u - \varrho \partial_x u = 0 \\ \partial_t \varrho u + \partial_x \varrho u^2 - \varrho u \partial_x u = 0 \\ \partial_t \varrho E + \partial_x \varrho E u - \varrho E \partial_x u = 0 \end{cases}$$

- Eigenvalues are given by u
- Usual CFL conditions for time-explicit schemes write

$$\frac{\Delta t}{h} \max(|u|) \leq \frac{1}{2}$$

The idea is then to propose a standard time-explicit scheme to keep accuracy on the slow phenomenon ($\Delta t = O(h)$ in all regime)

Operator splitting strategy

We will consider the following three-step numerical scheme :

First step $(t^n \to t^{Lag})$: solve implicitly the acoustic system with the solution at time t^n as initial solution

Second step $(t^{Lag} \to t^{n+1-})$ solve implicitly the source terms (when present) with the solution at time t^{Lag} as initial solution

Third step $(t^{n+1-} \to t^{n+1})$ solve explicitly the transport system with the solution at time t^{n+1-} as initial solution

Solving implicitly the source terms avoid the time-step restriction $\Delta t = O(\epsilon)$ when $\epsilon << 1$ $(\Delta t = O(h)$ in all regime)

A few words about the relaxation approach

The gas dynamics equations in Lagrangian coordinates :

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m pu = 0 \end{cases}$$

with $p = p(\tau, e)$ and

$$e = E - \frac{1}{2}u^2$$

Due to the nonlinearities of p, the Riemann problem is difficult to solve. The relaxation strategy :

- Idea : to deal with a larger but simpler system
- Design principle : to understand $p(\tau, e)$ as a new unknown that we denote Π

A few words about the relaxation approach

The gas dynamics in Lagrangian coordinates

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m pu = 0 \end{cases}$$

The relaxation system

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = 0 \\ \partial_t E + \partial_m \Pi u = 0 \\ \partial_t \Pi + a^2 \partial_m u = \lambda(p - \Pi) \end{cases}$$

At least formally, observe that

$$\lim_{\lambda \to +\infty} \Pi = p \quad (\text{if} \quad a > \rho c(\tau, e))$$

(see e.g. Chalons-Coulombel for a rigorous proof)

A few words about the relaxation approach

The time-explicit Godunov scheme applied to the relaxation system with initial data at equilibrium writes

$$\begin{cases} \tau_j^{Lag} = \tau_j^n + \dfrac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\ u_j^{Lag} = u_j^n - \dfrac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) \\ \Pi_j^{Lag} = \Pi_j^n - a^2\dfrac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\ E_j^{Lag} = E_j^n - \dfrac{\Delta t}{\Delta m}(p^*_{j+1/2}u^*_{j+1/2} - p^*_{j-1/2}u^*_{j-1/2}) \end{cases}$$

with $\Pi_j^n = p(\tau_j^n, e_j^n)$ and

$$u^*_{j+1/2} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{2a}(\Pi_{j+1}^n - \Pi_j^n)$$

$$p^*_{j+1/2} = \frac{1}{2}(\Pi_j^n + \Pi_{j+1}^n) - \frac{a}{2}(u_{j+1}^n - u_j^n)$$

A few words about the relaxation approach

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$$\begin{cases} \tau_j^{Lag} = \tau_j^n + \dfrac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\ u_j^{Lag} = u_j^n - \dfrac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) \\ \Pi_j^{Lag} = \Pi_j^n - a^2\dfrac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\ E_j^{Lag} = E_j^n - \dfrac{\Delta t}{\Delta m}(p^*_{j+1/2}u^*_{j+1/2} - p^*_{j-1/2}u^*_{j-1/2}) \end{cases}$$

with $\Pi_j^n = p(\tau_j^n, e_j^n)$ and

$$u^*_{j+1/2} = \frac{1}{2}(u_j^{Lag} + u_{j+1}^{Lag}) - \frac{1}{2a}(\Pi_{j+1}^{Lag} - \Pi_j^{Lag})$$

$$p^*_{j+1/2} = \frac{1}{2}(\Pi_j^{Lag} + \Pi_{j+1}^{Lag}) - \frac{a}{2}(u_{j+1}^{Lag} - u_j^{Lag})$$

