A HIGH ORDER FINITE VOLUME/FINITE ELEMENT PROJECTION METHOD FOR LOW-MACH NUMBER FLOWS WITH TRANSPORT OF SPECIES

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Outline

• Low-Mach number equations

• Numerical discretization
  – A dual finite volume mesh
  – Finite volume discretization
  – Projection method
  – Coupling FVM and FEM

• Numerical results

Low-Mach number equations

- Assumption: the Mach number $M$ is sufficiently small so that the pressure $p$ can be written as the sum of a spatially constant function $\bar{\pi}$ and a small perturbation $\pi$,

$$p(x, y, z, t) = \bar{\pi}(t) + \pi(x, y, z, t), \quad \frac{\pi}{\bar{\pi}} = O(M^{-2}), \quad (1)$$

- $\bar{\pi}(t)$ is a data
- $\pi$ is neglected in the state equation but it has to be retained in the momentum equation.
Low-Mach number equations

The compressible Navier-Stokes equations reduce to the mass conservation law, the momentum equation and the equation of state:

\[
\begin{align*}
  \frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div}(\rho \mathbf{u}) &= 0, \quad (2) \\
  \frac{\partial F_1(u, \rho)}{\partial x} + \frac{\partial F_2(u, \rho)}{\partial y} + \frac{\partial F_3(u, \rho)}{\partial z} + \nabla \pi - \operatorname{div} \tau &= f(x, y, z, t), \quad (3) \\
  \pi &= \rho R \theta. \quad (4)
\end{align*}
\]

- Density: \( \rho = \rho(x, y, z, t) \)
- Velocity vector: \( \mathbf{u} = \mathbf{u}(x, y, z, t) = (u_1, u_2, u_3)' \)
- Flux tensor, \( \mathbf{F} \): \( F_i(u, \rho) = \rho u_i \mathbf{u}, \ i = 1, 2, 3 \)
- The viscous part of the Cauchy stress tensor: \( \tau \)
- Generic source term used in the analytical test problems: \( f \)
- The gas constant: \( R = \mathcal{R}/\mathcal{M} \), the universal constant \( \mathcal{R} = 8314 \ \text{J}/(\text{kmol} \ \text{K}) \), the molecular mass \( \mathcal{M} \) and the absolute temperature \( \theta \).
Low-Mach number equations

- From (2) and (4) we get the following divergence condition:

\[
\text{div}(\rho \mathbf{u}) = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t} \left( \frac{\pi}{R\theta} \right). \tag{5}
\]

- **Conservative variables:** \( \mathbf{w}(x, y, z, t) = \rho(x, y, z, t)\mathbf{u}(x, y, z, t) \)

- Rewriting the flux in terms of \( \mathbf{w} \),

\[
\mathcal{F}_i(\mathbf{w}, \rho) = \frac{w_i}{\rho} \mathbf{w}, \quad i = 1, 2, 3.
\]

- Now, the system of equations to be solved read

\[
\frac{\partial \mathbf{w}}{\partial t} + \text{div}(\mathcal{F}(\mathbf{w}, \rho)) + \nabla \pi - \text{div}(\tau) = \mathbf{f}(x, y, z, t), \tag{6}
\]

\[
\text{div} \mathbf{w} = q, \tag{7}
\]

where

\[
q := -\frac{\partial}{\partial t} \left( \frac{\pi}{R\theta} \right).
\]
Numerical discretization

We start by considering $W^n$, approximations of the conservative variables $w(x, y, z, t^n)$ and $\pi^n$ of the pressure $\pi(x, y, z, t^n)$, and define $W^{n+1}$ and $\pi^{n+1}$ from the following system of equations:

$$
\frac{\tilde{W}^{n+1} - W^n}{\Delta t} + \text{div}(F(W^n, \rho^n)) + \nabla\pi^n - \text{div}(\tau^n) = f^n, \tag{8}
$$

$$
\frac{W^{n+1} - \tilde{W}^{n+1}}{\Delta t} + \nabla(\pi^{n+1} - \pi^n) = 0, \tag{9}
$$

$$
\text{div}W^{n+1} = q^{n+1}. \tag{10}
$$
Numerical discretization

Summarizing, the overall method consists of:

- **Transport-diffusion stage**: momentum equation (8) is solved by a FVM. The values $\tilde{\mathbf{W}}^{n+1}$ computed at this stage do not necessarily satisfy the divergence constraint at $t^{n+1}$.

- **Projection stage**: equations (9) and (10) are solved with a FEM to obtain the pressure correction $\delta^{n+1} := \pi^{n+1} - \pi^n$.

- **Post-projection stage**: the $\tilde{\mathbf{W}}^{n+1}$ computed at the first stage is updated by using $\delta^{n+1}$ in order to obtain another approximation $\mathbf{W}^{n+1}$ satisfying the divergence condition (10).
A dual finite volume mesh

- \( \{ N_i, \ i = 1, M \} = \) barycentre of the “faces” of the tetrahedron.
- \( K_i = \) set of all the neighbors of \( N_i \).
- \( vol( C_i ) = \) Volume of \( C_i \).
- \( \Gamma_i = \partial V_i = \bigcup_{j \in K_i} \Gamma_{ij} \), boundary of \( C_i \).

- \( \mathbf{n}_j = \) outward normal to \( \Gamma_{ij} \), \( \mathbf{n}_j / ||\mathbf{n}_j|| = \mathbf{n}_j ||\mathbf{n}_j|| \), \( \times \)
- \( S_i = \) Area(\( \Gamma_{ij} \)) = \( \bigcup_{j \in K_i} ||\mathbf{n}_j|| = \) Area of \( \Gamma_i \)
- \( i \neq j \)
- If \( N_i \not\in \Gamma \): \( \mathbf{n}_f = \) outward normal

\[ \Gamma_{if} = \Gamma_i \]
Finite volume discretization

- By integrating the momentum equation over the finite volume \( C_i \) and applying the Gauss’ Theorem, we get

\[
\text{vol}(C_i) \frac{\mathbf{W}_i^{n+1} - \mathbf{W}_i^n}{\Delta t} + \int_{\Gamma_i} \mathbf{F}(\mathbf{W}^n, \rho^n) \mathbf{\tilde{n}} \, dS = \\
\int_{C_i} \left[ -\nabla \pi^n + \text{div}\tau^n + \mathbf{f}(x, y, z, t^n) \right] \, dV,
\]

(11)

- The integral of the normal flux:
  - Firstly, we split \( \Gamma_i \) into the cell interfaces \( \Gamma_{ij} \):

\[
\int_{\partial C_i} \mathbf{F}(\mathbf{W}^n, \rho^n) \mathbf{\tilde{n}} \, dS = \sum_{N_j \in K_i} \int_{\Gamma_{ij}} \mathbf{F}(\mathbf{W}^n, \rho^n) \mathbf{\tilde{n}}_{ij} \, dS.
\]

(12)

