Time compactness for approximate solutions of evolution problems

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Porto, may 1, 2014

- Parabolic equation with L^1 data
  Coauthors : Lucio Boccardo (continuous setting, 1989)
  Robert Eymard, Raphaèle Herbin (discrete setting, 2000)
  Aurélien Larcher, Jean-Claude Latché (discrete setting, 2011)
- Stefan problem
  Coauthors: R. Eymard, P. Féron, C. Guichard, R. Herbin
- Other examples : incompressible and compressible Stokes and Navier-Stokes equations
  Coauthors : E. Chénier, R. E., R.H. (2013) and A. Fettah

Example (coming from RANS model for turbulent flows)

$$\partial_t u + \mathrm{div}(vu) - \Delta u = f \text{ in } \Omega \times (0,T),$$
$$u = 0 \text{ on } \partial\Omega \times (0,T),$$
$$u(\cdot,0) = u_0 \text{ in } \Omega.$$

- Ω is a bounded open subset of $\mathbb{R}^d$ ($d = 2$ or 3) with a Lipschitz continuous boundary
- $v \in C^1(\overline{\Omega} \times [0,T], \mathbb{R})$
- $u_0 \in L^1(\Omega)$ (or u_0 is a Radon measure on Ω)
- $f \in L^1(\Omega \times (0,T))$ (or f is a Radon measure on $\Omega \times (0,T)$)

with possible generalization to nonlinear problems.

Non smooth solutions.

What is the problem ?

1. Existence of weak solution and (strong) convergence of "continuous approximate solutions", that is solutions of the continuous problem with regular data converging to f and u_0.
2. Existence of weak solution and (strong) convergence of the approximate solutions given by a full discretized problem.

In both case, we want to prove strong compactness (in L^p space) of a sequence of approximate solutions. This is the main subject of this talk.

Continuous approximation

$(f_n)_{n\in\mathbb{N}}$ and $(u_{0,n})_{n\in\mathbb{N}}$ are two sequences of regular functions such that

$$\int_0^T\int_\Omega f_n\varphi dxdt \to \int_0^T\int_\Omega f\varphi dxdt,\ \forall\varphi\in C_c^\infty(\Omega\times(0,T),\mathbb{R}),$$
$$\int_\Omega u_{0,n}\varphi dx \to \int_\Omega u_0\varphi dx,\ \forall\varphi\in C_c^\infty(\Omega,\mathbb{R}).$$

For $n\in\mathbb{N}$, it is well known that there exist u_n solution of the regularized problem

$$\partial_t u_n + \mathrm{div}(vu_n) - \Delta u_n = f_n \text{ in } \Omega\times(0,T),$$
$$u_n = 0 \text{ on } \partial\Omega\times(0,T),$$
$$u_n(\cdot,0) = u_{0,n} \text{ in } \Omega.$$

One has, at least, $u_n\in L^2((0,T),H_0^1(\Omega))\cap C([0,T],L^2(\Omega))$ and $\partial_t u_n\in L^2((0,T),H^{-1}(\Omega))$.

Continuous approximation, steps of the proof of convergence

1. Estimate on u_n (not easy). One proves that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in

$$L^q((0,T),W_0^{1,q}(\Omega)) \text{ for all } 1 \le q < \frac{d+2}{d+1}.$$

   (This gives, up to a subsequence, weak convergence in $L^q(\Omega \times (0,T))$ of u_n to some u and then, since the problem is linear, that u is a weak solution of the problem with f and u_0.)
2. Strong compactness of the sequence $(u_n)_{n\in\mathbb{N}}$
3. Regularity of the limit of the sequence $(u_n)_{n\in\mathbb{N}}$.
4. Passage to the limit in the approximate equation (easy).

Aubin-Simon' Compactness Lemma

X, B, Y are three Banach spaces such that

- $X \subset B$ with compact embedding,
- $B \subset Y$ with continuous embedding.

Let $T > 0$, $1 \leq p < +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),X)$,
- $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),Y)$.

Then there exists $u \in L^p((0,T),B)$ such that, up to a subsequence, $u_n \to u$ in $L^p((0,T),B)$.

Example: $p = 2$, $X = H_0^1(\Omega)$, $B = L^2(\Omega)$, $Y = H^{-1}(\Omega)$ (dual space of X).
As usual, $H_0^1(\Omega) \subset L^2(\Omega) = L^2(\Omega)' \subset H^{-1}(\Omega)$.

Aubin-Simon' Compactness Lemma

X, B, Y are three Banach spaces such that

- $X \subset B$ with compact embedding,
- $B \subset Y$ with continuous embedding.