A few words about the relaxation approach

The time-explicit scheme

- deals with (possibly strongly nonlinear) pressure laws
- is stable and satisfies a discrete entropy inequality provided that a is chosen sufficiently large and under a CFL restriction

$$\frac{\Delta t}{\Delta m} \max(\rho c) \leq \frac{1}{2}$$

In dimensionless form (low Mach regime), it writes

$$\frac{\Delta t}{\Delta m} \max(\rho \frac{c}{M}) \leq \frac{1}{2}$$

that is to say

$$\Delta t = O(hM)$$

A few words about the relaxation approach

The time-implicit scheme

- deals with (possibly strongly nonlinear) pressure laws
- is free of CFL restriction !
- is cheap in the sense that only a linear problem w.r.t. u and Π has to be solved

In 1D, the following two equations are decoupled

$$\begin{cases} \partial_t(\Pi + au) + a\partial_x(\Pi + au) = 0 \\ \partial_t(\Pi - au) - a\partial_x(\Pi - au) = 0 \end{cases}$$

Formulation on unstructured meshes

On unstructured meshes, the time-explicit ($\sharp = n$) and time-implicit ($\sharp = Lag$) schemes write

$$\begin{aligned}
\mathbf{u}_j^{Lag} &= \mathbf{u}_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} \Pi_{jk}^* \mathbf{n}_{jk} \\
\Pi_j^{Lag} &= \Pi_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} (a_{jk})^2 u_{jk}^* \\
\tau_j^{Lag} &= \tau_j^n + \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^* \\
E_j^{Lag} &= E_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} p_{jk}^* u_{jk}^*
\end{aligned}$$

$$u_{jk}^* = \frac{1}{2}\mathbf{n}_{jk}^T(\mathbf{u}_j^\sharp + \mathbf{u}_k^\sharp) - \frac{1}{2a_{jk}}(\Pi_k^\sharp - \Pi_j^\sharp), \quad p_{jk}^* = \frac{1}{2}(\Pi_j^\sharp + \Pi_k^\sharp) - \frac{a_{jk}}{2}\mathbf{n}_{jk}^T(\mathbf{u}_k^\sharp - \mathbf{u}_j^\sharp)$$

Source terms

The time-implicit point-wise scheme for the gravity terms and external forces writes

$$\begin{aligned} \tau_j^{n+1-} &= \tau_j^{Lag} \\ \mathbf{u}_j^{n+1-} &= \mathbf{u}_j^{Lag} + \Delta t(\mathbf{g} - \alpha \mathbf{u}_j^{n+1-}) \\ E_j^{n+1-} &= E_j^{Lag} + \Delta t\, \mathbf{u}_j^{n+1-}.(\mathbf{g} - \alpha \mathbf{u}_j^{n+1-}) \end{aligned}$$

It is free of CFL restriction

Transport step

In order to approximate the solutions of the transport step

$$\begin{aligned} \partial_t \rho + (\mathbf{u}\cdot\nabla)\rho &= 0 \\ \partial_t(\rho\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho\mathbf{u} &= 0 \\ \partial_t(\rho E) + (\mathbf{u}\cdot\nabla)\rho E &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \partial_t \rho + \nabla\cdot(\rho\mathbf{u}) - \rho\nabla\cdot\mathbf{u} &= 0 \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \rho\mathbf{u}\nabla\cdot\mathbf{u} &= 0 \\ \partial_t \rho E + \nabla\cdot(\rho E\mathbf{u}) - \rho E\nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

we simply use the time-explicit upwind finite-volume scheme

$$\varphi_j^{n+1} = \varphi_j^{n+1-} - \Delta t \sum_{k\in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^* \varphi_{jk}^{n+1-} + \Delta t \varphi_j^{n+1-} \sum_{k\in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^*$$

where $\varphi = \rho, \rho\mathbf{u}, \rho E$ and $\varphi_{jk}^{n+1-} = \begin{cases} \varphi_j^{n+1-} & \text{if } u_{jk}^* > 0 \\ \varphi_k^{n+1-} & \text{if } u_{jk}^* \leq 0 \end{cases}$

This scheme is stable under a <u>material</u> CFL condition ($\Delta t = O(h)$)

Objectives

Our objective is to propose a numerical scheme that is

- all-regime : uniform stability and uniform consistency w.r.t. ϵ
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

What about the first objective ?