  - Then, in order to obtain a stable discretization, the integral on \( \Gamma_{ij} \) is approximated by an upwind scheme using a numerical flux function \( \phi \):

\[
\int_{\Gamma_{ij}} \mathbf{F}(\mathbf{W}^n, \rho) \mathbf{\tilde{n}}_{ij} \, dS \approx \phi \left( (\mathbf{W}_i^n, \rho_i^n), (\mathbf{W}_j^n, \rho_j^n), \mathbf{\eta}_{ij} \right).
\]

(13)

- We will consider the \( Q \)-scheme of van Leer and the Rusanov scheme
Numerical flux: $Q$-scheme of van Leer

\[
\phi((\mathbf{W}_1, \rho_1), (\mathbf{W}_2, \rho_2), \bm{\eta}) = \frac{Z(\mathbf{W}_1, \rho_1, \bm{\eta}) + Z(\mathbf{W}_2, \rho_2, \bm{\eta})}{2} - \frac{1}{2} A \left( \frac{\mathbf{W}_1 + \mathbf{W}_2}{2}, \frac{\rho_1 + \rho_2}{2}, \bm{\eta} \right) (\mathbf{W}_2 - \mathbf{W}_1),
\]

- $A(\mathbf{W}, \rho, \bm{\eta})$ Jacobian matrix of normal flux $Z(\mathbf{W}, \rho, \bm{\eta})$
- Eigenvalues $\lambda_1 = \lambda_2 = \frac{1}{\rho} \mathbf{W} \cdot \bm{\eta} = \mathbf{U} \cdot \bm{\eta}$, $\lambda_3 = \frac{2}{\rho} \mathbf{W} \cdot \bm{\eta} = 2\mathbf{U} \cdot \bm{\eta} = 2\lambda_1$,
- $\text{sign}(\lambda_1)=\text{sign}(\lambda_2)=\text{sign}(\lambda_3)$,
  - if $\text{sign}(\lambda_k) = 1 \implies |A(\mathbf{W}, \rho, \bm{\eta})| = A(\mathbf{W}, \rho, \bm{\eta})$
  - if $\text{sign}(\lambda_k) = -1 \implies |A(\mathbf{W}, \rho, \bm{\eta})| = -A(\mathbf{W}, \rho, \bm{\eta})$.

Numerical flux: Rusanov scheme

\[
\phi((\mathbf{W}_1, \rho_1), (\mathbf{W}_2, \rho_2), \bm{\eta}) = \frac{Z(\mathbf{W}_1, \rho_1, \bm{\eta}) + Z(\mathbf{W}_2, \rho_2, \bm{\eta})}{2} - \frac{1}{2} \alpha_{RS} (\mathbf{W}_2 - \mathbf{W}_1),
\]

\[
\alpha_{RS} = \alpha_{RS}((\mathbf{W}_1, \rho_1), (\mathbf{W}_2, \rho_2), \bm{\eta}) := \max \{2|\mathbf{U}_1 \cdot \bm{\eta}|, 2|\mathbf{U}_2 \cdot \bm{\eta}| \}.
\]
Pressure explicit integral approximation

\[ \int_{C_i} -\nabla \pi^n \, dV = - \sum_{N_j \in \mathcal{K}_i} \int_{\Gamma_{ij}} \pi^n \tilde{\eta}_{ij} \, dS. \]

- The pressure is computed at the vertices of the original tetrahedral mesh, at the projection stage.

- Its value on face \( \Gamma_{ij} \) is approximated by the arithmetic mean of its values at the three vertices of face, namely, \( V_1, V_2 \) and the barycenter \( B \),

\[ \int_{\Gamma_{ij}} \pi \tilde{\eta}_{ij} \, dS = \tilde{\eta}_{ij} \frac{1}{3} \left( \pi(V_1) + \pi(V_2) + \pi(B) \right) \text{Area}(\Gamma_{ij}) \]

with

\[ \pi(B) = \frac{1}{4} \left( \pi(V_1) + \pi(V_2) + \pi(V_3) + \pi(V_4) \right) \]

- Finally the approximation of the integral reads

\[ \int_{\Gamma_{ij}} \pi \tilde{\eta}_{ij} \, dS = \eta_{ij} \frac{1}{3} \left[ \pi(V_1) + \pi(V_2) + \frac{1}{4} \left( \pi(V_1) + \pi(V_2) + \pi(V_3) + \pi(V_4) \right) \right] \]
Viscous term discretization

- Explicit discretization

\[ \int_{C_i} \operatorname{div} \tau^n \, dV \approx \sum_{j \in \mathcal{K}_i} \nu_{ij} \left( \nabla U^n + (\nabla U^n)^T \right)_{ij} \eta_{ij} \]

\[ (\nabla U^n)_{ij} \approx \frac{1}{2} \left[ \nabla U^n_{T_{ij}^1} + \nabla U^n_{T_{ij}^2} \right] \]

\[ (\nabla U^n)_{ij} \approx \nabla U^n_{T_{ij}} \]

- \( \nu_{ij} \) is an approximation of the dynamic viscosity coefficient on face \( \Gamma_{ij} \)

- \( U^n_{T_{ij}^1}, \ U^n_{T_{ij}^2}, \ \) and \( U^n_{T_{ij}} \) denote linear interpolations of the velocity \( U \) in tetrahedra \( T_{ij}^1, \ T_{ij}^2, \) and \( T_{ij} \), respectively

- As it is well known, for a linear scalar equation \( w_t + \lambda w_x = \nu w_{xx} \), the stability condition for the upwind explicit scheme is

\[ \Delta t \leq \frac{\Delta x}{\lambda + \frac{2\nu}{\Delta x}} = \frac{(\Delta x)^2}{\lambda \Delta x + 2\nu} \]
A new Kolgan-type scheme

\[ \frac{\partial w}{\partial t} (x, t) + \frac{\partial f(w)}{\partial x} (x, t) = 0, \]

New upwind x? order scheme

\[ w_{j}^{n+1} = w_{j}^{n} - \frac{\Delta t}{\Delta x} \left( \phi(w_{j}^{n}, w_{j+1}^{n}, w_{L,j+1/2}^{n}, w_{R,j+1/2}^{n}) - \phi(w_{j-1}^{n}, w_{j}^{n}, w_{L,j-1/2}^{n}, w_{R,j-1/2}^{n}) \right) \]

Numerical flux

\[ \phi(U, V; U_L, V_R) = \frac{f(U) + f(V)}{2} - \frac{1}{2} |Q(U_L, V_R)| (V_R - U_L) \]

where \( \Delta_{j}^{L} \), and \( \Delta_{j}^{R} \) are the left and right limited slopes at the node \( x_{j} \)

\[
|\Delta_{j}^{L}| = \begin{cases} 
\max \left[ 0, \min \left( \frac{w_{j-1} - w_{j-1}}{2}, w_{j+1} - w_{j} \right) \right] & \text{if } w_{j+1} - w_{j} > 0 \\
\min \left[ 0, \max \left( \frac{w_{j-1} - w_{j-1}}{2}, w_{j+1} - w_{j} \right) \right] & \text{if } w_{j+1} - w_{j} < 0 
\end{cases}
\]

\[
|\Delta_{j}^{R}^{\ast}| = \begin{cases} 
\max \left[ 0, \min \left( \frac{w_{j+1} - w_{j}}{2}, w_{j+1} - w_{j} \right) \right] & \text{if } w_{j+1} - w_{j} > 0 \\
\min \left[ 0, \max \left( \frac{w_{j+1} - w_{j}}{2}, w_{j+1} - w_{j} \right) \right] & \text{if } w_{j+1} - w_{j} < 0 
\end{cases}
\]