Let $T > 0$, $1 \le p < +\infty$ and $(u_n)_{n \in \mathbb{N}}$ be a sequence such that

- $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^p((0,T),X)$,
- $(\partial_t u_n)_{n \in \mathbb{N}}$ is bounded in $L^p((0,T),Y)$.

Then there exists $u \in L^p((0,T),B)$ such that, up to a subsequence, $u_n \to u$ in $L^p((0,T),B)$.

Example: $p = 1$, $X = W_0^{1,1}(\Omega)$, $B = L^1(\Omega)$, $Y = W_\star^{-1,1}(\Omega) = (W_0^{1,\infty}(\Omega))'$. As usual, we identify an L^1-function with the corresponding linear form on $W_0^{1,\infty}(\Omega)$.

Classical Lions' lemma

X, B, Y are three Banach spaces such that

- $X \subset B$ with compact embedding,
- $B \subset Y$ with continuous embedding.

Then, for any $\varepsilon > 0$, there exists C_ε such that, for $w \in X$,

$$\|w\|_B \leq \varepsilon \|w\|_X + C_\varepsilon \|w\|_Y.$$

Proof: By contradiction
Improvment : "$B \subset Y$ with continuous embedding" can be replaced by the weaker hypothesis
"$(w_n)_{n\in\mathbb{N}}$ bounded in X, $w_n \to w$ in B, $w_n \to 0$ in Y implies $w = 0$"

Classical Lions' lemma, another formulation

X, B, Y are three Banach spaces such that, $X \subset B \subset Y$,

- If $(\|w_n\|_X)_{n\in\mathbb{N}}$ is bounded, then, up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- If $w_n \to w$ in B and $\|w_n\|_Y \to 0$, then $w = 0$.

Then, for any $\varepsilon > 0$, there exists C_ε such that, for $w \in X$,

$$\|w\|_B \le \varepsilon\|w\|_X + C_\varepsilon\|w\|_Y.$$

The hypothesis $B \subset Y$ is not necessary.

Classical Lions' lemma, improvment

X, B, Y are three Banach spaces such that, $X \subset B$,

If $(\|w_n\|_X)_{n\in\mathbb{N}}$ is bounded, then,

- up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- if $w_n \to w$ in B and $\|w_n\|_Y \to 0$, then $w = 0$.

Then, for any $\varepsilon > 0$, there exists C_ε such that, for $w \in X$,

$$\|w\|_B \le \varepsilon \|w\|_X + C_\varepsilon \|w\|_Y.$$

The hypothesis $B \subset Y$ is not necessary.

Classical Lions' lemma, a particular case, simpler

B is a Hilbert space and X is a Banach space $X \subset B$. We define on X the dual norm of $\| \cdot \|_X$, with the scalar product of B, namely

$$\|u\|_Y = \sup\{(u/v)_B,\ v \in X, \|v\|_X \leq 1\}.$$

Then, for any $\varepsilon > 0$ and $w \in X$,

$$\|w\|_B \leq \varepsilon \|w\|_X + \frac{1}{\varepsilon}\|w\|_Y.$$

The proof is simple since

$$\|u\|_B = (u/u)_B^{\frac{1}{2}} \leq (\|u\|_Y \|u\|_X)^{\frac{1}{2}} \leq \varepsilon \|w\|_X + \frac{1}{\varepsilon}\|w\|_Y.$$

Compactness of X in B is not needed here (but this compactness is needed for Aubin-Simon' Lemma, next slide...).

Aubin-Simon' Compactness Lemma

X, B, Y are three Banach spaces such that

- $X \subset B$ with compact embedding,
- $B \subset Y$ with continuous embedding.

Let $T > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),X)$,
- $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),Y)$.

Then there exists $u \in L^1((0,T),B)$ such that, up to a subsequence, $u_n \to u$ in $L^1((0,T),B)$.

Example: $X = W_0^{1,1}(\Omega)$, $B = L^1(\Omega)$, $Y = W_\star^{-1,1}(\Omega)$.

Aubin-Simon' Compactness Lemma, improvment

X, B, Y are three Banach spaces such that, $X \subset B$,
If $(\|w_n\|_X)_{n\in\mathbb{N}}$ is bounded, then,

- up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- if $w_n \to w$ in B and $\|w_n\|_Y \to 0$, then $w = 0$.

Let $T > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),X)$,
- $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),Y)$.

Then there exists $u \in L^1((0,T),B)$ such that, up to a subsequence, $u_n \to u$ in $L^1((0,T),B)$.