Uniform consistency in the large friction regime

Let us first focus on the first two steps of the time-explicit scheme (the transport step is not a problem)

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2})$$
$$u_j^{n+1-} = u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) + \Delta t(g - \frac{\alpha}{\epsilon}u_j^{n+1-})$$
$$E_j^{n+1-} = E_j^n - \frac{\Delta t}{\Delta m}((pu)^*_{j+1/2} - (pu)^*_{j-1/2}) + \Delta t\, u_j^{n+1-}.(g - \frac{\alpha}{\epsilon}u_j^{n+1-})$$

with

$$u^*_{j+1/2} = \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2a}(p_{j+1} - p_j)$$
$$p^*_{j+1/2} = \frac{1}{2}(p_j + p_{j+1}) - \frac{a}{2}(u_{j+1} - u_j)$$

Uniform consistency in the large friction regime

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2})$$

$$u_j^{n+1-} = u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) + \Delta t(g - \frac{\alpha}{\epsilon} u_j^{n+1-})$$

$$u^*_{j+1/2} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{2a}(p_{j+1}^n - p_j^n)$$

$$p^*_{j+1/2} = \frac{1}{2}(p_j^n + p_{j+1}^n) - \frac{a}{2}(u_{j+1}^n - u_j^n)$$

Numerical asymptotic analysis. $u_j = u_j^{(0)} + \epsilon u_j^{(1)} + \mathcal{O}(\epsilon^2)$

- Multiply the second equation by ϵ and let $\epsilon \to 0$: $u_j^{(0)} = 0$
- Let $\epsilon \to 0$ in the second equation : $\dfrac{p_{j+1} - p_{j-1}}{2\Delta m} = (g - \alpha u_j^{(1)})$
- Let then insert $u_j = 0 + \epsilon u_j^{(1)} + \mathcal{O}(\epsilon^2)$ in the first equation :

Uniform consistency in the large friction regime

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}\epsilon(u_{j+1/2}^{(1)} - u_{j-1/2}^{(1)}) + \mathcal{O}(\epsilon^2)$$

$$u_{j+1/2}^* = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{2a}(p_{j+1}^n - p_j^n)$$

Numerical asymptotic analysis. $u_j = u_j^{(0)} + \epsilon u_j^{(1)} + \mathcal{O}(\epsilon^2)$

- Multiply the second equation by ϵ and let $\epsilon \to 0$: $u_j^{(0)} = 0$
- Let $\epsilon \to 0$ in the second equation : $\dfrac{p_{j+1} - p_{j-1}}{2\Delta m} = (g - \alpha u_j^{(1)})$
- Let then $\epsilon \to 0$ in the first equation :

$$u_{j+1/2}^{(1)} = \frac{u_j^{(1)} + u_{j+1}^{(1)}}{2} - \frac{\Delta m}{\epsilon}\frac{p_{j+1} - p_j}{2a\Delta m} = \frac{u_j^{(1)} + u_{j+1}^{(1)}}{2} + \mathcal{O}(\frac{\Delta m}{\epsilon})$$

which is clearly not consistent with $\partial_t \tau - \epsilon \partial_m u_1 = O(\epsilon^2)$,

Uniform consistency in the large friction regime

The problem comes from the numerical diffusion in $u^*_{j+1/2}$

In order to obtain an uniform consistency with respect to ϵ we introduce the parameter $\theta_{j+1/2}$ and simply consider the following definition of $u^*_{j+1/2}$

$$u^*_{j+1/2} = \frac{1}{2}(u^n_j + u^n_{j+1}) - \frac{\theta_{j+1/2}}{2a}(p^n_{j+1} - p^n_j)$$