Viscous term discretization
Diffusive flux $D_{\text{tot}}$ decomposition on each face $\Gamma_{ij}$:

- **Orthogonal diffusion** $D_{\perp}^{ij}$
- **Non-orthogonal diffusion** $D_{\parallel}^{ij}$

- The approximation for each component of $\nabla U$ on $\Gamma_{ij}$ is given by

$$
\left( \frac{\partial U_m}{\partial x_k} \right)_{ij} \approx \frac{1}{\text{vol}(\mathcal{V}_{ij})} \int_{\mathcal{V}_{ij}} \frac{\partial U_m}{\partial x_k} \, dV = \frac{1}{\text{vol}(\mathcal{V}_{ij})} \int_{\partial \mathcal{V}_{ij}} U_m n_k \, dS
$$

$$
= \frac{1}{\text{vol}(\mathcal{V}_{ij})} \sum_{l=1}^{5} U_{ml} \text{area}(\Gamma_l) n_k^l, \quad k = 1, 3
$$

$$
\text{vol}(\mathcal{V}_{ij}) = \| \mathbf{n}_{ij} \| \cdot | \tilde{\mathbf{n}}_{ij} \cdot \mathbf{N}_{ij} | = | \mathbf{n}_{ij} \cdot \mathbf{N}_{ij} |, \quad \mathbf{N}_{ij} = \mathbf{N}_i - \mathbf{N}_j.
$$

- Using (14), the approximation for the normal gradient component is:

$$
\nu_{ij} \frac{\| \mathbf{n}_{ij} \|^2}{| \mathbf{n}_{ij} \cdot \mathbf{N}_{ij} |} [\mathbf{U}_j - \mathbf{U}_i] + \nu_{ij} \left( \frac{1}{| \mathbf{n}_{ij} \cdot \mathbf{N}_{ij} |} \left( \begin{array}{c} \mathbf{U}(V_1) \\ \mathbf{U}(V_2) \\ \mathbf{U}(B) \end{array} \right) \right) r_{\parallel ij},
$$

$$
r_{\parallel ij} = \langle (\mathbf{n}_{ij}, \mathbf{V}_{B2}, \mathbf{N}_{ij}), (\mathbf{n}_{ij}, \mathbf{V}_{1B}, \mathbf{N}_{ij}), (\mathbf{n}_{ij}, \mathbf{V}_{21}, \mathbf{N}_{ij}) \rangle^T
$$
Viscous term discretization

- **Orthogonal diffusion** \( D_{ij} \)

  In order to make implicit the main diagonal of the diffusion term, we introduce the component of the conservative variable \( W_i = \rho_i U_i \) in \( D_{ij} \):

  \[
  D_{ij} = \nu_{ij} \frac{\| \eta_{ij} \|^2}{\eta_{ij} \cdot \mathbf{N}_{ij}} \left[ \mathbf{U}_j - \mathbf{U}_i \right] = \nu_{ij} \frac{\| \eta_{ij} \|^2}{\eta_{ij} \cdot \mathbf{N}_{ij}} \left[ \mathbf{U}_j - \frac{1}{\rho_i} \mathbf{W}_i \right], \tag{15}
  \]

- Semi-implicit discretization

  \[
  D_{ij} = \nu_{ij} \frac{\| \eta_{ij} \|^2}{\eta_{ij} \cdot \mathbf{N}_{ij}} \left[ \mathbf{U}_j^n - \frac{1}{\rho_i} \mathbf{W}_i^{n+1} \right], \tag{16}
  \]

  * The coefficient in the \( i \)-th row of the main diagonal is

  \[
  \frac{\Delta t}{\text{vol}(C_i)} \frac{1}{\rho_i^n} \sum_{j \in K_i} \nu_{ij} \frac{\| \eta_{ij} \|^2}{\eta_{ij} \cdot \mathbf{N}_{ij}}. \tag{17}
  \]

- Explicit discretization

  \[
  D_{ij} = \nu_{ij} \frac{\| \eta_{ij} \|^2}{\eta_{ij} \cdot \mathbf{N}_{ij}} \left[ \mathbf{U}_j^n - \mathbf{U}_i^n \right], \tag{18}
  \]
Boundary conditions for the conservative variable

- **Transport-diffusion stage**: calculating $\tilde{W}^{n+1}$.
  - **Integral of the normal flux on a boundary face**
    \[
    \int_{\Gamma_F} \mathcal{F}(\mathbf{W}^n, \rho) \cdot \tilde{\eta}_F dS \approx \phi \left( (\mathbf{W}_F^n, \rho_F^n), (\tilde{\mathbf{W}}_F^n, \tilde{\rho}_F^n), \eta_F \right) = Z(\mathbf{W}_F^n, \rho_F^n, \eta_F).
    \]
  - **Integral of the pressure term on a boundary face**
    \[
    \int_{\Gamma_F} \pi^n \tilde{\eta}_F dS = \frac{1}{3} \left( \pi^n (V_{1F}) + \pi^n (V_{2F}) + \pi^n (V_{3F}) \right) \text{area}(\Gamma_F) \tilde{\eta}_F.
    \]
  - **Dirichlet boundary conditions**
    * Inviscid fluids. $\mathbf{w} \cdot \tilde{\eta}_{\partial\Omega} = g$
      \[
      \tilde{W}_F^{n+1} \leftarrow \tilde{W}_F^{n+1} - \tilde{W}_F^{n+1} \tilde{\eta}_F \tilde{\eta}_F + g(\mathbf{N}_F, t^{n+1}, \tilde{\eta}_F) \tilde{\eta}_F.
      \]
    * Viscous fluids. The conservative variable $\mathbf{w}$ is imposed at the boundary.
  - **Neumann boundary conditions** The computation of $\tilde{W}_F^{n+1}$ takes into account the inflow/outflow boundary condition.