Example: $X = W_0^{1,1}(\Omega)$, $B = L^1(\Omega)$, $Y = W_\star^{-1,1}(\Omega)$.

Continuous approx., compactness of the sequence $(u_n)_{n\in\mathbb{N}}$

u_n is solution of he continuous problem with data f_n and $u_{0,n}$.

$X = W_0^{1,1}(\Omega)$, $B = L^1(\Omega)$, $Y = W_\star^{-1,1}(\Omega)$.

In order to apply Aubin-Simon' lemma we need

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),X)$,
- $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),Y)$.

The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^q((0,T),W_0^{1,q}(\Omega))$ (for $1 \le q < (d+2)/(d+1)$) and then is bounded in $L^1((0,T),X)$, since $W_0^{1,q}(\Omega)$ is continuously embedded in $W_0^{1,1}(\Omega)$.

$\partial_t u_n = f_n - \mathrm{div}(v u_n) - \Delta u_n$. Is $(\partial_t u_n)_{n\in\mathbb{N}}$ bounded in $L^1((0,T),Y)$?

Continuous approx., Compactness of the sequence $(u_n)_{n\in\mathbb{N}}$

Bound of $(\partial_t u_n)_{n\in\mathbb{N}}$ in $L^1((0,T), W_\star^{-1,1}(\Omega))$?
$\partial_t u_n = f_n - \mathrm{div}(v u_n) - \Delta u_n$.

- $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^1(0,T), L^1(\Omega))$ and then in $L^1((0,T), W_\star^{-1,1}(\Omega))$, since $L^1(\Omega)$ is continously embedded in $W_\star^{-1,1}(\Omega)$,
- $(\mathrm{div}(v u_n))_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), W_\star^{-1,1}(\Omega))$ since $(v u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), (L^1(\Omega))^d$ and div is a continuous operator from $(L^1(\Omega))^d$ to $W_\star^{-1,1}(\Omega)$,
- $(\Delta u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), W_\star^{-1,1}(\Omega))$ since $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), W_0^{1,1}(\Omega))$ and Δ is a continuous operator from $W_0^{1,1}(\Omega)$ to $W_\star^{-1,1}(\Omega)$.

Finally, $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), W_\star^{-1,1}(\Omega))$.
Aubin-Simon' lemma gives (up to a subsequence) $u_n \to u$ in $L^1((0,T), L^1(\Omega))$.

Regularity of the limit

$u_n \to u$ in $L^1(\Omega \times (0,T))$ and $(u_n)_{n\in\mathbb{N}}$ bounded in $L^q((0,T), W_0^{1,q}(\Omega))$ for $1 \le q < (d+2)/(d+1)$. Then

$$u_n \to u \text{ in } L^q(\Omega \times (0,T))) \text{ for } 1 \le q < \frac{d+2}{d+1},$$

$$\nabla u_n \to \nabla u \text{ weakly in } L^q(\Omega \times (0,T))^d \text{ for } 1 \le q < \frac{d+2}{d+1},$$

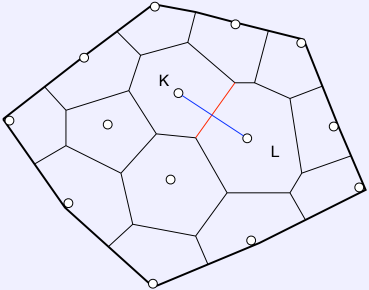
$$u \in L^q((0,T), W_0^{1,q}(\Omega)) \text{ for } 1 \le q < (d+2)/(d+1).$$

Remark: $L^q((0,T), L^q(\Omega)) = L^q(\Omega \times (0,T))$

An additional work is needed to prove the strong convergence of ∇u_n to ∇u.

Full approximation, FV scheme

Space discretization: Admissible mesh $\mathcal{M}$. Time step: k ($Nk = T$)



$T_{K,L} = m_{K,L} / d_{K,L}$

$\mathrm{size}(\mathcal{M}) = \sup\{\mathrm{diam}(K), K \in \mathcal{M}\}$
Unknowns: $u_K^{(p)} \in \mathbb{R}$, $K \in \mathcal{M}$, $p \in \{1, \ldots, N\}$.
Discretization: Implicit in time, upwind for convection, classical 2-points flux for diffusion. (Well known scheme.)

