A locally refined-staggered scheme for the variable density Navier-Stokes equations

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Outline

- Context and motivations.
- Variable density Navier-Stokes equations.
- General form of the discretization.
- Discrete kinetic energy balance.
- Non-conforming local refinement.
- Numerical applications.
- Error analysis.
- A convergence result.
- Conclusion and Perspectives.

Context and motivations

A CFD code for "industrial" purpose:

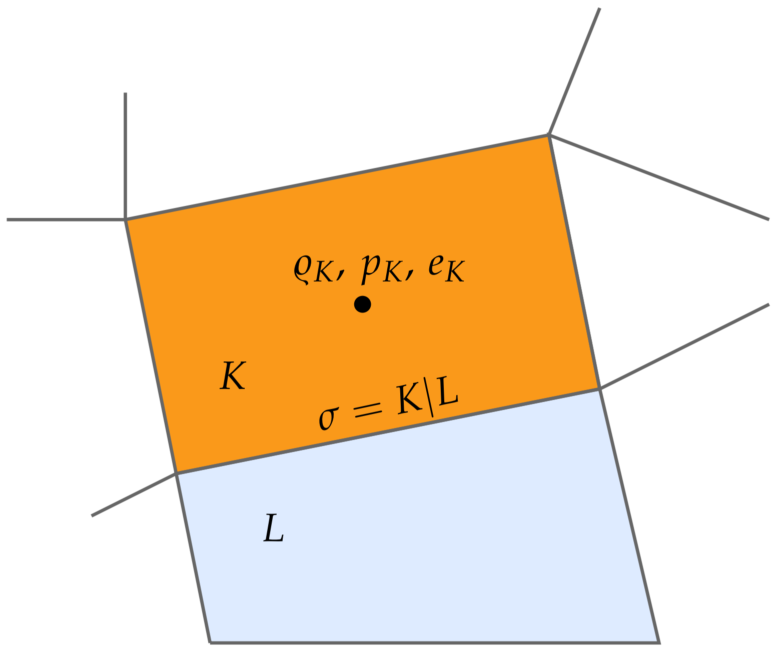
- Preserve some stability, (as far as possible) independently of the space and time steps,
- Accurate at any Mach number ?

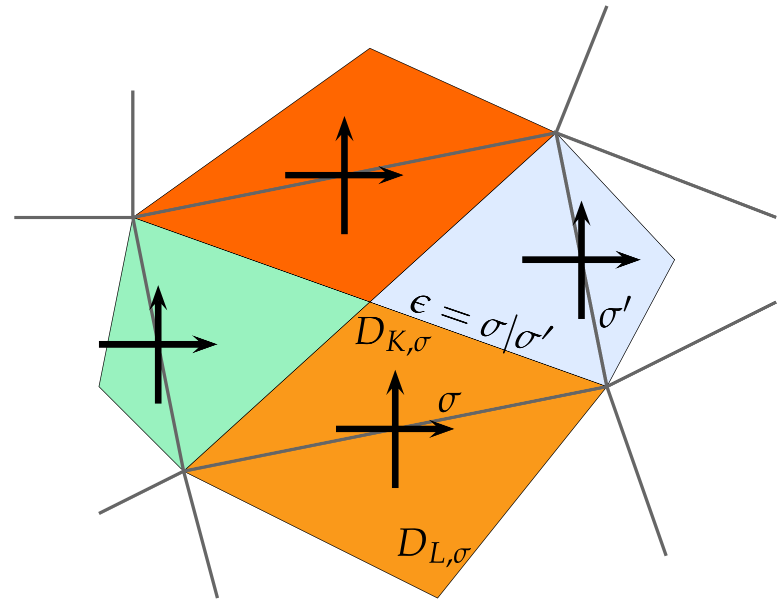
△ **Time discretization with a fractional step algorithm, pressure correction method.**

△ **Staggered space discretization (but not structured mesh).**

Schemes described hereafter are implemented in the free software ISIS, based on the software component library and framework PELICANS.

Space discretization





- Primal mesh : $\mathcal{M}$ = { set of control volumes K}.
- Scalar variables defined at cell centers: $(p_K)_{K\in\mathcal{M}}$, $(\varrho_K)_{K\in\mathcal{M}}$,$(\vartheta_K)_{K\in\mathcal{M}}$, ...
- Velocity components defined at the (some) edges : $(v_{\sigma,i})_{\sigma\in\mathcal{E}^{(i)}}$.
- Dual mesh(es) : $(D_\sigma)_{\sigma\in\mathcal{E}^{(i)}}$.

Variable density Navier-Stokes equations

On $(0,T)\times\Omega$ where Ω is a bounded connected domain of $\mathbb{R}^d$, $d\in\{2,3\}$:

$$\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u}) = 0,$$
$$\partial_t(\rho\boldsymbol{u}) + \mathrm{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \mathrm{div}(\boldsymbol{\tau}(\boldsymbol{u})) + \boldsymbol{\nabla} p = 0.$$

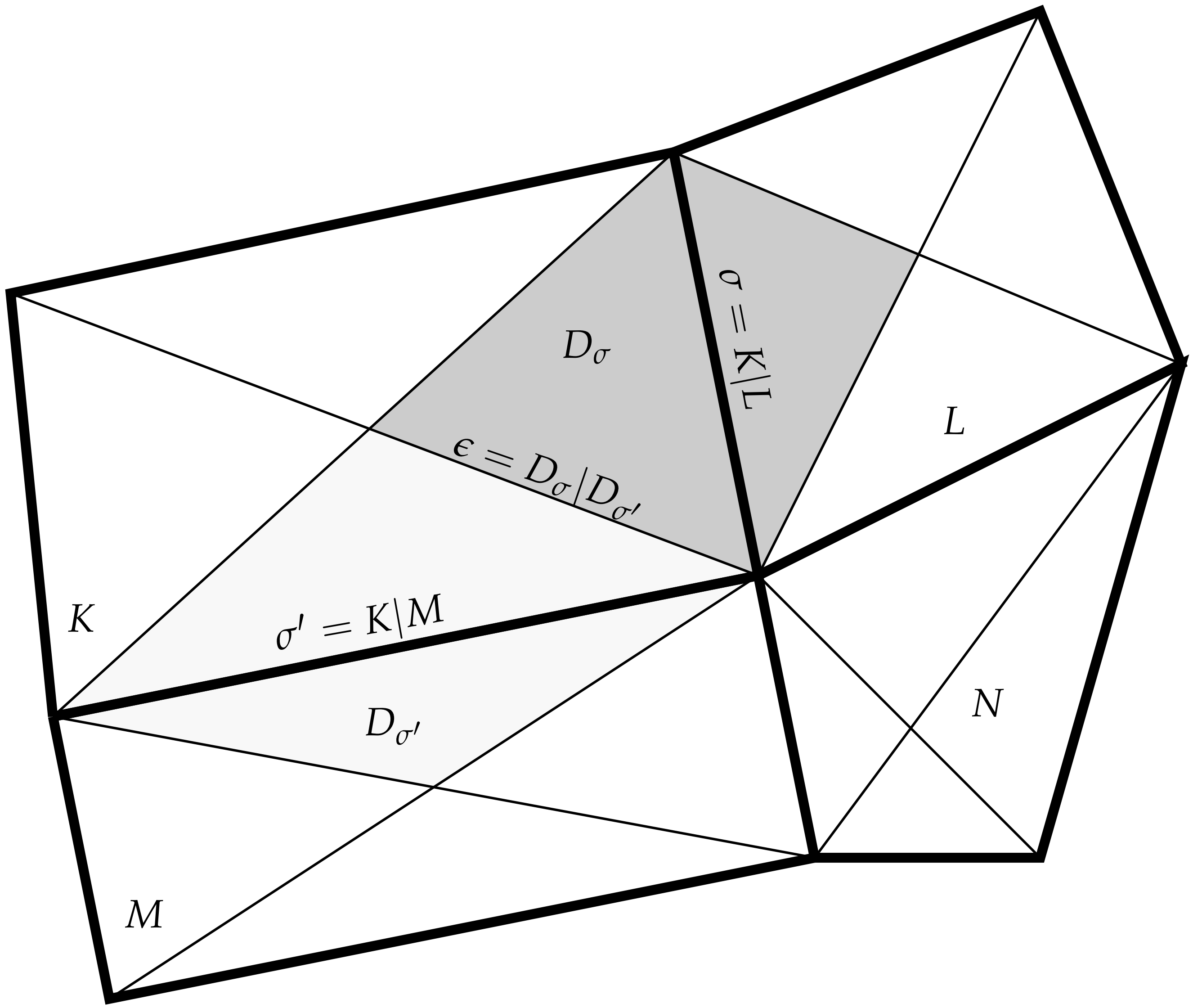
where :

- $\boldsymbol{\tau}(\boldsymbol{u}) = \mu(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^t\boldsymbol{u}) - \frac{2\mu}{3}\,\mathrm{div}\boldsymbol{u}\, I,$ where μ is a positive parameter, possibly depending on $\boldsymbol{x}$,
- $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_{\partial\Omega},\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0,$
- The density ρ is assumed to be a given positive known function, approximated by a piecewise constant function $\rho = \sum_K \rho_K 1_K$.