- **Post-projection stage**: calculating $\mathbf{W}^{n+1}$. After updating the conservative variables, depending on the existence of viscous terms it may be necessary to impose some boundary conditions.
Projection method

We have adapted the incremental projection method used in Guermond et al (2006).

\[
\frac{\tilde{W}^{n+1} - W^n}{\Delta t} + \text{div}(\mathcal{F}((\rho U)^n, \rho^n)) + \nabla \pi^n - \text{div} \tau^n = f^n \quad [\text{Output : } \tilde{W}^{n+1} \text{ Transport-diffusion stage}]
\]

\[
\frac{W^{n+1} - \tilde{W}^{n+1}}{\Delta t} + \nabla (\pi^{n+1} - \pi^n) = 0 \quad [\text{Output : } \pi^{n+1} - \pi^n \text{ Projection stage}]
\]

\[
\text{div}(W^{n+1}) = q^{n+1} \quad [\text{Projection stage}]
\]

\[\quad (*)\]

- We solve (*) and obtain \( \pi^{n+1} - \pi^n \) using a standard finite element method.
- Multiplying this equation by \( \nabla z \) and integrating over \( \Omega \) we obtain the following variational formulation:

\[
\int_{\Omega} \nabla (\pi^{n+1} - \pi^n) \cdot \nabla z = \frac{1}{\Delta t} \int_{\Omega} \tilde{W}^{n+1} \cdot \nabla z - \frac{1}{\Delta t} \int_{\Omega} W^{n+1} \cdot \nabla z \quad \forall z \in V_0,
\]

where \( V_0 := \{ z \in H^1(\Omega) : \int_{\Omega} z = 0 \} \).
- Using the divergence condition and the Green formula, \( \int_{\Omega} W^{n+1} \cdot \nabla z \) read:

\[
\int_{\Omega} W^{n+1} \cdot \nabla z = \int_{\partial \Omega} \left( \frac{W^{n+1}}{g^{n+1}} \right) \cdot n z - \int_{\Omega} \text{div}(W^{n+1}) z \quad \text{for } \quad g^{n+1} z.
\]
Projection method

- Replacing the previous expression in the variational formulation and introducing the variable $\delta^{n+1} := \pi^{n+1} - \pi^n$, to obtain $\pi^{n+1}$ we need to solve the weak problem:

  Find $\delta^{n+1} \in V_0$ verifying

  $$\int_\Omega \nabla \delta^{n+1} \cdot \nabla z = \frac{1}{\Delta t} \int_\Omega \widetilde{W}^{n+1} \cdot \nabla z + \frac{1}{\Delta t} \int_\Omega q^{n+1} z - \frac{1}{\Delta t} \int_{\partial \Omega} g^{n+1} z$$

- This weak problem can bee seen as corresponding to the following Laplace problem with Neumann conditions

  $$\Delta \delta^{n+1} = \frac{1}{\Delta t} (\text{div}(\widetilde{W}^{n+1}) - q^{n+1}) \text{ in } \Omega,$$

  $$\frac{\partial \delta^{n+1}}{\partial n} = \frac{1}{\Delta t} (\widetilde{W}^{n+1} \cdot n - g^{n+1}) \text{ in } \Gamma.$$

Post-projection stage

- Finally, with $\delta^{n+1}$, the value of $W^{n+1}$ at each finite volume is obtained from (9) at the post-projection stage:

  $$W^{n+1}_i = \widetilde{W}^{n+1}_i - \Delta t \nabla \delta^{n+1}_i,$$

  where $\nabla \delta^{n+1}_i$ denotes the arithmetic mean of $\nabla \delta^{n+1}$ at the two tetrahedra having $N_i$ as barycenter of one common face.
Coupling between the *Transport-diffusion stage* and *Projection stage*

- **Projection stage**: it is necessary to compute the integral \( \int_T \widetilde{W} \cdot \nabla p^T_i \), where \( T \) is an element of the finite element mesh and \( p^T_i \) a basis function in \( T \). We assume \( \widetilde{W} \) is constant per finite element \( (\widetilde{W}_T) \)

- **Transport-diffusion stage**: the values of \( \widetilde{W} \) are constant per finite volume.

- The difficulty of the coupling lies in the computation of \( \widetilde{W}_T \).

- The solutions:
  - **Unstable coupling**: \( \widetilde{W}_T \) is computed via the vertices of the tetrahedron
    \[
    \widetilde{W} \text{(Nodes FV mesh)} \rightarrow \widetilde{W} \text{(Vertices FE mesh)} \rightarrow \widetilde{W} \text{(tetrahedron)}
    \]
    \[
    \widetilde{W}_T = \frac{1}{4} \left( \widetilde{W}(V_1) + \widetilde{W}(V_2) + \widetilde{W}(V_3) + \widetilde{W}(V_4) \right),
    \]
  - **Stable coupling**: \( \widetilde{W}_T \) is computed via the nodes of the tetrahedron
    \[
    \widetilde{W} \text{(Nodes FV mesh)} \rightarrow \widetilde{W} \text{(tetrahedron)},
    \]
    \[
    \widetilde{W}_T = \frac{1}{4} \left( \widetilde{W}(N_1) + \widetilde{W}(N_2) + \widetilde{W}(N_3) + \widetilde{W}(N_4) \right).
    \]
Numerical Results

1. Academic test for Euler equations

- The flow is defined by

\[
\mathbf{w}(x, y, z, t) = \rho(x, y, z, t)\mathbf{u}(x, y, z, t) = e^{xt}(e^x, y^2 + 1, z^2)^T
\]
\[
\pi(x, y, z, t) = \frac{1}{2}e^t(x^2 + y^2 + z^2),
\]

The corresponding source term \( f \) is:

\[
f(x, y, z, t) := \begin{pmatrix}
xe^{x(t+1)} + (2 + t)e^{x(t+2)} + 2(y + z)e^{x(t+1)} + 2ze^{x(t+1)} + xe^t \\
x e^{xt} + ye^t + (t + 1)(y^2 + 1)e^{x(t+1)} + 2(y^2 + 1)(2y + z)e^{xt} \\
xz^2e^{xt} + z^2(t + 1)e^{x(t+1)} + 2yz^2e^{xt} + 4z^3e^{xt} + ze^t
\end{pmatrix}
\]

- Features of the meshes

<table>
<thead>
<tr>
<th>Mesh</th>
<th>( N = 1/h )</th>
<th>Elements (( nel ))</th>
<th>Vertices (( M ))</th>
<th>Nodes</th>
<th>( V_h^m ) (( m^3 ))</th>
<th>( V_h^M ) (( m^3 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M1 )</td>
<td>4</td>
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<td>125</td>
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<td>4913</td>
<td>50688</td>
<td>1.02E-05</td>
<td>2.03E-05</td>
</tr>
</tbody>
</table>

- To check the order of the error we compute the norms

\[
E(\mathbf{w}) = \| \mathbf{w} - \mathbf{w}_h \|_{L^2(\Omega)}, \quad E(\pi) = \| \pi - \pi_h \|_{L^2(\Omega)},
\]
1. Academic test for Euler equations

\[ \mu^n = \max_{i=1,\text{nod}} \Delta t^2 \|U^n_i\| \frac{S_i}{\text{vol}(C_i)} , \]

\[ \Delta t^n = \min_{i=1,\text{nod}} \frac{\mu \text{vol}(C_i)}{2 \|U^n_i\| S_i} . \]
1. Academic test for Euler equations

![Graph showing errors and CFL values]