Full approximation, approximate solution

- $H_{\mathcal{M}}$ the space of functions from Ω to $\mathbb{R}$, constant on each K, $K \in \mathcal{M}$.
- The discrete solution u is constant on $K \times ((p-1)k, pk)$ with $K \in \mathcal{M}$ and $p \in \{1, \ldots, N\}$.
  $u(\cdot, t) = u^{(p)}$ for $t \in ((p-1)k, pk)$ and $u^{(p)} \in H_{\mathcal{M}}$.
- Discrete derivatives in time, $\partial_{t,k} u$, defined by:

$$\partial_{t,k} u(\cdot, t) = \partial_{t,k}^{(p)} u = \frac{1}{k}(u^{(p)} - u^{(p-1)}) \text{ for } t \in ((p-1)k, pk),$$

for $p \in \{2, \ldots, N\}$ (and $\partial_{t,k} u(\cdot, t) = 0$ for $t \in (0, k)$).

Full approximation, steps of the proof of convergence

Sequence of meshes and time steps, $(\mathcal{M}_n)_{n\in\mathbb{N}}$ and k_n.
$\mathrm{size}(\mathcal{M}_n) \to 0$, $k_n \to 0$, as $n \to \infty$.
For $n \in \mathbb{N}$, u_n is the solution of the FV scheme.

1. Estimate on u_n.
2. Strong compactness of the sequence $(u_n)_{n\in\mathbb{N}}$.
3. Regularity of the limit of the sequence $(u_n)_{n\in\mathbb{N}}$.
4. Passage to the limit in the approximate equation.

Discrete norms

Admissible mesh: $\mathcal{M}$.

$u \in H_{\mathcal{M}}$ (that is u is a function constant on each K, $K \in \mathcal{M}$).

- $1 \leq q < \infty$. Discrete $W_0^{1,q}$-norm:

$$\|u\|_{1,q,\mathcal{M}}^q = \sum_{\sigma \in \mathcal{E}_{int}, \sigma = K|L} m_\sigma d_\sigma \left|\frac{u_K - u_L}{d_\sigma}\right|^q + \sum_{\sigma \in \mathcal{E}_{ext}, \sigma \in \mathcal{E}_K} m_\sigma d_\sigma \left|\frac{u_K}{d_\sigma}\right|^q$$

- $q = \infty$. Discrete $W_0^{1,\infty}$-norm: $\|u\|_{1,\infty,\mathcal{M}}^q = \max\{M_i, M_e, M\}$ with

$$M_i = \max\{\frac{|u_K - u_L|}{d_\sigma},\ \sigma \in \mathcal{E}_{int}, \sigma = K|L\},$$

$$M_e = \max\{\frac{|u_K|}{d_\sigma},\ \sigma \in \mathcal{E}_{ext}, \sigma \in \mathcal{E}_K\},$$

$$M = \max\{|u_K|,\ K \in \mathcal{M}\}.$$

Discrete dual norms

Admissible mesh: $\mathcal{M}$.
For $r \in [1, \infty]$, $\| \cdot \|_{-1,r,\mathcal{M}}$ is the dual norm of the norm $\| \cdot \|_{1,q,\mathcal{M}}$ with $q = r/(r-1)$. That is, for $u \in H_{\mathcal{M}}$,

$$\|u\|_{-1,r,\mathcal{M}} = \max\{\int_\Omega uv\,dx,\ v \in H_{\mathcal{M}}, \|v\|_{1,q,\mathcal{M}} \leq 1\}.$$

Example: $r = 1$ $(q = \infty)$.

Full discretization, estimate on the discrete solution

For $1 \leq q < (d+2)/(d+1)$, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^q((0,T),W_{q,n})$, where $W_{q,n}$ is the space $H_{\mathcal{M}_n}$, endowed with the norm $\|\cdot\|_{1,q,\mathcal{M}_n}$. That is

$$\sum_{p=1}^{N_n} k\|u_n^{(p)}\|_{1,q,\mathcal{M}_n}^q \leq C.$$

Discrete Lions' lemma (improved)

B is a Banach space, $(B_n)_{n\in\mathbb{N}}$ is a sequence of finite dimensional subspaces of B. $\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$ are two norms on B_n such that:
If $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then,

- up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- If $w_n \to w$ in B and $\|w_n\|_{Y_n} \to 0$, then $w = 0$.

Then, for any $\varepsilon > 0$, there exists C_ε such that, for $n \in \mathbb{N}$ and $w \in B_n$

$$\|w\|_B \le \varepsilon\|w\|_{X_n} + C_\varepsilon\|w\|_{Y_n}.$$

Example: $B = L^1(\Omega)$. $B_n = H_{\mathcal{M}_n}$ (the finite dimensional space given by the mesh $\mathcal{M}_n$). We have to choose $\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$.