Kinetic energy balance : (Momentum balance)$\cdot\boldsymbol{u}$ & mass balance:

$$\partial_t(\frac{1}{2}\rho|\boldsymbol{u}|^2) + \mathrm{div}(\frac{1}{2}\rho|\boldsymbol{u}|^2\boldsymbol{u}) - \mathrm{div}(\boldsymbol{\tau}(\boldsymbol{u}))\cdot\boldsymbol{u} + \boldsymbol{\nabla} p\cdot\boldsymbol{u} = 0.$$

General form of the discretization



General form of the discretization

Implicit scheme :

$$\frac{1}{\delta t}(\rho_K - \rho_K^\star) + \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = 0, \qquad K \in \mathcal{M},$$

$$\frac{1}{\delta t}(\rho_\sigma \boldsymbol{u}_\sigma - \rho_\sigma^\star \boldsymbol{u}_\sigma^\star) + \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} \boldsymbol{u}_\epsilon + (T_{\text{diff}}(\boldsymbol{u}))_\sigma + (\boldsymbol{\nabla} p)_\sigma = 0, \quad \sigma \in \mathcal{E}_{\text{int}}.$$

with :

$$F_{K,\sigma} = |\sigma| \, \hat{\rho}_\sigma \, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}, \quad \hat{\rho}_\sigma = \frac{\rho_K + \rho_L}{2},$$

$$(\boldsymbol{\nabla} p)_\sigma = \frac{|\sigma|}{|D_\sigma|}(p_L - p_K)\boldsymbol{n}_{K,\sigma},$$

$(T_{\text{diff}}(\boldsymbol{u}))_\sigma$ discretized by the Rannacher-Turek finite element.

The dual densities ρ_σ and dual fluxes $F_{\sigma,\epsilon}$ are built so that a discrete kinetic energy holds.

Second order centered discretization of the velocity on the dual faces: $\boldsymbol{u}_\epsilon = \frac{\boldsymbol{u}_\sigma + \boldsymbol{u}_{\sigma'}}{2}$.

General form of the discretization

Rannacher-Turek finite element for the diffusion :

Case of the Stokes problem :

$$\begin{aligned} &-\boldsymbol{\Delta} \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f}, && \Omega \\ &\operatorname{div} \boldsymbol{u} = 0, && \Omega, \\ &\boldsymbol{u} = 0, && \partial\Omega. \end{aligned}$$

Discrete problem :

- $\widetilde{\mathbb{Q}}_1(K) = \operatorname{span}\{1, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}^2 - \boldsymbol{x}_{i+1}^2, 1 \leqslant i \leqslant d-1\}$
- Discrete spaces :

$$\boldsymbol{v} \in V_h \Longleftrightarrow \begin{cases} \boldsymbol{v}|_K \in \widetilde{\mathbb{Q}}_1(K), & \forall K \in \mathcal{M} \\ \displaystyle\int_\sigma [v] = 0, & \forall \sigma \in \mathcal{E}_{\text{int}}. \end{cases} \qquad q \in Q_h \Longleftrightarrow q|_K = cst.$$

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- Find $(\boldsymbol{u}_h, p_h)$ in $V_h^d \times Q_h$ such that for all $(\boldsymbol{v}_h, q_h)$ in $V_h^d \times Q_h$:

$$\sum_K \int_K \boldsymbol{\nabla u}_h : \boldsymbol{\nabla v}_h + \sum_K \int_K p_h \mathrm{div}\boldsymbol{v}_h = \int_\Omega f \cdot v_h,$$
$$\sum_K \int_K q_h \mathrm{div}\boldsymbol{u}_h = 0$$

General form of the discretization

Properties of the scheme :

- Gradient-divergence duality :

$$\sum_{K\in\mathcal{M}} |K|\; p_K\; (\mathrm{div}\boldsymbol{v})_K + \sum_{\sigma\in\mathcal{E}} |D_\sigma|\; \boldsymbol{v}_\sigma \cdot (\boldsymbol{\nabla} p)_\sigma = 0, \qquad \text{for all } (p,v) \in Q_h \times V_h^d.$$

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- Positivity/coercivity of the diffusion term (Rannacher-Turek FE) :

$$\sum_{\sigma\in\mathcal{E}} |D_\sigma|\; (T_{\mathrm{diff}}(\boldsymbol{v}))_\sigma \cdot \boldsymbol{v}_\sigma \geqslant \|\boldsymbol{v}\|_{1,h}^2, \qquad \text{for all } v \in V_h^d.$$

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- LBB (inf-sup) condition (*via* a Fortin interpolation operator) :

$$\inf_{p\in Q_h} \sup_{\boldsymbol{u}\in V_h^d} \frac{\displaystyle\int_\Omega p\,\mathrm{div}_h u}{\|p\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{u}\|_{1,h}} \geqslant \beta.$$

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- Discrete kinetic energy balance :

$$\frac{1}{2\delta t}\Big(\rho_\sigma\,|\boldsymbol{u}_\sigma|^2 - \rho_\sigma^\star\,|\boldsymbol{u}_\sigma^\star|^2\Big) + \frac{1}{2|D_\sigma|} \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon=D_\sigma|D_{\sigma'}}} F_{\sigma,\epsilon}\; \boldsymbol{u}_\sigma\cdot\boldsymbol{u}_{\sigma'} - (T_{\mathrm{diff}}(\boldsymbol{u}))_\sigma\cdot\boldsymbol{u}_\sigma + (\boldsymbol{\nabla} p)_\sigma\cdot\boldsymbol{u}_\sigma \leqslant 0.$$

Discrete kinetic energy balance

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

$$\partial_t (\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}(\boldsymbol{\tau}(\boldsymbol{u})) + \boldsymbol{\nabla} p = 0.$$

$$\Downarrow$$

$$\rho \left(\partial_t \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{u} . \boldsymbol{u}\right) - \operatorname{div}(\boldsymbol{\tau}(\boldsymbol{u})) + \boldsymbol{\nabla} p = 0.$$

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$$\boldsymbol{u} \cdot \Big\{ \rho \left(\partial_t \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{u} . \boldsymbol{u} \right) - \mathrm{div}(\boldsymbol{\tau}(\boldsymbol{u})) + \boldsymbol{\nabla} p = 0. \Big\}$$

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$$\rho \left(\partial_t \frac{|\boldsymbol{u}|^2}{2} + \boldsymbol{\nabla} \frac{|\boldsymbol{u}|^2}{2} \right) - \operatorname{div}(\boldsymbol{\tau}(\boldsymbol{u})) \cdot \boldsymbol{u} + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = 0.$$

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- The mass balance equation was used twice.

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- The mass balance equation was used twice.
- The mass and momentum balances are not discretized at the same locations !

$$\frac{|K|}{\delta t}(\rho_K - \rho_K^\star) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = 0, \qquad K \in \mathcal{M},$$

$$\frac{1}{\delta t}(\rho_\sigma \boldsymbol{u}_\sigma - \rho_\sigma^\star \boldsymbol{u}_\sigma^\star) + \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} \boldsymbol{u}_\epsilon - (T_{\text{diff}}(\boldsymbol{u}))_\sigma + (\boldsymbol{\nabla} p)_\sigma = 0, \quad \sigma \in \mathcal{E}_{\text{int}}.$$

Discrete kinetic energy balance

Theorem (Herbin, Kheriji, Latché) :

Assume that a discrete mass balance holds over dual mesh :

$$\frac{|D_\sigma|}{\delta t}(\rho_\sigma - \rho^\star_\sigma) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon} = 0, \qquad \forall \sigma \in \mathcal{E}_{\text{int}}.$$

Then a discrete kinetic energy balance holds :

$$\frac{1}{2\delta t}\Big(\rho_\sigma\,|\boldsymbol{u}_\sigma|^2 - \rho^\star_\sigma\,|\boldsymbol{u}^\star_\sigma|^2\Big) + \frac{1}{2|D_\sigma|}\sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) \\ \epsilon = D_\sigma|D_{\sigma'}}} F_{\sigma,\epsilon}\;\boldsymbol{u}_\sigma \cdot \boldsymbol{u}_{\sigma'} - (T_{\text{diff}}(\boldsymbol{u}))_\sigma \cdot \boldsymbol{u}_\sigma + (\boldsymbol{\nabla} p)_\sigma \cdot \boldsymbol{u}_\sigma \leqslant 0,$$

+ summing over $\sigma \in \mathcal{E}$ and $k = 1,..,n$:

$$\frac{1}{2}\sum_{\sigma \in \mathcal{E}} |D_\sigma|\rho^n_\sigma\,|\boldsymbol{u}^n_\sigma|^2 + \sum_{k=1}^{\star} \delta t \|\boldsymbol{u}^k\|^2_{1,h} \leqslant \frac{1}{2}\sum_{\sigma \in \mathcal{E}} |D_\sigma|\rho^0_\sigma\,|\boldsymbol{u}^0_\sigma|^2, \qquad \forall n \in \mathbb{N}.$$

Discrete kinetic energy balance

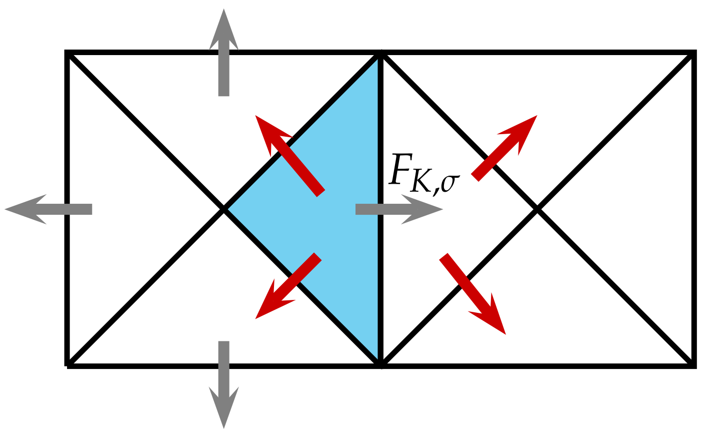
We want :

$$\frac{|D_\sigma|}{\delta t}(\rho_\sigma - \rho_\sigma^\star) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon} = 0.$$

We have :

$$\frac{|K|}{\delta t}(\rho_K - \rho_K^\star) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = 0,$$

$$\frac{|L|}{\delta t}(\rho_L - \rho_L^\star) + \sum_{\sigma \in \mathcal{E}(L)} F_{L,\sigma} = 0.$$



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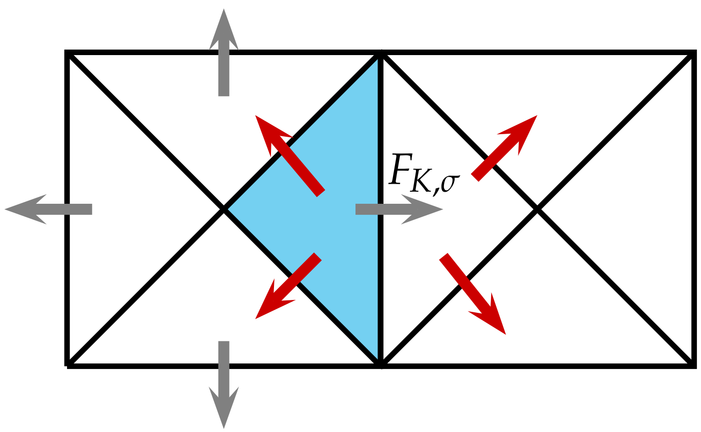

Let $\boldsymbol{w}$ such that $\mathrm{div}\boldsymbol{w} = cst$ and for all face σ of K:

$$\int_\sigma \boldsymbol{w}\cdot\boldsymbol{n}_{K,\sigma} = F_{K,\sigma}$$ (lifting of the primal fluxes).