Then we get $u^{(1)}_{j+1/2} = \dfrac{u^{(1)}_j + u^{(1)}_{j+1}}{2} + \mathcal{O}(\dfrac{\theta_{j+1/2}\Delta m}{\epsilon})$

Which gives the uniform consistency provided that $\theta_{j+1/2} = \mathcal{O}(\epsilon)$

Uniform consistency in the low Mach regime

Let us focus on the first step of the time-explicit scheme (the transport step is not a problem)

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2})$$
$$u_j^{n+1-} = u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2})$$
$$E_j^{n+1-} = E_j^n - \frac{\Delta t}{\Delta m}((pu)^*_{j+1/2} - (pu)^*_{j-1/2})$$

with

$$u^*_{j+1/2} = \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2a}(p_{j+1} - p_j)$$
$$p^*_{j+1/2} = \frac{1}{2}(p_j + p_{j+1}) - \frac{a}{2}(u_{j+1} - u_j)$$

Uniform consistency in the low Mach regime

In dimensionless form we get

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2})$$
$$u_j^{n+1-} = u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2})$$
$$E_j^{n+1-} = E_j^n - \frac{\Delta t}{\Delta m}((pu)^*_{j+1/2} - (pu)^*_{j-1/2})$$

with, since $p_{j+1} - p_j = \mathcal{O}(M^2)$

$$u^*_{j+1/2} = \frac{u_j + u_{j+1}}{2} - \frac{M\Delta m}{2aM^2}\frac{(p_{j+1} - p_j)}{\Delta m} = \frac{u_j + u_{j+1}}{2} + \mathcal{O}(M\Delta m)$$

$$p^*_{j+1/2} = \frac{p_j + p_{j+1}}{2M^2} - \frac{a\Delta m}{2M}\frac{(u_{j+1} - u_j)}{\Delta m} = \frac{p_j + p_{j+1}}{2M^2} + \mathcal{O}(\frac{\Delta m}{M})$$

Uniform consistency in the low Mach regime

The problem comes from the numerical diffusion in $p^*_{j+1/2}$

In order to obtain an uniform consistency with respect to M we introduce the parameter $\theta_{j+1/2}$ and simply consider the following definition of $p^*_{j+1/2}$

$$p^*_{j+1/2} = \frac{1}{2}(p^n_j + p^n_{j+1}) - \theta_{j+1/2}\frac{a}{2}(u^n_{j+1} - u^n_j)$$

Then we get $p^*_{j+1/2} = \dfrac{p_j + p_{j+1}}{2M^2} + \mathcal{O}(\dfrac{\theta_{j+1/2}\Delta m}{M})$

Which gives the uniform consistency provided that $\theta_{j+1/2} = \mathcal{O}(M)$

Remarks

The modifications give the uniform consistency and we recover the classical scheme provided that $\theta_{j+1/2} = 1$

The modifications apply directly on unstructured meshes

Considering the time-implicit treatment of the Lagrangian step gives the uniform stability

The relaxation approach allows to consider any given pressure law

Recall that the unstructured mesh is fixed (not moving)

All the objectives are reached

Remarks

Interestingly, we proved that operator splitting strategies are compatible with asymptotic-preserving and all-regime properties !

How does the modifications affect the stability properties ?

One are able to prove that the schemes are

- conservative (with no source terms and external forces)
- positive
- uniformly stable and uniformly consistent w.r.t. ϵ
- entropy satisfying under a suitable definition of θ

$\theta = 0$ is also possible ! (numerical diffusion in the transport step)

High-order extension under progress using DG methods

Outline

Implementation of the numerical scheme

The numerical schemes have been implemented in YAFiVoC (Yet Another Finite Volume Code)

A code that was developed by Mathieu Girardin and Samuel Kokh to implement finite volume methods on unstructured meshes

Programming Language : C
Compilation : CMake
Linear problem solver : Petsc

Numerical results

We want to assess the following properties of the numerical scheme :