<table>
<thead>
<tr>
<th>CFL</th>
<th>$E(w)_{M_1}$</th>
<th>$E(w)_{M_2}$</th>
<th>$E(w)_{M_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2.90E–01</td>
<td>1.21E–01</td>
<td>4.96E–02</td>
</tr>
<tr>
<td>4</td>
<td>6.77E–03</td>
<td>3.30E–03</td>
<td>1.63E–03</td>
</tr>
<tr>
<td>3</td>
<td>9.03E–02</td>
<td>4.85E–02</td>
<td>2.59E–02</td>
</tr>
<tr>
<td>2</td>
<td>3.38E–03</td>
<td>1.65E–02</td>
<td>8.14E–04</td>
</tr>
<tr>
<td>1.5</td>
<td>9.04E–02</td>
<td>4.85E–02</td>
<td>2.59E–02</td>
</tr>
<tr>
<td>1</td>
<td>1.69E–03</td>
<td>8.26E–04</td>
<td>4.07E–04</td>
</tr>
<tr>
<td>0.75</td>
<td>9.05E–02</td>
<td>4.86E–02</td>
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</tr>
</tbody>
</table>

![Table showing ratios of spatial errors]

<table>
<thead>
<tr>
<th>CFL</th>
<th>$E(w)<em>{M_1}/E(w)</em>{M_2}$</th>
<th>$E(w)<em>{M_2}/E(w)</em>{M_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2.4012</td>
<td>2.4367</td>
</tr>
<tr>
<td>4</td>
<td>1.8610</td>
<td>1.8745</td>
</tr>
<tr>
<td>3</td>
<td>1.8618</td>
<td>1.8749</td>
</tr>
<tr>
<td>2</td>
<td>1.8626</td>
<td>1.8752</td>
</tr>
<tr>
<td>1.5</td>
<td>1.8631</td>
<td>1.8754</td>
</tr>
<tr>
<td>1</td>
<td>1.8635</td>
<td>1.8756</td>
</tr>
<tr>
<td>0.75</td>
<td>1.8638</td>
<td>1.8638</td>
</tr>
</tbody>
</table>
2. Euler flow around a sphere

This problem was discussed by Landau and Lifshitz

- A sphere of radius $R = 1$ centered at the origin is surrounded by the computational domain of the flow, consisting in a cube of dimensions $\Omega = [-4,4]^3$.

- The parameters of the flow

\[ M_\infty = 10^{-2}, \quad \gamma = 1.4, \quad \rho_\infty = 1, \quad \pi_\infty = \frac{1}{\gamma}, \quad c_\infty = 1, \quad u_\infty = M_\infty c_\infty = 10^{-2} \]

- The analytic solution, expressed in cartesian coordinates

\[
\begin{align*}
\mathbf{u}(x, y, z, t) &= \mathbf{u}_\infty + \frac{1}{2} \left( \frac{R}{r(x, y, z)} \right)^3 \left[ -3 \frac{\mathbf{r}(x, y, z)}{r^2(x, y, z)} \mathbf{u}_\infty \cdot \mathbf{r}(x, y, z) + \mathbf{u}_\infty \right], \\
\pi(x, y, z, t) &= \pi_\infty + \rho_0 \left( \frac{1}{2} u_\infty^2 - \frac{1}{2} \| \mathbf{u}(x, y, z) \|^2 \right).
\end{align*}
\]

$r$ is the position vector, $r = \| \mathbf{r} \|$ and $\mathbf{u}_\infty = u_\infty \mathbf{e}_1$. 
2. Euler flow around a sphere

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Surface</th>
<th>Type (Transport-diffusion stage)</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_i$, inlet flow</td>
<td>$x = -4$</td>
<td>Dirichlet</td>
<td>$w \cdot \mathbf{n} = g$</td>
</tr>
<tr>
<td>$\Gamma_w$, wall</td>
<td>$r = 1, y = -4, y = 4, z = -4 \text{ y } z = 4$</td>
<td>Dirichlet</td>
<td>$w \cdot \mathbf{n} = g$</td>
</tr>
<tr>
<td>$\Gamma_o$, outlet flow</td>
<td>$x = 4$</td>
<td>Neumann</td>
<td>$\frac{\partial w}{\partial \mathbf{n}} = 0$</td>
</tr>
</tbody>
</table>

Table 3: Boundary conditions for the flow around a sphere

- The algorithm is started with the following values for velocity and pressure:

\[
\mathbf{u}(x, y, z)_0 = \begin{cases} 
\mathbf{u}_\infty \mathbf{e}_1 & \text{if } (x, y, z) \in \Gamma_i \\
0 & \text{in other case}
\end{cases}
\]

\[
\pi(x, y, z)_0 = \begin{cases} 
\pi_\infty & \text{if } (x, y, z) \in \Gamma_i \\
0 & \text{in other case}
\end{cases}
\]
The mesh M employed

- 318684 tetrahedra,
- 55022 vertices
- 641149 finite volumes

Finite volume mesh section in $z = 0$
Steady state

\[ \frac{\Delta W_{M}^{out}}{\Delta t} := \frac{1}{\Delta t} \| W_{M}^{out} - W_{M}^{out-1} \|_{L^{2}(\Omega)^{3}}, \]

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>time iterations</th>
<th>( \mu^{out} )</th>
<th>CPU time</th>
<th>( \Delta W_{M}^{out}/\Delta t )</th>
<th>( E(w)_{M} )</th>
<th>( E(\pi)_{M} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>25</td>
<td>4.12</td>
<td>00:02:21</td>
<td>5.37E-05</td>
<td>7.46E-4</td>
<td>1.22E-5</td>
</tr>
<tr>
<td>0.3</td>
<td>34</td>
<td>3.09</td>
<td>00:03:03</td>
<td>4.33E-05</td>
<td>7.16E-4</td>
<td>1.17E-5</td>
</tr>
<tr>
<td>0.2</td>
<td>50</td>
<td>2.06</td>
<td>00:04:06</td>
<td>3.78E-05</td>
<td>6.93E-4</td>
<td>1.14E-5</td>
</tr>
<tr>
<td>0.1</td>
<td>100</td>
<td>1.03</td>
<td>00:07:42</td>
<td>3.33E-05</td>
<td>6.79E-4</td>
<td>1.12E-5</td>
</tr>
<tr>
<td>0.05</td>
<td>200</td>
<td>0.51</td>
<td>00:14:48</td>
<td>3.19E-05</td>
<td>6.75E-4</td>
<td>1.11E-5</td>
</tr>
</tbody>
</table>

- \( \Delta t = 0.45 \) the numerical method is unstable
- The computer code was written in Fortran 90 and ran on a node of a cluster with a processor Intel Xeon CPU X5570@2.93GHz
2. Euler flow around a sphere

Streamlines and pressure contours on the sphere

$\Delta t=0.2$ at $t=10$
2. Euler flow around a sphere

\[ w \]

\[ \pi \]

\[ Z=0 \Delta t=0.2 at t=10 \]
Numerical Results

Academic test for Navier-Stokes equations

- Exact solution:
  \[
  \mathbf{w}(x, y, z, t) = (\sin(\pi t y), -\cos(\pi t z), e^{-\pi x t})^T, \\
  \pi(x, y, z, t) = \cos(\pi t(x + y + z)) \\
  \rho(x, y, z, t) = 1
  \]