Then one defines : $F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w}\cdot\boldsymbol{n}_{\sigma,\epsilon}$.

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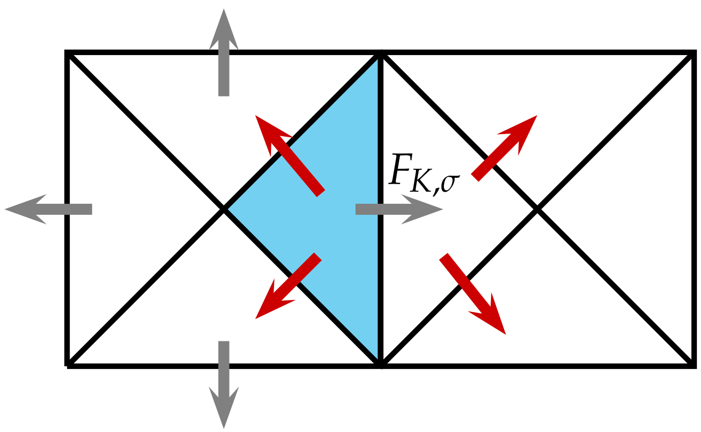

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Then one defines : $F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w} \cdot \boldsymbol{n}_{\sigma,\epsilon}$.

$$F_{K,\sigma} + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon} = \int_{\partial D_{K,\sigma}} \boldsymbol{w} \cdot \boldsymbol{n} = \int_{D_{K,\sigma}} \mathrm{div} \boldsymbol{w} = \frac{|D_{K,\sigma}|}{|K|} \int_K \mathrm{div} \boldsymbol{w} = \frac{|D_{K,\sigma}|}{|K|} \int_{\partial K} \boldsymbol{w} \cdot \boldsymbol{n}$$

$$= \frac{|D_{K,\sigma}|}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = -\frac{|D_{K,\sigma}|}{|K|} \frac{|K|}{\delta t}(\rho_K - \rho_K^\star)$$

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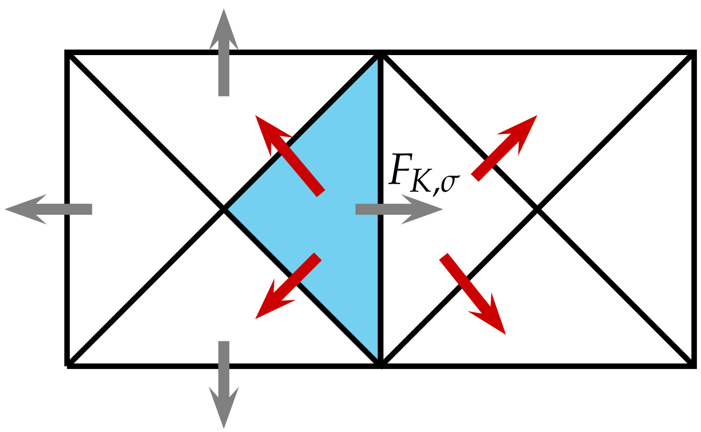

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Then one defines : $F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w} \cdot \boldsymbol{n}_{\sigma,\epsilon}$.

$$\frac{|D_{K,\sigma}|}{\delta t}(\rho_K - \rho_K^\star) + F_{K,\sigma} + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon} = 0,$$

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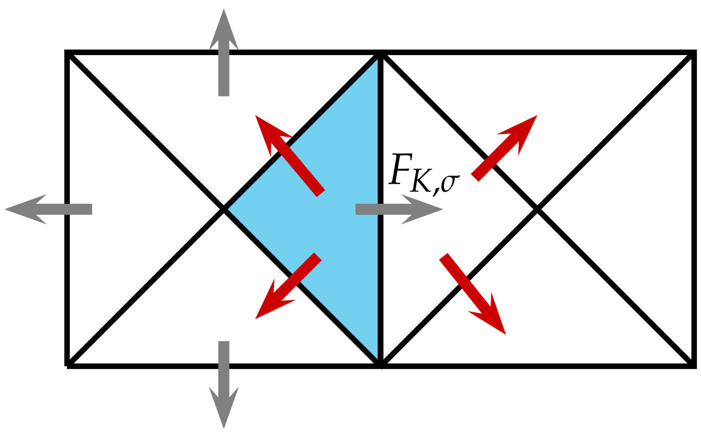

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$$\frac{|D_{K,\sigma}|}{\delta t}(\rho_K - \rho_K^\star) + F_{K,\sigma} + \sum_{\substack{\epsilon\in\mathcal{E}(D_\sigma)\\ \epsilon\subset K}} F_{\sigma,\epsilon} = 0,$$ and $$\frac{|D_{L,\sigma}|}{\delta t}(\rho_L - \rho_L^\star) + F_{L,\sigma} + \sum_{\substack{\epsilon\in\mathcal{E}(D_\sigma)\\ \epsilon\subset L}} F_{\sigma,\epsilon} = 0.$$

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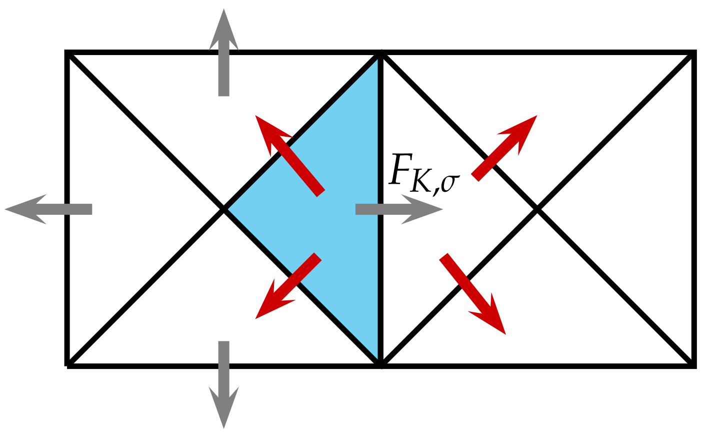


Let $\boldsymbol{w}$ such that $\mathrm{div}\boldsymbol{w} = cst$ and for all face σ of K:

$$\int_\sigma \boldsymbol{w} \cdot \boldsymbol{n}_{K,\sigma} = F_{K,\sigma} \quad \text{(lifting of the primal fluxes).}$$

Then one defines : $F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w} \cdot \boldsymbol{n}_{\sigma,\epsilon}$.

$$\frac{1}{\delta t}\Big(\underbrace{|D_{K,\sigma}|\rho_K + |D_{L,\sigma}|\rho_L}_{=:|D_\sigma|\rho_\sigma} - |D_{K,\sigma}|\rho_K^\star - |D_{L,\sigma}|\rho_L^\star\Big) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon} = 0.$$

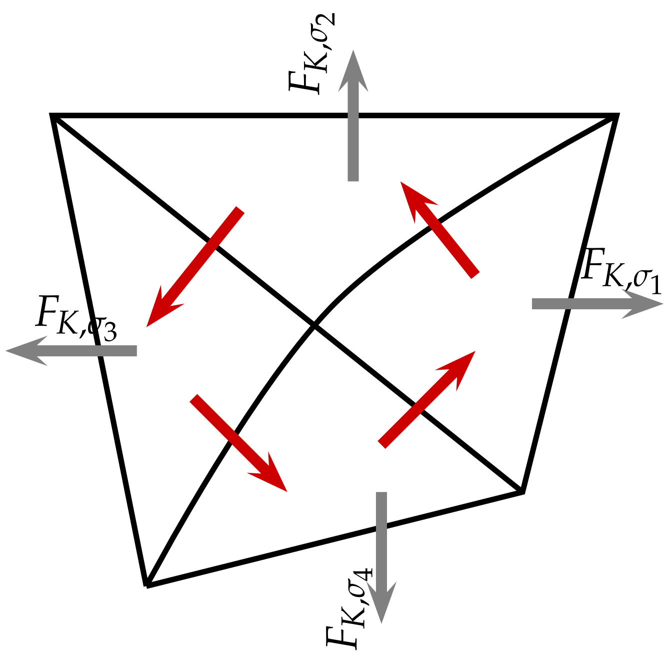
Discrete kinetic energy balance

A construction which does not depend on the element K :

$$F_{K,\sigma} + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon} = \int_{\partial D_{K,\sigma}} \boldsymbol{w} \cdot \boldsymbol{n} = \int_{D_{K,\sigma}} \mathrm{div} \boldsymbol{w} = \frac{|D_{K,\sigma}|}{|K|} \int_K \mathrm{div} \boldsymbol{w} = \frac{|D_{K,\sigma}|}{|K|} \int_{\partial K} \boldsymbol{w} \cdot \boldsymbol{n}$$

$$= \frac{|D_{K,\sigma}|}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = - \frac{|D_{K,\sigma}|}{|K|} \frac{|K|}{\delta t} (\rho_K - \rho_K^\star)$$

where $\xi_K^\sigma = \frac{|D_{K,\sigma}|}{|K|}$ is independent of K and σ.