- Accuracy of the numerical scheme in the large friction regime if $\tilde{\theta} = O(\epsilon)$
- Accuracy of the numerical scheme in the low Mach regime if $\theta = O(M)$
- Robustness of the numerical scheme with respect to the choice of θ (resp. $\tilde{\theta}$) in and outside the low Mach regime (resp. large friction regime)
- Performance in terms of CPU time of the mixed implicit-explicit numerical scheme

Large friction modification

Large friction modification

test case : sensitivity w.r.t. the space step

The fluid is equipped with a perfect gas equation of state

$$p = (\gamma - 1)\rho e, \quad \gamma = 1.4$$

We consider the domain $\Omega = (0, 1)$.
The initial condition is given by

$$\begin{cases} (\rho, u, p) & = (1.0, 0, 10000.0), \quad \text{if } x \in \quad [0, 0.35] \cap [0.65, 1], \\ (\rho, u, p) & = (2.0, 0, 26390.2), \quad \text{if } x \in \quad [0.35, 0.65]. \end{cases}$$

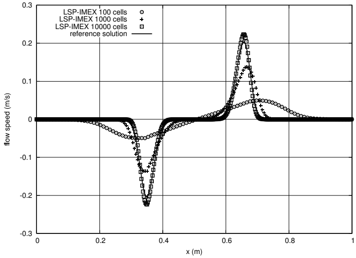
We impose periodic boundary conditions.

The friction parameter is given by $\alpha = 10^6 s^{-1}$, so that we are in the large fraction regime.

test case : sensitivity w.r.t. the space step

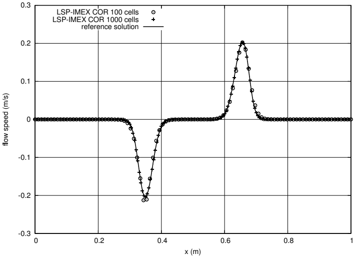
We compute approximate solutions with a 100-cell, 1000-cell and a 10 000-cell grid, with $\beta = n$

$\tilde{\theta} = 1$



flow speed

$\tilde{\theta} = \min\left(\frac{2a}{\alpha \Delta x}, 1\right)$

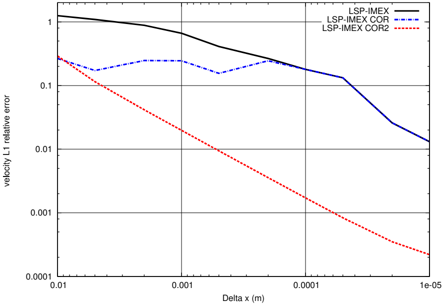


flow speed

test case : sensitivity w.r.t. the space step

We plot convergence curves in L^1 norm for

$$\tilde{\theta} = 1 \text{ (black)}, \quad \tilde{\theta} = \min\left(\frac{2a}{\alpha \Delta x}, 1\right) \text{ (blue)}, \quad \tilde{\theta} = \frac{1}{\alpha} \text{ (red)}$$



Low Mach modification

Low Mach modification

Vortex in a box : test case

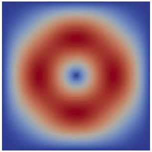
The fluid is equipped with a perfect gas equation of state

$$p = (\gamma - 1)\rho e, \quad \gamma = 1.4$$

We consider the domain $\Omega = (0, 1)^2$.
The initial condition is given by

$$\begin{cases} \rho_0(x, y) = 1 - \frac{1}{2} tanh\left(y - \frac{1}{2}\right), & u_0(x, y) = 2 sin^2(\pi x) sin(\pi y) cos(\pi y)), \\ p_0(x, y) = 1000, & v_0(x, y) = -2 sin(\pi x) cos(\pi x) sin^2(\pi y). \end{cases}$$

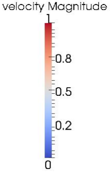
We impose a no-slip boundary condition.

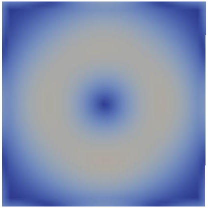


This configuration leads to a Mach number of order 0.026, so that we are in the low Mach regime.