  The corresponding source term \( f \) is:
  \[
  f(x, y, z, t) := \begin{pmatrix} 
  \pi y \cos(\pi t y) - \pi t \cos(\pi t z) \cos(\pi t y) - \pi t \sin(\pi t(x + y + z)) + \nu \pi^2 t^2 \sin(\pi t y) \\
  \pi z \sin(\pi t z) + \pi t \sin(\pi t z)e^{-\pi t x} - \pi t \sin(\pi t(x + y + z)) - \nu \pi^2 t^2 \cos(\pi t z) \\
  -\pi x e^{-\pi t x} - \pi t \sin(\pi t y)e^{-\pi t x} - \pi t \sin(\pi t(x + y + z)) - \nu \pi^2 t^2 \cos(\pi t x)
  \end{pmatrix}
  \]

- Features of the meshes

<table>
<thead>
<tr>
<th>Mesh</th>
<th>( N = 1/h )</th>
<th>Elements (( nel ))</th>
<th>Vertices (( M ))</th>
<th>Nodes</th>
<th>( V_h^m ) (( m^3 ))</th>
<th>( V_h^M ) (( m^3 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M1 )</td>
<td>4</td>
<td>384</td>
<td>125</td>
<td>864</td>
<td>6.51E-04</td>
<td>1.30E-03</td>
</tr>
<tr>
<td>( M2 )</td>
<td>8</td>
<td>3072</td>
<td>729</td>
<td>6528</td>
<td>8.14E-05</td>
<td>1.63E-04</td>
</tr>
<tr>
<td>( M3 )</td>
<td>16</td>
<td>24576</td>
<td>4913</td>
<td>50688</td>
<td>1.02E-05</td>
<td>2.03E-05</td>
</tr>
</tbody>
</table>

- To check the order of the error we compute the norms
  \[
  E(\mathbf{w}) = \| \mathbf{w} - \mathbf{w}_h \|_{L^2(L^2(\Omega))^3}, \quad E(\pi) = \| \pi - \pi_h \|_{L^2(L^2(\Omega))},
  \]
Errors $E(w)$ (left) and $E(\pi)$ (right) obtained setting the time step to $t = 4.88E-4$
The explicit discretization

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$E(w)<em>{M1}/E(w)</em>{M2}$</th>
<th>$E(w)<em>{M2}/E(w)</em>{M3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.95E-03</td>
<td>2.18E+00</td>
<td>2.12E+00</td>
</tr>
<tr>
<td>9.77E-04</td>
<td>2.18E+00</td>
<td>2.12E+00</td>
</tr>
<tr>
<td>4.88E-04</td>
<td>2.18E+00</td>
<td>2.12E+00</td>
</tr>
</tbody>
</table>

Table 7: Ratios of spatial errors $E(w)_{M1}/E(w)_{M2}$ and $E(w)_{M2}/E(w)_{M3}$.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$E(\pi)_{M1}$</th>
<th>$E(\pi)_{M2}$</th>
<th>$E(\pi)_{M3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.95E-03</td>
<td>8.55E-02</td>
<td>3.92E-02</td>
<td>2.11E-02</td>
</tr>
<tr>
<td>9.77E-04</td>
<td>8.60E-02</td>
<td>3.97E-02</td>
<td>2.15E-02</td>
</tr>
<tr>
<td>4.88E-04</td>
<td>8.62E-02</td>
<td>4.00E-02</td>
<td>2.18E-02</td>
</tr>
</tbody>
</table>

Table 8: Errors $E(\pi)_{M_k} = \| \pi - \pi_{M_k} \|_{L^2(\Omega)}$, $k = 1, 2, 3$.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$E(\pi)<em>{M1}/E(\pi)</em>{M2}$</th>
<th>$E(\pi)<em>{M2}/E(\pi)</em>{M3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.95E-03</td>
<td>2.18E+00</td>
<td>1.85E+00</td>
</tr>
<tr>
<td>9.77E-04</td>
<td>2.16E+00</td>
<td>1.85E+00</td>
</tr>
<tr>
<td>4.88E-04</td>
<td>2.15E+00</td>
<td>1.84E+00</td>
</tr>
</tbody>
</table>

Table 9: Ratios of spatial errors $E(\pi)_{M1}/E(\pi)_{M2}$ and $E(\pi)_{M2}/E(\pi)_{M3}$.
The semi-implicit discretization

\[
\begin{array}{|c|c|c|c|}
\hline
\Delta t & E(w)_{M_1} & E(w)_{M_2} & E(w)_{M_3} \\
\hline
1.95E-03 & 4.67E-02 & 2.33E-02 & 3.56E-02 \\
9.77E-04 & 4.66E-02 & 2.20E-02 & 2.11E-02 \\
4.88E-04 & 4.66E-02 & 2.16E-02 & 1.42E-02 \\
\hline
\end{array}
\]

Table 10: Errors $E(w)_{M_k} = \|w - w_{M_k}\|_{L^2(L^2(\Omega))}$, $k = 1, 2, 3$.

\[
\begin{array}{|c|c|c|}
\hline
\Delta t & E(w)_{M_1}/E(w)_{M_2} & E(w)_{M_2}/E(w)_{M_3} \\
\hline
1.95E-03 & 2.01E+00 & 6.54E-01 \\
9.77E-04 & 2.12E+00 & 1.04E+00 \\
4.88E-04 & 2.16E+00 & 1.52E+00 \\
\hline
\end{array}
\]

Table 11: Ratios of spatial errors $E(w)_{M_1}/E(w)_{M_2}$ and $E(w)_{M_2}/E(w)_{M_3}$. 
The semi-implicit discretization

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$E(\pi)_{M_1}$</th>
<th>$E(\pi)_{M_2}$</th>
<th>$E(\pi)_{M_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.95E-03</td>
<td>8.51E-02</td>
<td>5.32E-02</td>
<td>3.56E-02</td>
</tr>
<tr>
<td>9.77E-04</td>
<td>8.56E-02</td>
<td>4.32E-02</td>
<td>7.75E-02</td>
</tr>
<tr>
<td>4.88E-04</td>
<td>8.60E-02</td>
<td>4.07E-02</td>
<td>4.28E-02</td>
</tr>
</tbody>
</table>

Table 12: Errors $E(\pi)_{M_k} = ||\pi - \pi_{M_k}||_{L^2(L^2(\Omega))}$, $k = 1, 2, 3$.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$E(\pi)<em>{M_1}/E(\pi)</em>{M_2}$</th>
<th>$E(\pi)<em>{M_2}/E(\pi)</em>{M_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.95E-03</td>
<td>1.60E+00</td>
<td>1.49E+00</td>
</tr>
<tr>
<td>9.77E-04</td>
<td>1.98E+00</td>
<td>5.57E-01</td>
</tr>
<tr>
<td>4.88E-04</td>
<td>2.11E+00</td>
<td>9.50E-01</td>
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</tbody>
</table>