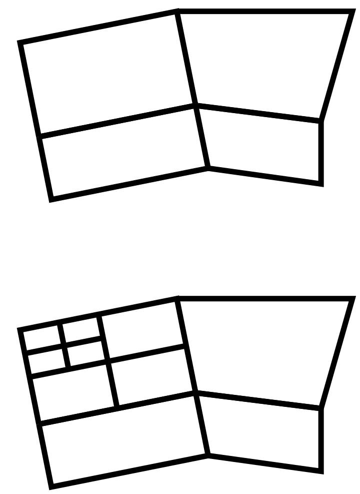
Solve the linear system :

$$F_{K,\sigma} + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon} = \xi_K^\sigma \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}, \qquad \forall \sigma \in \mathcal{E}(K).$$

Non-conforming local refinement

Admissible refined meshes :

- start from a regular triangulation
- recursive application of a refinement pattern to some cells
- maximum difference of refinement levels of adjacent cells: 1



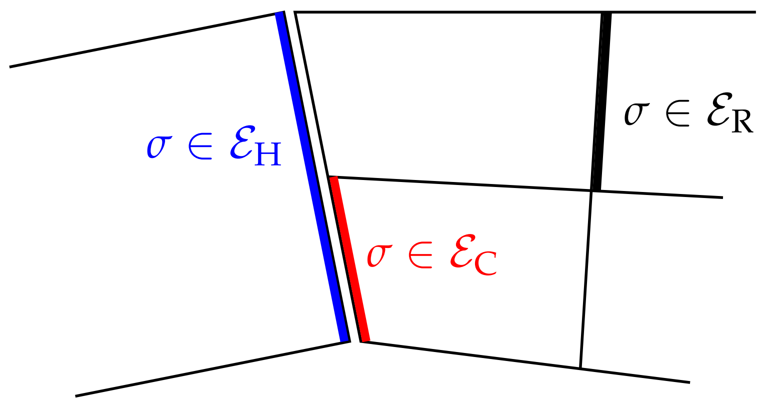
Hanging Faces :

$\mathcal{E} = \mathcal{E}_{\mathrm{R}} \cup \mathcal{E}_{\mathrm{H}} \cup \mathcal{E}_{\mathrm{C}}$

$\mathcal{E}_{\mathrm{R}}$: regular faces

$\mathcal{E}_{\mathrm{H}}$: hanging faces

$\mathcal{E}_{\mathrm{C}}$: child faces



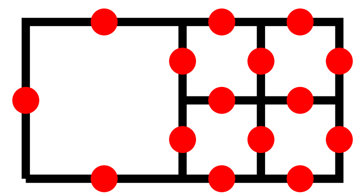
Non-conforming local refinement

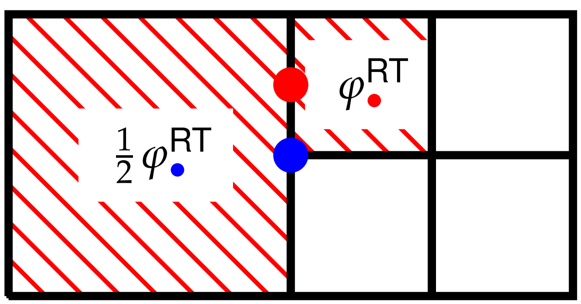
Rannacher-Turek element adapted to locally refined meshes :

$$V_h = \left\{ v \subset \mathrm{L}^2(\Omega);\ v|_K \in \widetilde{\mathrm{Q}}_1(K),\ \forall K \in \mathcal{M} \text{ and } \int_\sigma [v] = 0,\ \forall \sigma \in \mathcal{E}_R \cup \mathcal{E}_C \right\}.$$

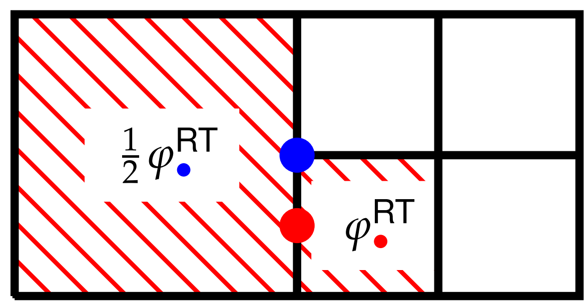
Basis functions :

$$V_h = \mathrm{span}\left\{ \varphi_\sigma \,;\ \sigma \in \mathcal{E}_R \cup \mathcal{E}_C \right\}$$





piecewise definition of $\varphi_\bullet$



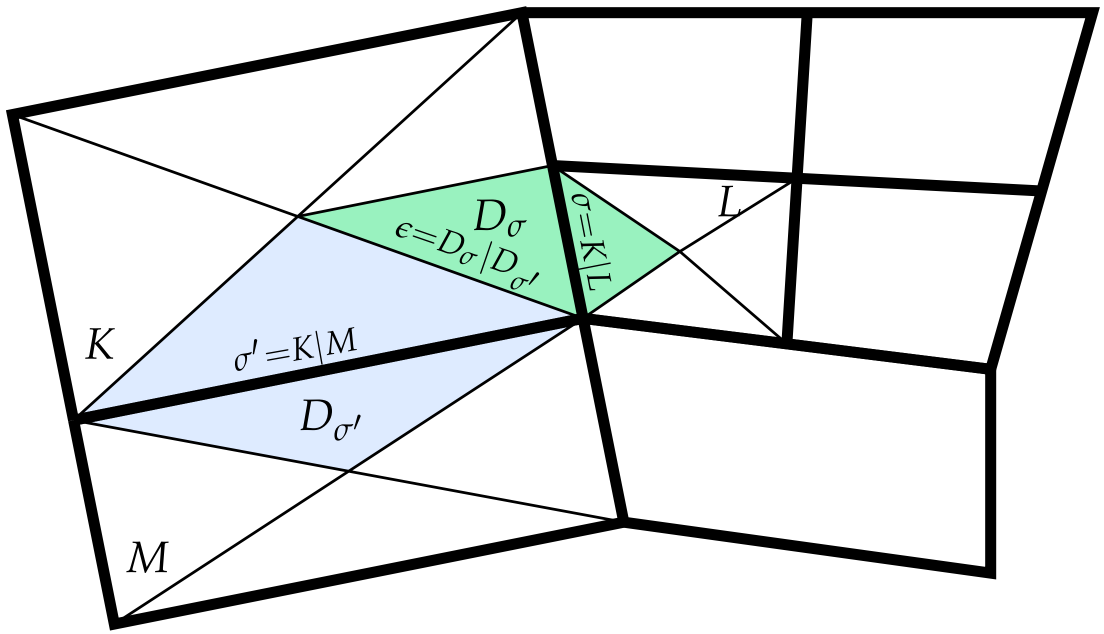
piecewise definition of $\varphi_\bullet$

Practical Implementation :

$$(T_{\mathrm{diff}}(\boldsymbol{u}))_{\sigma,i} = \frac{1}{|D_\sigma|} \sum_{K \in \mathcal{M}} \int_K \sum_{\sigma' \in \mathcal{E}(K)} \sum_{j=1}^{d} \boldsymbol{u}_{\sigma',j}\ \boldsymbol{\tau}(\varphi_{\sigma'} \boldsymbol{e}^{(j)}) : \boldsymbol{\nabla}(\varphi_\sigma \boldsymbol{e}^{(i)})\, \mathrm{d}\boldsymbol{x},$$

Non-conforming local refinement

Dual fluxes for locally refined meshes :



Find a solution to the system :

$$\forall \sigma \in \mathcal{E}(K), \qquad F_{K,\sigma} + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon} = \xi_K^\sigma \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}, \qquad \forall \sigma \in \mathcal{E}(K). \qquad (H1)$$

Non-conforming local refinement

Properties of the scheme :

- Gradient-divergence duality :

$$\sum_{K\in\mathcal{M}} |K|\ p_K\ (\mathrm{div}\boldsymbol{v})_K + \sum_{\sigma\in\mathcal{E}} |D_\sigma|\ \boldsymbol{v}_\sigma \cdot (\boldsymbol{\nabla} p)_\sigma = 0, \qquad \text{for all } (p,v) \in Q_h \times V_h^d.$$

- Positivity/coercivity of the diffusion term (Rannacher-Turek FE) :

$$\sum_{\sigma\in\mathcal{E}} |D_\sigma|\ (T_{\mathrm{diff}}(\boldsymbol{v}))_\sigma \cdot \boldsymbol{v}_\sigma = \|\boldsymbol{v}\|_{1,h}^2, \qquad \text{for all } v \in V_h^d.$$

- LBB (inf-sup) condition (*via* a Fortin interpolation operator) :

$$\inf_{p\in Q_h} \sup_{\boldsymbol{u}\in V_h^d} \frac{\int_\Omega p\,\mathrm{div}_h u}{\|p\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{u}\|_{1,h}} \geqslant \beta.$$

- Discrete kinetic energy balance :

$$\frac{1}{2\delta t}\Big(\rho_\sigma\,|\boldsymbol{u}_\sigma|^2 - \rho_\sigma^\star\,|\boldsymbol{u}_\sigma^\star|^2\Big) + \frac{1}{2|D_\sigma|} \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon=D_\sigma|D_{\sigma'}}} F_{\sigma,\epsilon}\ \boldsymbol{u}_\sigma \cdot \boldsymbol{u}_{\sigma'} - (T_{\mathrm{diff}}(\boldsymbol{u}))_\sigma \cdot \boldsymbol{u}_\sigma + (\boldsymbol{\nabla} p)_\sigma \cdot \boldsymbol{u}_\sigma \leqslant 0.$$