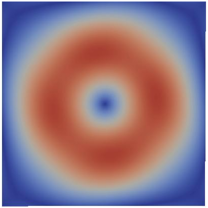
Vortex in a box $(M\#0.026)$: explicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.

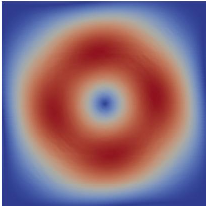




explicit scheme
$(\theta = 1)$
Cartesian Mesh
$50 * 50 cells$



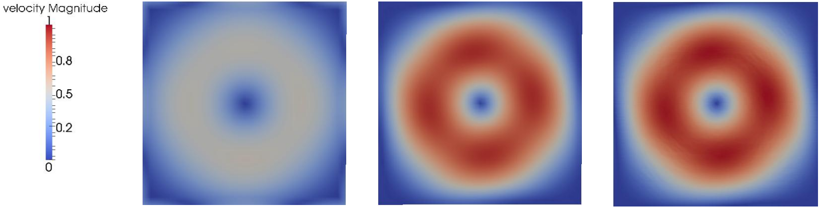
explicit scheme
$(\theta = 1)$
Cartesian Mesh
$400 * 400 cells$



reference solution
explicit scheme
$(\theta = 1)$
Triangular Mesh

Vortex in a box ($M\#0.026$) : modified explicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.



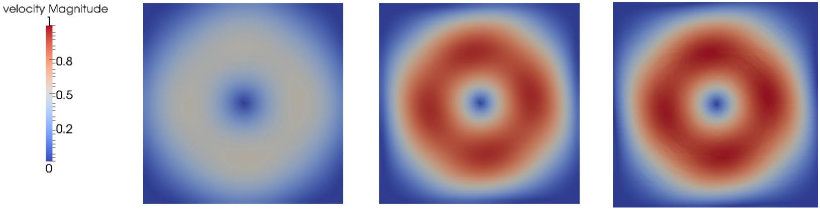
explicit scheme
$(\theta = 1)$
Cartesian Mesh
$50 * 50 cells$

explicit scheme
$(\theta_{ij} = M_{ij}^n)$
Cartesian Mesh
$50 * 50 cells$

reference solution
explicit scheme
$(\theta = 1)$
Triangular Mesh

Vortex in a box ($M\#0.026$) : modified implicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.



implicit-explicit scheme ($\theta = 1$)
Cartesian Mesh
$50 * 50 cells$

implicit-explicit scheme ($\theta_{ij} = M_{ij}^n$)
Cartesian Mesh
$50 * 50 cells$

reference solution
explicit scheme
($\theta = 1$)
Triangular Mesh

Vortex in a box ($M\#0.026$) : CPU Time

EX : $\beta = n$,   IMEX : $\beta = Lag$.

Numerical scheme	EX($\theta = 1$) (Mesh $400 * 400$)	EX($\theta = 1$) (Mesh $50 * 50$)	EX($\theta_{ij} = M_{ij}$) (Mesh $50 * 50$)
Number of iterations	18 457	2 306	2 305
CPU time (s)	9 263.04 ($2h34min$)	17.14	19.3

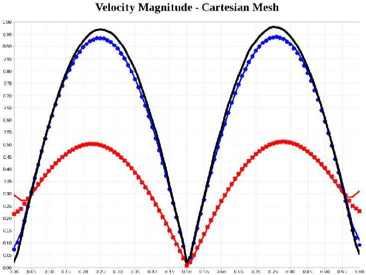
Speed up ($\theta = 1 \rightarrow \theta_{ij} = M_{ij}$) $= 480$

Numerical scheme	IMEX($\theta = 1$) (Mesh $50 * 50$)	IMEX($\theta_{ij} = M_{ij}$) (Mesh $50 * 50$)
Number of iterations	43	56
CPU time (s)	3.75	5.77

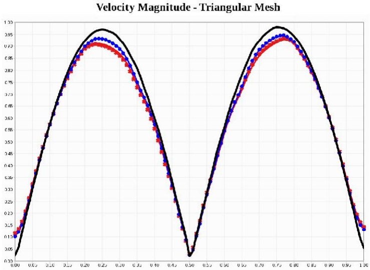
Speed up (explicit$\rightarrow$ implicit-explicit)$= 3.3$

Vortex in a box $(M\#0.026)$: Influence of the cell geometry

We plot a 1D-cut at $x = 0.5$ of the flow speed magnitude at time $T = 0.125s$.