Discrete Lions' lemma, proof

Proof by contradiction. There exists $\varepsilon > 0$ and $(w_n)_{n\in\mathbb{N}}$ such that, for all n, $w_n \in B_n$ and

$$\|w_n\|_B > \varepsilon \|w_n\|_{X_n} + C_n \|w_n\|_{Y_n},$$

with $\lim_{n\to\infty} C_n = +\infty$.

It is possible to assume that $\|w_n\|_B = 1$. Then $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded and, up to a subsequence, $w_n \to w$ in B (so that $\|w\|_B = 1$). But $\|w_n\|_{Y_n} \to 0$, so that $w = 0$, in contradiction with $\|w\|_B = 1$.

Discrete Aubin-Simon' Compactness Lemma

B a Banach, $(B_n)_{n\in\mathbb{N}}$ family of finite dimensional subspaces of B.
$\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$ two norms on B_n such that:
If $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then,

- up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- If $w_n \to w$ in B and $\|w_n\|_{Y_n} \to 0$, then $w = 0$.

$X_n = B_n$ with norm $\|\cdot\|_{X_n}$, $Y_n = B_n$ with norm $\|\cdot\|_{Y_n}$. Let $T > 0$, $k_n > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- for all n, $u_n(\cdot, t) = u_n^{(p)} \in B_n$ for $t \in ((p-1)k_n, pk_n)$
- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), X_n)$,
- $(\partial_{t,k_n} u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), Y_n)$.

Then there exists $u \in L^1((0,T), B)$ such that, up to a subsequence, $u_n \to u$ in $L^1((0,T), B)$.

Example: $B = L^1(\Omega)$. $B_n = H_{\mathcal{M}_n}$. What choice for $\|\cdot\|_{X_n}$, $\|\cdot\|_{Y_n}$?

Full approx., compactness of the sequence $(u_n)_{n\in\mathbb{N}}$

u_n is solution of the fully discretized problem with mesh $\mathcal{M}_n$ and time step k_n.

$B = L^1(\Omega)$, $B_n = H_{\mathcal{M}_n}$,
$\|\cdot\|_{X_n} = \|\cdot\|_{1,1,\mathcal{M}_n}$, $\|\cdot\|_{Y_n} = \|\cdot\|_{-1,1,\mathcal{M}_n}$

In order to apply the discrete Aubin-Simon' lemma we need to verify the hypotheses of the discrete Lions' lemma and that

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),X_n)$,
- $(\partial_{t,k_n} u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),Y_n)$.

The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^q((0,T),W_{q,n}(\Omega))$ (for $1 \le q < (d+2)/(d+1)$) and then is bounded in $L^1((0,T),X_n)$ since $\|\cdot\|_{1,1,\mathcal{M}_n} \le C_q\|\cdot\|_{1,q,\mathcal{M}_n}$ for $q > 1$.

Using the scheme, it is quite easy to prove (similarly to the continuous approximation) that $(\partial_{t,k_n} u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),Y_n)$.

Full approx., Compactness of the sequence $(u_n)_{n\in\mathbb{N}}$

It remains to verify the hypotheses of the discrete Lions' lemma.

- If $w_n \in H_{\mathcal{M}_n}$, $(\|w_n\|_{1,1,\mathcal{M}_n})_{n\in\mathbb{N}}$ is bounded, there exists $w \in L^1(\Omega)$ such that $w_n \to w$ in $L^1(\Omega)$?
  Yes, this is classical now...
- If $w_n \in H_{\mathcal{M}_n}$, $w_n \to w$ in $L^1(\Omega)$ and $\|w_n\|_{-1,1,\mathcal{M}_n} \to 0$, then $w = 0$? Yes... Proof :
  Let $\varphi \in W_0^{1,\infty}(\Omega)$ and its "projection" $\pi_n\varphi \in H_{\mathcal{M}_n}$. One has $\|\pi_n\varphi\|_{1,\infty,\mathcal{M}_n} \le \|\varphi\|_{W^{1,\infty}(\Omega)}$ and then

  $$|\int_\Omega w_n(\pi_n\varphi)dx| \le \|w_n\|_{-1,1,\mathcal{M}_n}\|\varphi\|_{W^{1,\infty}(\Omega)} \to 0,$$

  and, since $w_n \to w$ in $L^1(\Omega)$ and $\pi_n\varphi \to \varphi$ uniformly,

  $$\int_\Omega w_n(\pi_n\varphi)dx \to \int_\Omega w\varphi dx.$$

  This gives $\int_\Omega w\varphi dx = 0$ for all $\varphi \in W_0^{1,\infty}(\Omega)$ and then $w = 0$ a.e.