Pressure correction scheme

Knowing $\rho^\star$, $\boldsymbol{u}^\star$, $p^\star$ and $\rho^{\star\star}$ *s.t.* $\dfrac{1}{\delta t}(\rho_K^\star - \rho_K^{\star\star}) + \dfrac{1}{|K|}\displaystyle\sum_{\sigma\in\mathcal{E}} F_{K,\sigma}^\star,\ \forall K \in \mathcal{M},$

0-Preparation step – Compute $(\rho_\sigma^{\star\star})_{\sigma\in\mathcal{E}_{\rm int}}$, $(\rho_\sigma^{\star})_{\sigma\in\mathcal{E}_{\rm int}}$ and $(F_{\sigma,\epsilon}\star)_{\sigma\in\mathcal{E}_{\rm int}}$ such that:

$$\frac{1}{\delta t}(\rho_\sigma^\star - \rho_\sigma^{\star\star}) + \frac{1}{|D_\sigma|}\sum_{\epsilon\in\mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^\star = 0, \qquad \sigma \in \mathcal{E}_{\rm int}.$$

1-Prediction step – Find $(\widetilde{\boldsymbol{u}}_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$ such that:

$$\frac{1}{\delta t}(\rho_\sigma^\star \widetilde{\boldsymbol{u}}_\sigma - \rho_\sigma^{\star\star}\boldsymbol{u}_\sigma^\star) + \frac{1}{|D_\sigma|}\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^\star \widetilde{\boldsymbol{u}}_\epsilon + (T_{\rm diff}(\widetilde{\boldsymbol{u}}))_\sigma + \left(\frac{\rho_\sigma^\star}{\rho_\sigma^{\star\star}}\right)^{\frac{1}{2}} (\boldsymbol{\nabla} p)_\sigma^\star = 0, \qquad \sigma \in \mathcal{E}_{\rm int}.$$

2-Correction step – Find $(\boldsymbol{u}_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$ and $(p_K)_{K\in\mathcal{M}}$ such that:

$$\frac{1}{\delta t}\rho_\sigma^\star(\boldsymbol{u}_\sigma - \widetilde{\boldsymbol{u}}_\sigma) + (\boldsymbol{\nabla} p)_\sigma - \left(\frac{\rho_\sigma^\star}{\rho_\sigma^{\star\star}}\right)^{\frac{1}{2}} (\boldsymbol{\nabla} p)_\sigma^\star = 0, \qquad \sigma \in \mathcal{E}_{\rm int},$$

$$\frac{1}{\delta t}(\rho_K - \rho_K^\star) + \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma} = 0, \qquad K \in \mathcal{M}.$$

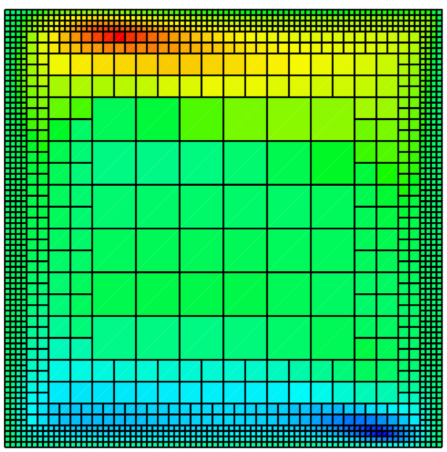
Numerical applications (1/2)

A low Mach number model for buoyant flows :

$$\begin{aligned}
&\partial_t(\rho c_p \vartheta) + \operatorname{div}(\rho c_p \vartheta \boldsymbol{u}) - \operatorname{div}(\lambda \boldsymbol{\nabla} \vartheta) = 0,\\
&P_{th}(t) = \rho R \vartheta,\\
&\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0,\\
&\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}(\boldsymbol{\tau}(\boldsymbol{u})) + \boldsymbol{\nabla} p = \rho g.
\end{aligned}$$

The evolution of P_{th} with time must be given by and additional relation: the conservation of the total mass in the domain:

$$\int_\Omega \rho(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = \frac{P_{th}(t)}{R} \int_\Omega \frac{1}{\vartheta(\boldsymbol{x},t)}\,\mathrm{d}\boldsymbol{x} = |\Omega|\,\rho_0, \text{ with } \rho_0 = \frac{P_{th}(0)}{R\,\vartheta_0}.$$



The velocity is set to zero on the boundary and the border is heated according to: $\vartheta(\boldsymbol{x}) = \frac{L-x_1}{L}\,\vartheta_h + \frac{x_1}{L}\,\vartheta_c$, $\vartheta_h = (1+\varepsilon)\,\vartheta_0$, $\vartheta_c = (1-\varepsilon)\vartheta_0$.

Numerical applications (1/2)

A low Mach number model for buoyant flows :

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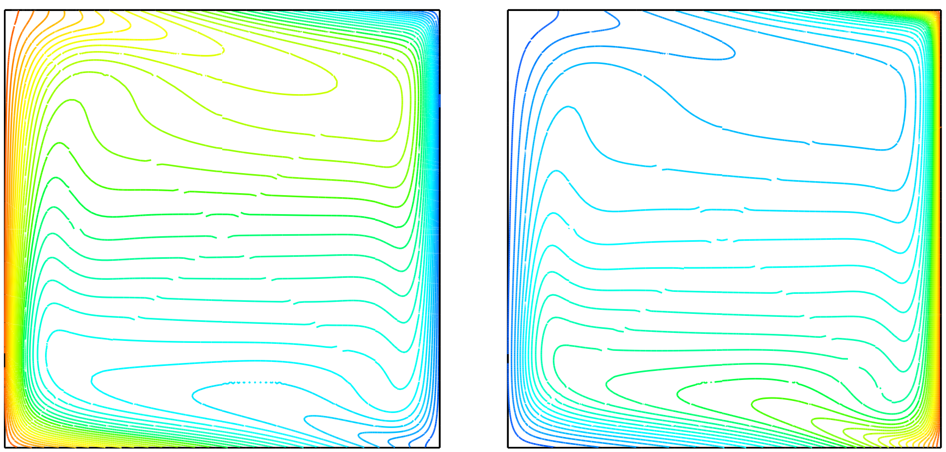


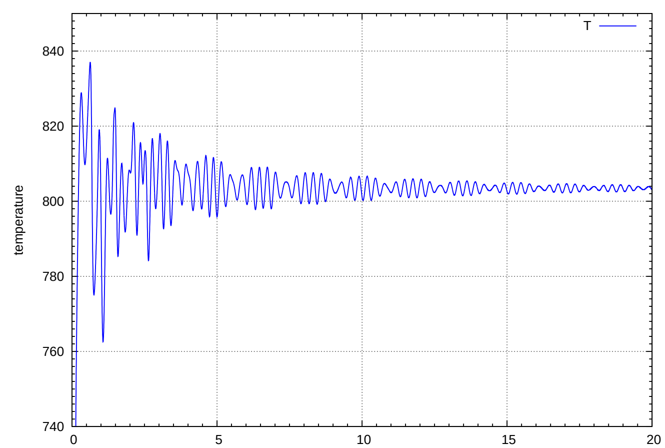
Numerical applications (1/2)

A low Mach number model for buoyant flows :

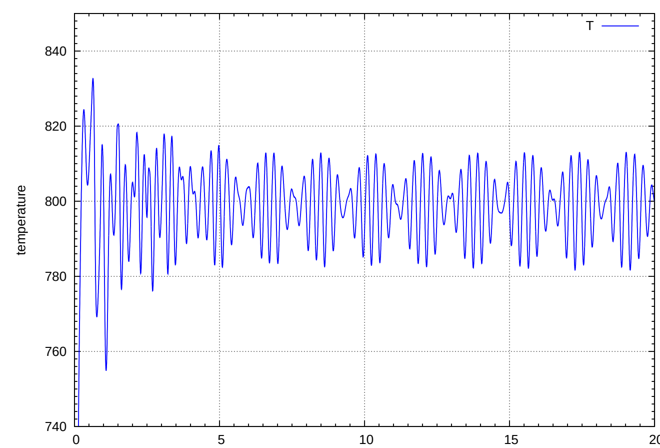
Dimensional analysis $\Longrightarrow$ the flow is governed by the Prandtl and the Rayleigh numbers :

$$\mathrm{Pr} = \frac{\mu\, c_p}{\lambda}, \qquad \mathrm{Ra} = \frac{\rho_0^2\, c_p\, g\, (\vartheta_h - \vartheta_c)\, L^3}{\mu\, \lambda\, \vartheta_0}.$$

S. Xin and P. Le Quéré (Physics of Fluids, 2001), provide a stability analysis that shows that the flow reaches a steady state up to a critical value of the Rayleigh number approximately equal to $\mathrm{Ra} = 2.1\, 10^6$. Beyond this value, the flow remains time-dependent.