Table 13: Ratios of spatial errors $E(\pi)_{M_1}/E(\pi)_{M_2}$ and $E(\pi)_{M_2}/E(\pi)_{M_3}$. 
<table>
<thead>
<tr>
<th>Explicito-Cea</th>
<th>CFL VELVIS</th>
<th>Error M1</th>
<th>last dt</th>
<th>Error M2</th>
<th>last dt</th>
<th>Error M3</th>
<th>last dt</th>
<th>ErrorM1/ errorM2</th>
<th>ErrorM2/errorM3</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Err t final</td>
<td>7.00E+00</td>
<td>8.42E-02</td>
<td>1.63E-02</td>
<td>3.94E-02</td>
<td>5.02E-03</td>
<td>2.00E-02</td>
<td>1.89E-03</td>
<td>2.14E+00</td>
<td>1.97E+00</td>
<td>1.095</td>
</tr>
<tr>
<td>Err t final</td>
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<td>8.42E-02</td>
<td>1.17E-02</td>
<td>3.95E-02</td>
<td>4.14E-03</td>
<td>2.00E-02</td>
<td>2.13E+00</td>
<td>1.97E+00</td>
<td>1.97E+00</td>
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<td>8.42E-02</td>
<td>9.32E-03</td>
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<td>1.97E+00</td>
<td>1.092</td>
</tr>
<tr>
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<td>2.00E-02</td>
<td>8.09E-04</td>
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<td>1.97E+00</td>
<td>1.091</td>
</tr>
<tr>
<td>Err t final</td>
<td>2.00E+00</td>
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<td>4.67E-03</td>
<td>3.96E-02</td>
<td>1.66E-03</td>
<td>2.01E-02</td>
<td>5.39E-04</td>
<td>2.13E+00</td>
<td>1.97E+00</td>
<td>1.090</td>
</tr>
<tr>
<td>Err l2 tiempo</td>
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<td>1.01E-02</td>
<td>5.39E-04</td>
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</table>

<table>
<thead>
<tr>
<th>Explicito-Cea</th>
<th>CFL VELVIS</th>
<th>Error M1</th>
<th>Error M2</th>
<th>Error M3</th>
<th>ErrorM1/ errorM2</th>
<th>ErrorM2/errorM3</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Err t final</td>
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<tr>
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<td>1.78E-01</td>
<td>8.04E-02</td>
<td>4.36E-02</td>
<td>2.21E+00</td>
<td>1.84E+00</td>
<td>1.145</td>
</tr>
<tr>
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<td>8.73E-02</td>
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<tr>
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<td>1.73E-01</td>
<td>7.84E-02</td>
<td>4.39E-02</td>
<td>2.21E+00</td>
<td>1.78E+00</td>
<td>1.146</td>
</tr>
<tr>
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<td>1.74E-01</td>
<td>8.12E-02</td>
<td>4.39E-02</td>
<td>2.15E+00</td>
<td>1.85E+00</td>
<td>1.103</td>
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<tr>
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<td>8.46E-02</td>
<td>3.94E-02</td>
<td>2.17E-02</td>
<td>2.15E+00</td>
<td>1.81E+00</td>
<td>1.104</td>
</tr>
</tbody>
</table>
### Orden2 Cea-Vc convectivo

**Explicito-Ceva Viscosidad**

#### Velocidad

<table>
<thead>
<tr>
<th>CFL VELVIS</th>
<th>Error M1</th>
<th>Last dt</th>
<th>Error M2</th>
<th>Last dt</th>
<th>Error M3</th>
<th>Last dt</th>
<th>Error M1/errorM2</th>
<th>ErrorM2/errorM3</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Err t final</td>
<td>7.00E+00</td>
<td>9.15E-02</td>
<td>1.63E-02</td>
<td>3.15E-02</td>
<td>5.81E-03</td>
<td>1.25E-02</td>
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<td>2.91E+00</td>
<td>2.52E+00</td>
</tr>
<tr>
<td>Err L2 tiempo</td>
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<td>1.84E-02</td>
<td>7.10E-03</td>
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<td>2.003</td>
<td>1.377</td>
<td></td>
</tr>
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</table>

#### Presión

<table>
<thead>
<tr>
<th>CFL VELVIS</th>
<th>Error M1</th>
<th>Error M2</th>
<th>Error M3</th>
<th>ErrorM1/errorM2</th>
<th>ErrorM2/errorM3</th>
<th>Order</th>
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<tbody>
<tr>
<td>Err t final</td>
<td>7.00E+00</td>
<td>1.28E-01</td>
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<td>1.19E-02</td>
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<td>1.78E-02</td>
<td>6.56E-03</td>
<td>3.67E+00</td>
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### Orden2 convectivo + viscosidad

**Explicito-Ceva Viscosidad**

#### Velocidad

<table>
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<tr>
<th>CFL VELVIS</th>
<th>Error M1</th>
<th>Last dt</th>
<th>Error M2</th>
<th>Last dt</th>
<th>Error M3</th>
<th>Last dt</th>
<th>Error M1/errorM2</th>
<th>ErrorM2/errorM3</th>
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#### Presión

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<th>DT VV tfinal-1</th>
<th>Error M3</th>
<th>Last dt</th>
<th>ErrorM1/errorM2</th>
<th>ErrorM2/errorM3</th>
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### Presión

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<th>ErrorM2/errorM3</th>
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<td>2.43E+00</td>
</tr>
</tbody>
</table>
3. Flow around a cylinder


- The computational domain consists of a solid cylinder surrounded by a rectangular channel in which the flow evolves.

- The kinematic viscosity of the fluid is \( \nu = 10^{-3} m^2/s \) and the inlet velocity has the form

\[
\mathbf{u}(x, y, z, t) = \left(16Uy(z-H)H-H^4, 0, 0\right)^t,
\]

with \( U = 0.45 m/s, H = 0.41 m \).

- Based on \( \nu \), the cylinder diameter and an estimate of 0.2\( m/s \) for the mean inflow velocity, the flow has a Reynolds number of 20.

- The mesh employed to obtain the numerical solution consists of 449,746 finite elements and 992,280 finite volumes.
Approximated $w_x$ at $z=0$

Approximated pressure at $z=0$
Streamlines
• The drag and lift coefficients:

\[ c_d = \frac{500}{0.41} F_d, \quad c_l = \frac{500}{0.41} F_l, \]

- Drag force: \( F_d \)
- Lift force: \( F_l \)

• The pressure difference \( \Delta p \) between the points \( (0.45, 0.2, 0.205) \) and \( (0.55, 0.2, 0.205) \).