Regularity of the limit

As in the continuous approximation,
$u_n \to u$ in $L^1(\Omega \times (0,T))$ and $(u_n)_{n\in\mathbb{N}}$ bounded in
$L^q((0,T), W_{q,n}(\Omega))$ for $1 \le q < (d+2)/(d+1)$. Then

$$u_n \to u \text{ in } L^q(\Omega \times (0,T))) \text{ for } 1 \le q < \frac{d+2}{d+1},$$

$$u \in L^q((0,T), W_0^{1,q}(\Omega)) \text{ for } 1 \le q < (d+2)/(d+1).$$

Discrete Aubin-Simon' Compactness Lemma

B a Banach, $(B_n)_{n\in\mathbb{N}}$ family of finite dimensional subspaces of B.
$\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$ two norms on B_n such that:
If $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then,

- up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- If $w_n \to w$ in B and $\|w_n\|_{Y_n} \to 0$, then $w = 0$.

$X_n = B_n$ with norm $\|\cdot\|_{X_n}$, $Y_n = B_n$ with norm $\|\cdot\|_{Y_n}$. Let $T > 0$, $k_n > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- for all n, $u_n(\cdot, t) = u_n^{(p)} \in B_n$ for $t \in ((p-1)k_n, pk_n)$
- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), X_n)$,
- $(\partial_{t,k_n} u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), Y_n)$.

Then there exists $u \in L^1((0,T), B)$ such that, up to a subsequence, $u_n \to u$ in $L^1((0,T), B)$.

Stefan problem

$$\begin{aligned}
&\partial_t u - \Delta\varphi(u) = f \text{ in } \Omega \times (0,T),\\
&u = 0 \text{ on } \partial\Omega \times (0,T),\\
&u(\cdot,0) = u_0 \text{ in } \Omega.
\end{aligned}$$

- Ω is a polygonal (for $d=2$) or polyhedral (for $d=3$) open subset of $\mathbb{R}^d$ ($d = 2$ or 3), $T > 0$
- φ is a non decreasing function from $\mathbb{R}$ to $\mathbb{R}$, Lipschitz continuous and $\liminf_{s\to+\infty} \varphi(s)/s > 0$
- $u_0 \in L^2(\Omega)$
- $f \in L^2(\Omega \times (0,T))$

Mail difficulty : φ may be constant on some interval of $\mathbb{R}$

Objective : To present a general framework to prove the convergence of many different schemes (FE, NCFE, FV, HFV…)

Discrete unknown

Discretization parameters, $\mathcal{D}$: spatial mesh, time step (δt)

Discrete unknown at time $t_k = k\delta t$: $u^{(k)} \in X_{\mathcal{D},0}$.

- values at the vertices of the mesh (FE)
- values at the edges of the mesh (NCFE)
- values in the cells (FV)
- values in the cells and in the edges (HFV)

With an element v of $X_{\mathcal{D},0}$ (for instance $v = u^{(k)}$ or $v = \varphi(u^{(k)})$), one defines two functions

- $\bar{v}$ (reconstruction of the approximate solution)
- $\nabla_{\mathcal{D}} v$ (reconstruction of an approximate gradient)

with some natural properties of consistency.

A crucial property is $\overline{\varphi(u)} = \varphi(\bar{u})$

N.B. the functions $\bar{v}$ and $\nabla_{\mathcal{D}} v$ are piecewise constant functions, but not necessarily on the same mesh

Numerical scheme (Gradient schemes)

$\bar{u}^{(0)}$ given by the initial condition and for $k \geq 0$,
$u^{(k+1)} \in X_{\mathcal{D},0}$

$$\int_\Omega \frac{\bar{u}^{(k+1)} - \bar{u}^{(k)}}{\delta t} \bar{v} dx dt + \int_\Omega \nabla_{\mathcal{D}} \varphi(u^{(k+1)}) \cdot \nabla_{\mathcal{D}} v dx = \frac{1}{\delta t} \int_{t_k}^{t_{k+1}} f \bar{v} dx dt, \ \forall v \in X_{\mathcal{D},0}$$

Classical examples : FE with mass lumping, FV
but also many other schemes. . .