$\mathrm{Ra} = 2\, 10^6$



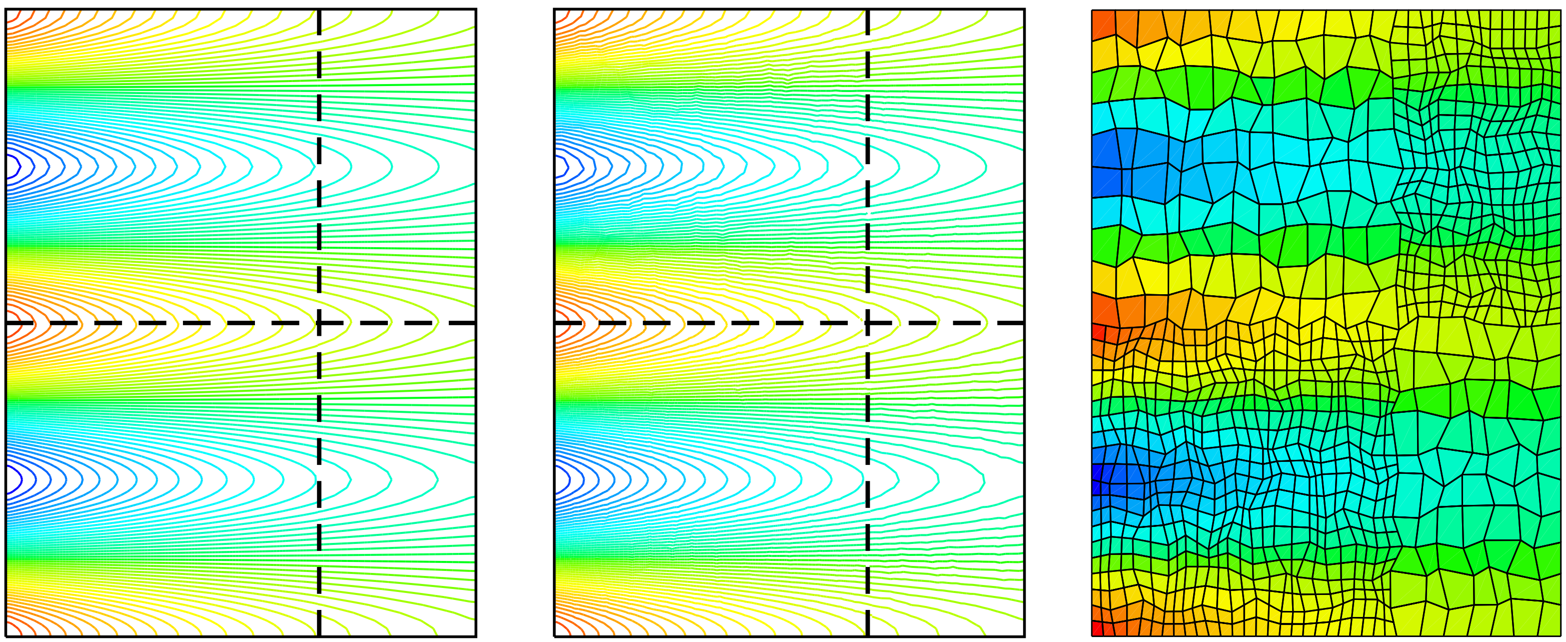
$\mathrm{Ra} = 2.2\, 10^6$

Numerical applications (2/2)

A stationary incompressible flow (Kovasznay flow) :

$$\boldsymbol{u} = \begin{bmatrix} 1 - e^{\lambda x} \cos(2\pi y) \\ \frac{\lambda}{2\pi} e^{\lambda x} \sin(2\pi y) \end{bmatrix}, \quad p = \frac{1}{2}\,(1 - e^{2\lambda x}), \quad \lambda = \frac{1}{2\mu} - \left(\frac{1}{4\mu^2} + 4\pi^2\right)^{1/2},$$

where μ stands for the viscosity of the flow, taken here as $\mu = 1/40$ is an analytic solution of the incompressible Navier-Stokes equations.



Numerical applications (2/2)

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where μ stands for the viscosity of the flow, taken here as $\mu = 1/40$ is an analytic solution of the incompressible Navier-Stokes equations.

n	$\|\boldsymbol{u} - \boldsymbol{u}_{\text{exact}}\|_{L^2(\Omega)^d}$	$\|p - p_{\text{exact}}\|_{L^2(\Omega)}$
20	0.0384	0.0334
40	0.00825	0.0158
80	0.00211	0.00782
160	0.000544	0.00390

Second order convergence for $\boldsymbol{u}$.

n	$\|\boldsymbol{u} - \boldsymbol{u}_{\text{exact}}\|_{L^2(\Omega)^d}$	$\|p - p_{\text{exact}}\|_{L^2(\Omega)}$
20	0.0617	0.0406
40	0.0119	0.0179
80	0.00281	0.0087
160	0.000718	0.0043

Second order convergence for $\boldsymbol{u}$.

Pressure correction scheme

Knowing $\rho^\star$, $\boldsymbol{u}^\star$, $p^\star$ and $\rho^{\star\star}$ *s.t.* $\dfrac{1}{\delta t}(\rho_K^\star - \rho_K^{\star\star}) + \dfrac{1}{|K|}\displaystyle\sum_{\sigma\in\mathcal{E}} F_{K,\sigma}^\star,\ \forall K \in \mathcal{M},$

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$$\frac{1}{\delta t}(\rho_\sigma^\star - \rho_\sigma^{\star\star}) + \frac{1}{|D_\sigma|}\sum_{\epsilon\in\mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^\star = 0, \qquad \sigma \in \mathcal{E}_{\rm int}.$$

1-Prediction step – Find $(\widetilde{\boldsymbol{u}}_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$ such that:

$$\frac{1}{\delta t}(\rho_\sigma^\star \widetilde{\boldsymbol{u}}_\sigma - \rho_\sigma^{\star\star}\boldsymbol{u}_\sigma^\star) + \frac{1}{|D_\sigma|}\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^\star \widetilde{\boldsymbol{u}}_\epsilon + (T_{\rm diff}(\widetilde{\boldsymbol{u}}))_\sigma + \left(\frac{\rho_\sigma^\star}{\rho_\sigma^{\star\star}}\right)^{\frac{1}{2}} (\boldsymbol{\nabla} p)_\sigma^\star = 0, \qquad \sigma \in \mathcal{E}_{\rm int}.$$

2-Correction step – Find $(\boldsymbol{u}_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$ and $(p_K)_{K\in\mathcal{M}}$ such that:

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$$\frac{1}{\delta t}(\rho_K - \rho_K^\star) + \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma} = 0, \qquad K \in \mathcal{M}.$$

Error analysis : convection-diffusion equation

Let $\boldsymbol{w} \in \mathcal{C}^1(\overline{\Omega})^d$ such that $\boldsymbol{w}|_{\partial\Omega} = 0,\ \mathrm{div}\boldsymbol{w} = 0$ and let $f \in \mathrm{L}^2(\Omega)$.

Strong form of the convection-diffusion equation :

$$\begin{aligned} & u + \mathrm{div}(u\boldsymbol{w}) - \boldsymbol{\Delta} u = f, && \text{on } \Omega, \\ & u = 0, && \text{on } \partial\Omega. \end{aligned}$$

Weak formulation of the convection-diffusion equation :

Let us define :

$$\begin{aligned} a(u,v) &= \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v, && u,v \in \mathrm{H}^1_0(\Omega), \\ b(\boldsymbol{w},u,v) &= \int_\Omega \mathrm{div}(u\boldsymbol{w})v, && u,v \in \mathrm{H}^1_0(\Omega). \end{aligned}$$

A weak solution is a function $u \in \mathrm{H}^1_0(\Omega)$ such that:

$$(u,v) + b(\boldsymbol{w},u,v) + a(u,v) = (f,v), \qquad \forall v \in \mathrm{H}^1_0(\Omega).$$

Lax-Milgram theorem $\Longrightarrow$ there exists a unique weak solution.

Error analysis : convection-diffusion equation

The staggered scheme for the convection-diffusion equation :

- Let $F_{K,\sigma}(\boldsymbol{w}) = \displaystyle\int_\sigma \boldsymbol{w} \cdot n_{K,\sigma} \,\mathrm{d}\gamma$ and compute the dual fluxes $F_{\sigma,\epsilon}(\boldsymbol{w})$ so that $(H1)$ holds.

Error analysis : convection-diffusion equation

The staggered scheme for the convection-diffusion equation :

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- The staggered scheme reads:

$$u_\sigma + \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\boldsymbol{w}) u_\epsilon + \sum_{K \in \mathcal{M}} \int_K \sum_{\sigma' \in \mathcal{E}} u_{\sigma'} \boldsymbol{\nabla} \varphi_{\sigma'} \cdot \boldsymbol{\nabla} \varphi_\sigma = f_\sigma.$$

Error analysis : convection-diffusion equation

The staggered scheme for the convection-diffusion equation :

- Let $F_{K,\sigma}(\boldsymbol{w}) = \int_\sigma \boldsymbol{w} \cdot n_{K,\sigma} \, \mathrm{d}\gamma$ and compute the dual fluxes $F_{\sigma,\epsilon}(\boldsymbol{w})$ so that $(H1)$ holds.
- The staggered scheme reads:

$$u_\sigma + \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\boldsymbol{w}) u_\epsilon + \sum_{K \in \mathcal{M}} \int_K \sum_{\sigma' \in \mathcal{E}} u_{\sigma'} \boldsymbol{\nabla} \varphi_{\sigma'} \cdot \boldsymbol{\nabla} \varphi_\sigma = f_\sigma.$$

- Multiplying by v_σ and summing over $\sigma \in \mathcal{E}_{\text{int}}$, the scheme is equivalent to:

Find $u_h \in V_h$ such that :

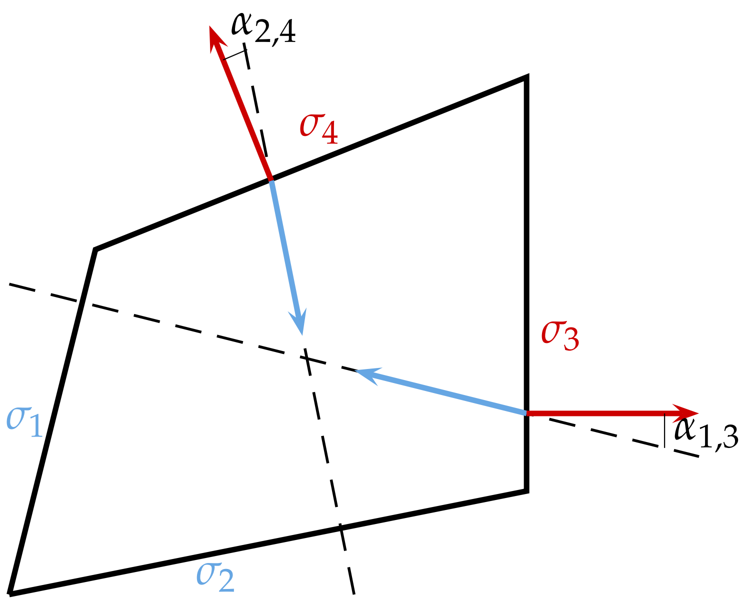
$$(u_h, v)_h + b_h(\boldsymbol{w}, u_h, v) + a_h(u_h, v) = (f, v), \qquad \text{for all } v \in V_h,$$

where $\forall (u, v) \in V_h \times V_h$,

$$\left| \begin{aligned} (u,v)_h &= \sum_{\sigma \in \mathcal{E}} |D_\sigma| u_\sigma v_\sigma, \\ b_h(\boldsymbol{w}, u, v) &= \sum_{\sigma \in \mathcal{E}} v_\sigma \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\boldsymbol{w}) u_\epsilon, \\ a_h(u,v) &= \int_\Omega \boldsymbol{\nabla}_h u \cdot \boldsymbol{\nabla}_h v. \end{aligned} \right.$$

Error analysis : convection-diffusion equation

Regularity of the mesh :



We assume convex control volumes K.