Cartesian Mesh



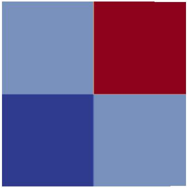
Triangular Mesh

2D-Riemann problem : test case

The fluid is equipped with a perfect gas equation of state

$$p = (\gamma - 1)\rho e, \quad \gamma = 1.4$$

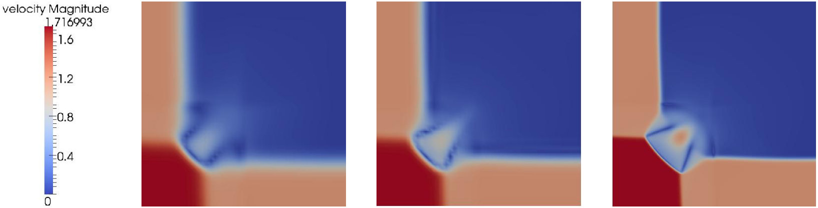
We consider the domain $\Omega = (0,1)^2$.
The initial condition corresponds to a 2D Riemann problem that consists of 4 shock waves. We impose Neumann boundary conditions.



This configuration leads to a Mach number that ranges from 10^{-5} to 3.15, so that we have both low Mach and order 1 Mach values.

2D-Riemann problem $M \in (10^{-5}, 3.15)$: modified explicit scheme

We plot the flow speed magnitude at time $T = 0.4s$.



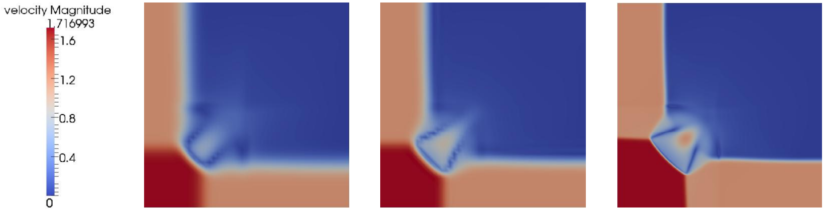
explicit scheme
$(\theta = 1)$
Cartesian Mesh
$50 * 50 cells$

explicit scheme
$(\theta = 0)$
Cartesian Mesh
$50 * 50 cells$

reference solution
explicit scheme
$(\theta = 1)$
Triangular Mesh

2D-Riemann problem $M \in (10^{-5}, 3.15)$: modified implicit scheme

We plot the flow speed magnitude at time $T = 0.4s$.



implicit-explicit scheme $(\theta = 1)$ Cartesian Mesh $50 * 50 cells$	implicit-explicit scheme $(\theta = 0)$ Cartesian Mesh $50 * 50 cells$	reference solution explicit scheme $(\theta = 1)$ Triangular Mesh

2D-Riemann problem $M \in (10^{-5}, 3.15)$: CPU time

Numerical scheme	EX($\theta = 1$) (Mesh $50 * 50$)	EX($\theta = 0$) (Mesh $50 * 50$)
Number of iterations	323	343
CPU time (s)	2.59	2.79

Speed up $(\theta = 1 \to \theta = 0) \approx 1$

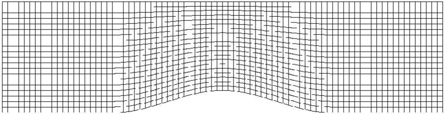
Numerical scheme	IMEX($\theta = 1$) (Mesh $50 * 50$)	IMEX($\theta = 0$) (Mesh $50 * 50$)
Number of iterations	216	218
CPU time (s)	10.28	10.33

Speed up (explicit$\rightarrow$ implicit-explicit)$= 0.25$

flow in a channel with bump

The fluid is equipped with a mixture of two perfect gas with different adiabatic coefficients equation of state : $\gamma_1 = 2$, $\gamma_2 = 1.4$.