Steps of the proof of convergence

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of approximate solutions (associated to $\mathcal{D}_n$ and δt_n with $\lim_{n\to\infty} \mathrm{size}(\mathcal{D}_n) = 0$ and $\lim_{n\to\infty} \delta t_n = 0$)

1. Estimates on the approximate solution
2. Compactness result on the sequence of approximate solutions
3. Passage to the limit in the approximate equation

Steps 2 and 3 are tricky due to the fact that φ may be constant on some interval of $\mathbb{R}$

Estimates

One mimics the estimates for the continuous equation

$$
\begin{aligned}
&\partial_t u - \Delta\varphi(u) = f \text{ in } \Omega \times (0, T),\\
&u = 0 \text{ on } \partial\Omega \times (0, T),\\
&u(\cdot, 0) = u_0 \text{ in } \Omega.
\end{aligned}
$$

Taking $\varphi(u)$ as test function one obtains

- an estimate on u in $L^\infty((0, T), L^2(\Omega))$
- an estimate on $\varphi(u)$ in $L^2((0, T), H_0^1(\Omega))$
- and therefore an estimate on $\partial_t u$ in $L^2((0, T), H^{-1}(\Omega))$

Estimates with corresponding discrete norms hold for the discrete setting of gradient schemes : $L^\infty((0, T), L^2(\Omega))$-estimate on $\bar{u}$, $L^2((0, T), L^2(\Omega))$-estimate on $\nabla_{\mathcal{D}}\varphi(u)$ and an estimate on the time discrete derivative for a dual norm

Estimates (2)

These estimates give only weak compactness on the sequences of approximate solutions $(u_n)_{n\in\mathbb{N}}$ and $(\varphi(u_n))_{n\in\mathbb{N}}$. Not sufficient to pass to the limit...

$$\lim_{n\to\infty}\varphi(u_n) = \varphi(\lim_{n\to\infty} u_n)?$$

Lions-Aubin-Simon Compactness Lemma

X, B, Y are three Banach spaces such that

- $X \subset B$ with compact embedding,
- $B \subset Y$ with continuous embedding.

Let $T > 0$, $1 \leq p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),X)$,
- $(\partial_t v_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),Y)$.

Then there exists $v \in L^p((0,T),B)$ such that, up to a subsequence, $v_n \to v$ in $L^p((0,T),B)$.

Example: $p = 2$, $X = H_0^1(\Omega)$, $B = L^2(\Omega)$, $Y = H^{-1}(\Omega)$.

A dicrete version with a family a spaces $(X_n)_{n\in\mathbb{N}}$ and a family a spaces $(Y_n)_{n\in\mathbb{N}}$ is possible.

The Lions-Aubin-Simon lemma is of no use here

- $(\partial_t u_n)_{n\in\mathbb{N}}$ bounded in $L^2((0,T),H^{-1}(\Omega))$
- $\varphi(u_n)_{n\in\mathbb{N}}$ bounded in $L^2((0,T),H_0^1(\Omega))$

Unfortunately,

- the estimate on $(\varphi(u_n))_{n\in\mathbb{N}}$ does not give an analogue estimate on $(u_n)_{n\in\mathbb{N}}$ (since φ may be constant on some interval). It gives only $(u_n)_{n\in\mathbb{N}}$ bounded in $L^2((0,T),L^2(\Omega))$
- the estimate on $(\partial_t u_n)_{n\in\mathbb{N}}$ does not give an analogue estimate on $(\partial_t\varphi(u_n))_{n\in\mathbb{N}}$ (the product of an $L^\infty(\Omega)$ function with a $H^{-1}(\Omega)$ element is not well defined)

One cannot use Lions-Aubin-Simon Compactness lemma on the sequence $(u_n)_{n\in\mathbb{N}}$ nor on the sequence $(\varphi(u_n))_{n\in\mathbb{N}}$

Between Kolmogorov and Aubin-Simon

X, B are two Banach spaces such that

- $X \subset B$ with compact embedding,

Let $T > 0$, $1 \leq p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),X)$,
- $\|v_n(\cdot + h) - v_n\|_{L^p((0,T-h),B)} \to 0$, as $h \to 0_+$, unif. w.r.t. n.

Then there exists $v \in L^p((0,T),B)$ such that, up to a subsequence, $v_n \to v$ in $L^p((0,T),B)$.

Example: $p = 2$, $X = H_0^1(\Omega)$, $B = L^2(\Omega)$

Here also, a dicrete version with a family a spaces $(X_n)_{n\in\mathbb{N}}$ is possible.