$h_K = \text{size}(K)$

$r_K =$ radius of the biggest ball included in K

$\alpha_K = \max(\alpha_{1,3}, \alpha_{2,4})$

$h = \max\{h_K, K \in \mathcal{M}\}$

$\alpha_h = \max\{\alpha_K, K \in \mathcal{M}\}$

$$\theta_h = \max\left\{\frac{h_K}{r_K},\ K \in \mathcal{M}\right\}$$

Error analysis : convection-diffusion equation

Reminder : Stokes problem :

$$
\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad & \Omega \\
\operatorname{div} \boldsymbol{u} = 0, \qquad & \Omega, \\
\boldsymbol{u} = 0, \qquad & \partial\Omega.
\end{aligned}
$$

Theorem : *Let $\theta_0 > 0$ and let $\mathcal{M}$ be a non-refined mesh of the computational domain Ω such that $\theta_h \leqslant \theta_0$. Let $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ be the Rannacher-Turek approximation of the Stokes. We assume that the solution $(p, \boldsymbol{u})$ of the Stokes problem belongs to $\mathrm{H}^1(\Omega) \times \mathrm{H}^1_0(\Omega)^d \cap \mathrm{H}^2(\Omega)^d$. Then the discrete solution $(p_h, \boldsymbol{u}_h)$ satisfies:*

$$
\|p_h - p\|_{\mathrm{L}^2(\Omega)} + \|\boldsymbol{u}_h - \boldsymbol{u}\|_{1,h} \leqslant C\,(h + \alpha_h),
$$

where C only depends on p, $\boldsymbol{u}$, Ω and θ_0.

Question : For the convection-diffusion equation, do we have the same accuracy of the staggered scheme on locally non-conforming refined meshes and knowing the particular discretization of the convection term ?

Error analysis : convection-diffusion equation

$$\begin{aligned} & u + \mathrm{div}(u\boldsymbol{w}) - \Delta u = f, && \text{on } \Omega, \\ & u = 0, && \text{on } \partial\Omega. \end{aligned}$$

Theorem (Jean-Claude Latché, Bruno Piar, K.S.) : *Let $\theta_0 > 0$ and let $\mathcal{M}$ be a locally refined mesh of the computational domain Ω such that $\theta_h \leqslant \theta_0$. Let $u_h \in V_h$ be the solution to the staggered scheme. We assume that the solution u of the continuous convection-diffusion problem belongs to $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. Then u_h satisfies:*

$$\|u_h - u\|_{1,h} \leqslant C\,(h + \alpha_h)\,\|u\|_{\mathrm{H}^2(\Omega)},$$

where C only depends on $\boldsymbol{w}$, Ω and θ_0.

Remark : Thanks to the gradient-divergence duality and the inf-sup stability properties, a similar result can be proved for the Oseen model:

$$\begin{aligned} & \boldsymbol{u} + \mathrm{div}(\boldsymbol{w} \otimes \boldsymbol{u}) - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, && \Omega \\ & \mathrm{div}\boldsymbol{u} = 0, && \Omega, \\ & \boldsymbol{u} = 0, && \partial\Omega. \end{aligned}$$

Error analysis : convection-diffusion equation

Sketch of the proof :

- By the triangle inequality : $\|u_h - u\|_{1,h} \leqslant \|u_h - r_h u\|_{1,h} + \|r_h u - u\|_{1,h}$ where r_h is the following interpolation operator:

$$r_h : \left| \begin{array}{lcl} \mathrm{H}^1_0(\Omega) & \longrightarrow & V_h \\ v & \mapsto & r_h v(\boldsymbol{x}) = \displaystyle\sum_{\sigma \in \mathcal{E}} |\sigma|^{-1} \left(\int_\sigma v \,\mathrm{d}\gamma \right) \varphi_\sigma(\boldsymbol{x}). \end{array} \right.$$

- Approximation property of r_h (Rannacher-Turek extended to refined meshes) :

$$\forall v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega),\ \forall K \in \mathcal{M},$$
$$\|v - r_h v\|_{\mathrm{L}^2(K)} + h_K \,\|\boldsymbol{\nabla}(v - r_h v)\|_{\mathrm{L}^2(K)^d} \leqslant C_2\, h_K (h_K + \alpha_K)\, |v|_{\mathrm{H}^2(K)}.$$

Hence:

$$\|r_h u - u\|_{1,h} = \Big(\sum_{K \in \mathcal{M}} \|\boldsymbol{\nabla}(u - r_h u)\|^2_{\mathrm{L}^2(K)^d} \Big)^{\frac{1}{2}} \leqslant C_2 (h + \alpha_h) |u|_{\mathrm{H}^2(\Omega)}.$$

Error analysis : convection-diffusion equation

Sketch of the proof :

- Dual formulation of the norm :

$$\|u_h - r_h u\|_{1,h} \leqslant \sup_{v \in V_h} \frac{\mathcal{A}_h(u_h - r_h u, v)}{\|v\|_h}.$$

where $\mathcal{A}_h(u,v) := (u,v)_h + b_h(\boldsymbol{w}, u, v) + a_h(u,v)$.

Error analysis : convection-diffusion equation

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- As u_h is the numerical solution, one has $\mathcal{A}_h(u_h, v) = (f, v)$.

Error analysis : convection-diffusion equation

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- The exact solution u of the convection-diffusion is assumed to be in $\mathrm{H}^2(\Omega)$, hence, the equation is satisfied in the strong sense in $\mathrm{L}^2(\Omega)$:

$$(u,v) + (\mathrm{div}(u\boldsymbol{w}), v) - (\boldsymbol{\Delta} u, v) = (f, v), \qquad \forall v \in \mathrm{L}^2(\Omega)$$

$$\Longrightarrow \quad \mathcal{A}_h(u_h, v) = (u,v) + (\mathrm{div}(u\boldsymbol{w}), v) - (\boldsymbol{\Delta} u, v), \quad \forall v \in V_h.$$

Error analysis : convection-diffusion equation

Sketch of the proof :

- Dual formulation of the norm :

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- The exact solution u of the convection-diffusion is assumed to be in $\mathrm{H}^2(\Omega)$, hence, the equation is satisfied in the strong sense in $\mathrm{L}^2(\Omega)$:

$$(u,v) + (\mathrm{div}(u\boldsymbol{w}), v) - (\boldsymbol{\Delta} u, v) = (f, v), \qquad \forall v \in \mathrm{L}^2(\Omega)$$

$$\Longrightarrow \quad \mathcal{A}_h(u_h, v) = (u,v) + (\mathrm{div}(u\boldsymbol{w}), v) - (\boldsymbol{\Delta} u, v), \quad \forall v \in V_h.$$

- The error decomposes in three terms :

$$\mathcal{A}_h(u_h - r_h u, v) = \underbrace{(u,v) - (r_h u, v)_h}_{\text{reaction error}} + \underbrace{(\mathrm{div}(u\boldsymbol{w}), v) - b_h(\boldsymbol{w}, r_h u, v)}_{\text{convection error}} + \underbrace{(-\boldsymbol{\Delta} u, v) - a_h(r_h u, v)}_{\text{diffusion error}}.$$

Error analysis : convection-diffusion equation

Sketch of the proof :

Convection error: $|(\mathrm{div}(u\boldsymbol{w}), v) - b_h(\boldsymbol{w}, r_h u, v)|$ with

$$b_h(\boldsymbol{w}, u, v) = \sum_{\sigma \in \mathcal{E}} v_\sigma \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\boldsymbol{w}) u_\epsilon.$$

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- The expressions of the dual fluxes $F_{\sigma,\epsilon}(\boldsymbol{w})$ w.r.t $\boldsymbol{w}$ is complicated, contrary to the expressions of $F_{K,\sigma}(\boldsymbol{w}) = \int_\sigma \boldsymbol{w} \cdot \boldsymbol{n}_{K,\sigma} d\gamma \;\; (= |\sigma| \hat{\rho}_\sigma \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma})$.

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- In the analysis, replace the implemented convection term by a simpler convection term on the primal mesh:

$$\widetilde{b}_h(\boldsymbol{w}, u, v) = \sum_{K \in \mathcal{M}} v_K \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}(\boldsymbol{w})\ u_\sigma, \qquad \text{with } v_K = \sum_{\sigma \in \mathcal{E}(K)} \xi_K^\sigma\, v_\sigma,$$

and prove the first-order bound

$$|\widetilde{b}_h(\boldsymbol{w}, u, v) - b_h(\boldsymbol{w}, u, v)| \leqslant C\,h\,\|\boldsymbol{w}\|_{\mathrm{L}^\infty(\Omega)^d} \|u\|_{1,h} \|v\|_{1,h}, \qquad \forall u, v \in V_h.$$

Error analysis : convection-diffusion equation

Sketch of the proof :

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- Prove

$$|(\mathrm{div}(u\boldsymbol{w}), v) - \widetilde{b}_h(\boldsymbol{w}, r_h u, v)| \leqslant Ch.$$

Error analysis : convection-diffusion equation

Sketch of the proof :

Diffusion error: $|(-\Delta u, v) - a_h(r_h u, v)|$.