We consider for the domain a channel with a 20% sinusoidal bump.



The initial condition corresponds to a constant state

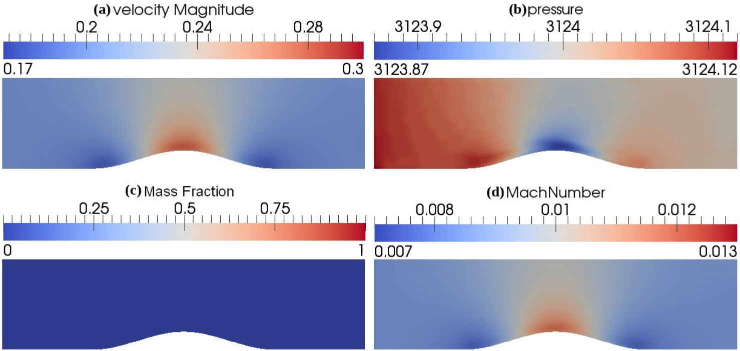
$$(\rho, Y, p, u, v) = (7.81, 0, 3124, 0, 0).$$

We impose inlet/outlet and Wall boundary conditions.

This configuration leads to a subsonic flow for $u_{in} = 0.2$ and a transonic flow for $u_{in} = 12$.

flow in a channel with bump : subsonic flow

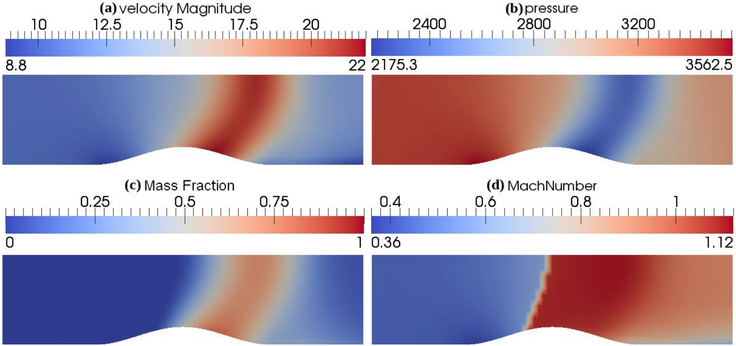
We plot the results obtained for the subsonic test case ($u_{in} = 0.2$) on a 80×20 quadrangular mesh at time $T = 2s$ with $\beta = Lag$ and $\theta_{ij} = M_{ij}$



Flow speed animation

flow in a channel with bump : transonic flow

We plot the results obtained for the transonic test case ($u_{in} = 12$) on a 80×20 quadrangular mesh at time $T = 2s$ with $\beta = n$ and $\theta_{ij} = 0$



Flow speed animation

Publications

- C. Chalons, M. Girardin and S. Kokh, Large time step and asymptotic preserving numerical schemes for the gas dynamics equations with source terms, SIAM J. Sci. Comput., 35(6) (2013)
- C. Chalons, M. Girardin and S. Kokh, Operator-splitting-based asymptotic preserving scheme for the gas dynamics equations with stiff source terms, AIMS on Applied Mathematics, Proceedings of the 2012 International Conference on Hyperbolic Problems : Theory, Numerics, Applications, 8 (2014)
- C. Chalons, M. Girardin and S. Kokh, An all-regime Lagrange-Projection like scheme for the gas dynamics equations on unstructured meshes, submitted to CICP
- C. Chalons, M. Girardin and S. Kokh, An all-regime Lagrange-Projection like scheme for 2D homogeneous models for two-phase flows on unstructured meshes, submitted to JCP
- M. Girardin, Méthodes numériques tout-régime et préservant l'asymptotique de type Lagrange-Projection. Application aux écoulements diphasiques en régime bas Mach, Thèse de l'Université Paris 6 (2014)