<table>
<thead>
<tr>
<th>Test case</th>
<th>Time iterations</th>
<th>CPU time</th>
<th>( c_d ) (min, max)</th>
<th>( c_l ) (min, max)</th>
<th>( D\pi ) (min, max)</th>
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</thead>
<tbody>
<tr>
<td>T1 (1st Order)</td>
<td>1841</td>
<td>01:35:33</td>
<td>(6.05, 6.25)</td>
<td>(0.0053)</td>
<td>(0.165, 0.175)</td>
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<tr>
<td>T2 (1st Order)</td>
<td>1745</td>
<td>01:36:20</td>
<td>(6.79)</td>
<td>(0.0062)</td>
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<td>T1 (2nd Order)</td>
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<td>01:35:33</td>
<td>(6.33)</td>
<td>(0.0006)</td>
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<td>T2 (2nd Order)</td>
<td>1745</td>
<td>01:36:20</td>
<td>(6.34)</td>
<td>(0.0015)</td>
<td>(0.16350)</td>
</tr>
</tbody>
</table>

Q-Scheme of van Leer (T1) and Rusanov (T2). 2nd Order Cea-VC method.
\[ F_d = \int_S \left( \rho \nu \frac{\partial \mathbf{u}_r}{\partial n_S} n_y - \pi n_x \right) dS, \quad F_l = \int_S \left( -\rho \nu \frac{\partial \mathbf{u}_r}{\partial n_S} n_x - \pi n_y \right) dS \] (58)

with \( n_S = (n_x, n_y, 0)^t \) the inward pointing unit normal with respect to \( \Omega \), \( S \) is the surface of the cylinder, and \( \mathbf{n}_r = (n_y, -n_x, 0)^t \) is one of the tangential vectors, the other one being \( (0, 0, 1)^t \). The computation of the integrals in (58) is done taking into account the relations

\[
\begin{align*}
\frac{\partial \mathbf{u}_r}{\partial n_S} n_y &= \nabla u_1 \cdot \mathbf{n}_S, \\
\frac{\partial \mathbf{u}_r}{\partial n_S} n_x &= -\nabla u_2 \cdot \mathbf{n}_S,
\end{align*}
\] (59) (60)

obtained from the boundary condition \( \mathbf{u}|_S = 0 \). The approximate value of the the drag and lift forces is given by

\[
(F_d^{n+1}, F_l^{n+1})^t = \sum_{k=1}^{N_{BC}} \rho_k^{n+1} \nu_k^{n+1} \frac{\text{area}(\Gamma_k)}{|\vec{\eta}_k \cdot \mathbf{V}_{kB}|} u^{n+1}_B + \text{area}(\Gamma_k) \pi_k^{n+1} \vec{\eta}_k,
\] (61)
Pressure difference $\Delta \pi$ between points $(0.45, 0.2, 0.205)$ and $(0.55, 0.2, 0.205)$
Low-Mach number equations with transport and energy equations

- The system of equations to be solved read

$$\frac{\partial \mathbf{w}}{\partial t} + \text{div}(\mathcal{F}(\mathbf{w}, \rho)) + \nabla \pi - \text{div}(\tau) = \mathbf{f}(x, y, z, t),$$

$$\text{div} \mathbf{w} = q,$$

$$\frac{\partial \mathbf{\Delta w}}{\partial t} + \text{div}(\mathcal{F}(\mathbf{\Delta w}, \mathbf{w}, \rho)) - \text{div} \left( \rho \mathcal{D} \text{grad} \frac{\mathbf{\Delta w}}{\rho} \right) = \mathbf{f}(x, y, z, t),$$

- The flux $\mathcal{F}$ and $\mathbf{\Delta F}$ are given by:

$$\mathcal{F}_i(\mathbf{w}, \rho) = \frac{w_i}{\rho} \mathbf{w}, \ i = 1, 2, 3.$$  

$$\mathbf{\Delta F}_i(\mathbf{\Delta w}, \mathbf{w}, \rho) = \frac{w_i \mathbf{\Delta w}}{\rho} \mathbf{w}, \ i = 1, ..., N + 1$$

Spatial dimension: 3 → $\mathbf{w} = (w_1, w_2, w_3)^T$

Number of species: $N$ → $\mathbf{\Delta w}_i = \rho Y_i, \ i = 1, ..., N$

Energy equation: 1 → $\mathbf{\Delta w}_{N+1} = \rho h,$
• We define a “generalized conservative variable”, $\mathbf{w}^g$, as a combination of the previous conservative variables vector $\mathbf{w}$ and the new vector $\hat{\mathbf{w}}$, so that

$$\mathbf{w}^g = \left( \begin{array}{c} \mathbf{w} \\ \hat{\mathbf{w}} \end{array} \right)_{3+N+1}$$

• We define a “generalized conservative variable”, $\mathcal{F}^g$

$$\mathcal{F}^g(\mathbf{w}^g, \rho) = \frac{1}{\rho} \begin{pmatrix} w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 \end{pmatrix}_{(3+N+1) \times 3}$$

• More precisely, the expression for this tensor is as follows:

$$\mathcal{F}^g(\mathbf{w}^g, \rho) = \frac{1}{\rho} \begin{pmatrix} w_1 & w_1 \triangle & w_1 \triangle & w_1 \triangle \\ \vdots & \vdots & \vdots \\ w_1 \triangle & w_N \triangle & w_N \triangle & w_N \triangle \\ w_1 \triangle & w_{N+1} \triangle & w_{N+1} \triangle & w_{N+1} \triangle \end{pmatrix}_{(3+N+1) \times 3}$$
Test with a transport equation: Gaussian sphere

- The computational domain of the flow, consisting in a cube of dimensions \( \Omega = (-0.9, 0.9) \times (-0.9, 0.9) \times (-0.3, 0.3) \).
- The analytic solution:

\[
\begin{align*}
\mathbf{u}(x, y, z, t) &= (-y, x, 0) \\
\pi(x, y, z, t) &= 1 \\
\rho(x, y, z, t) &= 1 \\
f(x, y, z, t) &= (-x, -y, 0)
\end{align*}
\]

\[
Y(x, y, t) = \left(\frac{\sigma_0}{\sqrt{\sigma_0^2 + 2\nu t}}\right)^3 \exp \left\{-\frac{(\bar{x}(t) + 0.25)^2 + (\bar{y}(t))^2 + (\bar{z}(t))^2}{2(\sigma_0^2 + 2\nu t)}\right\}
\]

\[
\begin{align*}
\bar{x}(t) &= x \cos t + y \sin t, \\
\bar{y}(t) &= -x \sin t + y \cos t, \\
\bar{z}(t) &= z \\
\sigma_0 &= 0.08,
\end{align*}
\]

L. Saavedra (2012)
Conclusions

• The transport-diffusion stage is done by a time-explicit face finite volume discretization on a unstructured mesh, upwinded by classical approximate Riemann solvers.
• The projection stage consists of boundary-value problem for the Laplace operator that is solved by a standard finite element method.
• The use of face-type finite volumes facilitates the imposition of flux boundary conditions and leads to a staggered-like spatial discretization of velocity and pressure.
• In the case of steady-state problems a semi-implicit discretization is also proposed for the viscous term allowing for larger time steps than those needed for the fully explicit scheme.
• The method is applied to several different test problems allowing us to assess its accuracy. In general, first order is achieved, both in time and in space.
• Since the transport-diffusion stage is explicit in time and the matrix for the projection stage can be factorized only once because it is time independent, the overall method is cheap. Moreover, the transport-diffusion stage can be easily parallelized leading to a considerable improvement in the computational efficiency.
Obrigada!

elena.vazquez-cendon@usc.es