Two main ingredients:

- The approximation property of the Fortin operator for refined meshes:

$$\forall v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega),\ \forall K \in \mathcal{M},$$
$$\|v - r_h v\|_{\mathrm{L}^2(K)} + h_K \|\nabla(v - r_h v)\|_{\mathrm{L}^2(K)^d} \leqslant C_2\, h_K(h_K + \alpha_K)\, |v|_{\mathrm{H}^2(K)}.$$

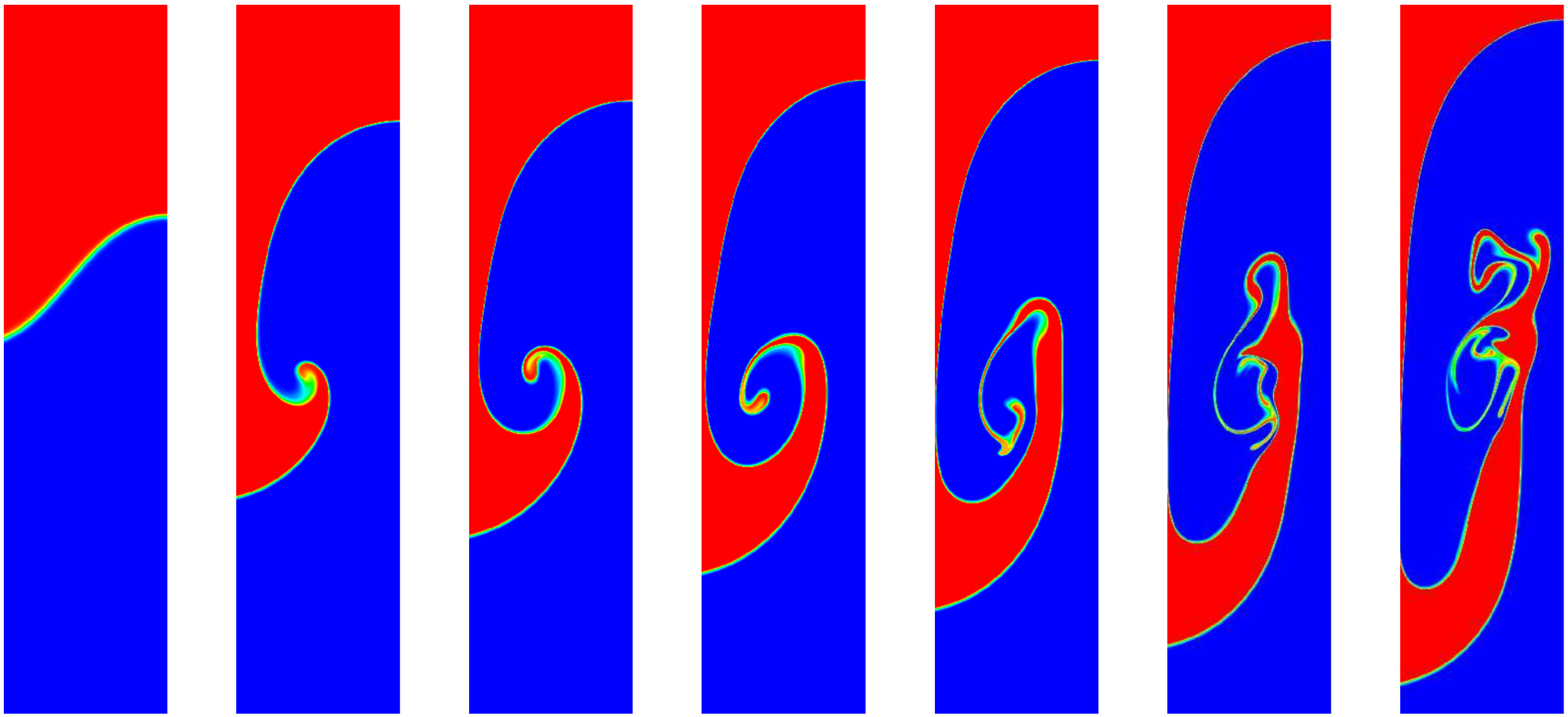
- A bound on the interface jumps of the velocity:

$$\sum_{\Sigma \in \mathcal{E}_R \cup \mathcal{E}_H} \left| \int_\Sigma [v]_\Sigma\, \partial_n u \right| \leqslant C\, h \|v\|_{1,h}\, \|u\|_{\mathrm{H}^1_0(\Omega)}, \qquad \forall v \in V_h, u \in \mathrm{H}^1_0(\Omega).$$

A convergence result

Incompressible variable density Navier-Stokes equations

$$\partial_t(\rho c_p \vartheta) + \operatorname{div}(\rho c_p \vartheta \boldsymbol{u}) = 0,$$
$$\rho = \mathscr{F}(\theta),$$
$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$
$$\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}(\boldsymbol{\tau}(\boldsymbol{u})) + \boldsymbol{\nabla} p = \rho g.$$



A convergence result

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\end{aligned}$$

This model is equivalent to :

$$\begin{aligned}
&\operatorname{div} \boldsymbol{u} = 0,\\
&\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0,\\
&\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \Delta \boldsymbol{u} + \boldsymbol{\nabla} p = 0.
\end{aligned}$$

On a bounded domain $(0,T) \times \Omega$, we prove the convergence of the staggered scheme towards the weak solutions of this model, with:

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \quad \rho|_{t=0} = \rho_0 \geqslant \rho_{\min} > 0.$$

A convergence result

The scheme:

$$(\mathrm{div}\,\boldsymbol{u})_K = 0, \qquad K \in \mathcal{M},$$

$$\frac{1}{\delta t}(\rho_K - \rho_K^\star) + \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma} = 0, \qquad K \in \mathcal{M},$$

$$\frac{1}{\delta t}(\rho_\sigma \boldsymbol{u}_\sigma - \rho_\sigma^\star \boldsymbol{u}_\sigma^\star) + \frac{1}{|D_\sigma|}\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}\boldsymbol{u}_\epsilon - (\Delta \boldsymbol{u})_\sigma + (\boldsymbol{\nabla} p)_\sigma = 0, \qquad \sigma \in \mathcal{E}_{\mathrm{int}}.$$

with:

$$(\mathrm{div}\,\boldsymbol{u})_K = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\; \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}$$

$F_{K,\sigma} = |\sigma|\;\rho_\sigma\;\boldsymbol{u}_\sigma^{\mathrm{up}} \cdot \boldsymbol{n}_{K,\sigma}$, $\quad \rho_\sigma^{\mathrm{up}}$: upwind approximation of ρ on σ.

$$(\boldsymbol{\nabla} p)_\sigma = \frac{|\sigma|}{|D_\sigma|}(p_L - p_K)\boldsymbol{n}_{K,\sigma},$$

centered approximation for $\boldsymbol{u}_\epsilon$,

$-(\Delta \boldsymbol{u})_\sigma$ discretized by the usual Rannacher-Turek expansion.

A convergence result

For a given mesh:

1 - *A priori* estimates.

2 - Existence of a solution.

For a sequence of meshes and time steps $(\mathcal{M}^{(m)}, \delta t^{(m)})$*:*

$$\left| \begin{array}{l} \rho^{(m)}(\boldsymbol{x}, t) \\ p^{(m)}(\boldsymbol{x}, t) \end{array} \right. = \sum_{n=1}^{N^{(m)}} \sum_{K \in \mathcal{M}^{(m)}} \left| \begin{array}{l} \rho_K^n \\ p_K^n \end{array} \right. \mathcal{X}_K(\boldsymbol{x}) \, \mathcal{X}_{(n-1,n]}(t)$$

$$\boldsymbol{u}^{(m)}(\boldsymbol{x}, t) = \sum_{n=1}^{N^{(m)}} \sum_{\sigma \in \mathcal{E}^{(m)}} \boldsymbol{u}_\sigma^n \, \mathcal{X}_{D_\sigma} \, \mathcal{X}_{(n-1,n]}(t).$$

3 - Compactness: (up to the extraction of a subsequence) strong convergence of $\boldsymbol{u}^{(m)}$ *c.f.* the talk of T. Gallouët) and weak convergence of $\rho^{(m)}$.

4 - Passage to the limit in the divergence-free constraint.

5 - Passage to the limit in the mass balance equation.

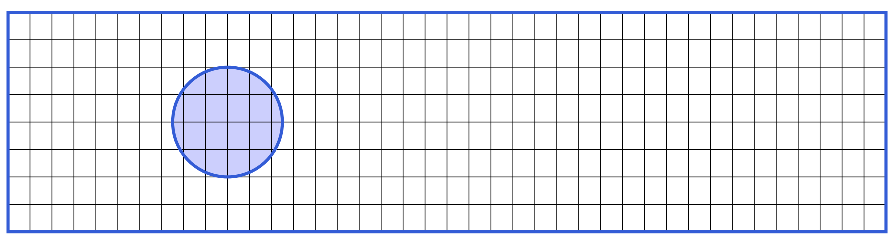
6 - Strong convergence of ρ.

7 - Passage to the limit in the momentum balance equation.

Theorem (Jean-Claude Latché, K.S.): *Up to the extraction of a subsequence, the scheme converges (in strong norms) to a weak solution of the problem.*

Perspectives

- Ongoig work : local refinement for Euler equations (avoid spurious oscillations at the refinement boundary).
- Improvement of numerical schemes: explicit MUSCL for Euler, implicit MUSCL for scalar transport and Euler (defect correction techniques), assessing the use of local refinement techniques for vanishing-diffusion flows.
- Run the code on massively parallel machines (Maison de la Simulation (Saclay) ?).
- Toward the generalization of LES approaches, on structured schemes + immersed boundary method.



obstacle, boundary and mesh

- New field of application: explosion (H_2, natural gas, dust).
  - solvers for reactive compressible flows,
  - and for the so-called G-equation (transport of a quantity by a velocity field colinear with its gradient).

Thank you !