Alt-Luckhaus method for the Stefan problem

One knows that $\varphi(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2((0,T),H_0^1(\Omega))$. To obtain compactness of $\varphi(u_n)_{n\in\mathbb{N}}$ in $L^2((0,T),L^2(\Omega))$ one has to prove that $\|\varphi(u_n)(\cdot+h)-\varphi(u_n)\|_{L^2((0,T-h),L^2(\Omega))}\to 0_+$, as $h\to 0$, uniformly w.r.t. n. (For simplicity, $f=0$.)

$$\partial_t u_n(s)-\Delta\varphi(u_n(s))=0,\ s\in(t,t+h).$$

One multiplies by $\varphi(u_n(t+h))-\varphi(u_n(t))$ and integrate between t and $t+h$ and on Ω

$$\int_t^{t+h}\int_\Omega \partial_t u_n(s)(\varphi(u_n(t+h))-\varphi(u_n(t)))dxds$$
$$+\int_t^{t+h}\int_\Omega \nabla\varphi(u_n(s))\cdot(\nabla\varphi(u_n(t+h))-\nabla\varphi(u_n(t)))dxds.$$

AL method for the Stefan problem (2)

$$\int_t^{t+h}\int_\Omega \partial_t u_n(s)(\varphi(u_n(t+h))-\varphi(u_n(t)))dxds$$
$$+\int_t^{t+h}\int_\Omega \nabla\varphi(u_n(s))\cdot(\nabla\varphi(u_n(t+h))-\nabla\varphi(u_n(t)))dxds=0.$$
$$\int_\Omega (u_n(t+h))-u_n(t))(\varphi(u_n(t+h))-\varphi(u_n(t)))dx \leq$$
$$\int_t^{t+h}\int_\Omega |\nabla\varphi(u_n(s))||\nabla\varphi(u_n(t+h))|+|\nabla\varphi(u_n(s))||\nabla\varphi(u_n(t))|dxds.$$

One now integrates on $t\in(0,T-h)$, uses a Lipschitz constant for φ (denoted L) and $ab\leq(a^2+b^2)/2$

$$\int_0^{T-h}\int_\Omega (\varphi(u_n(t+h))-\varphi(u_n(t)))^2dx \leq$$
$$L\int_0^{T-h}\int_\Omega (u_n(t+h))-u_n(t))(\varphi(u_n(t+h))-\varphi(u_n(t)))dx \leq$$
$$L\sum_{i=1}^3 T_i$$

AL method for the Stefan problem (3)

$$\int_0^{T-h} \int_\Omega (\varphi(u_n(t+h)) - \varphi(u_n(t)))^2 dx \leq L(T_1 + T_2 + T_3)$$

$$T_1 = \int_0^{T-h} \int_t^{t+h} \int_\Omega |\nabla \varphi(u_n(s))|^2 dxdsdt \leq h \||\nabla \varphi(u_n)|\|^2_{L^2(Q)}$$

$$T_2 = \int_0^{T-h} \int_t^{t+h} \int_\Omega |\nabla \varphi(u_n(t+h))|^2 dxdsdt \leq h \||\nabla \varphi(u_n)|\|^2_{L^2(Q)}$$

$$T_3 = \int_0^{T-h} \int_t^{t+h} \int_\Omega |\nabla \varphi(u_n(t))|^2 dxdsdt \leq h \||\nabla \varphi(u_n)|\|^2_{L^2(Q)}$$

where $Q = \Omega \times (0, T)$.

Thanks to the $L^2((0,T), H_0^1(\Omega))$ estimate on $(\varphi(u_n))_{n\in\mathbb{N}}$, one obtains the relative compactness of this sequence in $L^2(Q)$.

Translation (in time) of $\varphi(u_n)$, at the discrete level

At the discrete level, let u_n be the approximate solution associated to mesh $\mathcal{D}_n$ and time step δt_n. A very similar proof gives

$$\int_0^{T-h}\int_\Omega (\varphi(\bar{u}_n(t+h)) - \varphi(\bar{u}_n(t)))^2 dx \leq h\||\nabla_{\mathcal{D}}\varphi(u_n)|\|^2_{L^2(Q)}$$

The only difference is due to the fact that $\partial_t u$ is replaced by a differential quotient.
For this proof, the crucial property $\overline{\varphi(u)} = \varphi(\bar{u})$ is used

Compactness, for a sequence of approximate solutions

X, B are two Banach spaces such that

- $X \subset B$ with compact embedding,

Let $T > 0$, $1 \leq p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),X)$,
- $\|v_n(\cdot + h) - v_n\|_{L^p((0,T-h),B)} \to 0$, as $h \to 0_+$, unif. w.r.t. n.

Then there exists $v \in L^p((0,T),B)$ such that, up to a subsequence, $v_n \to v$ in $L^p((0,T),B)$.

Example: $p = 2$, $X = H^1_0(\Omega)$, $B = L^2(\Omega)$
One wants to take $v_n = \varphi(\bar{u}_n)$.

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Then there exists $v \in L^p((0,T),B)$ such that, up to a subsequence, $v_n \to v$ in $L^p((0,T),B)$.

Example: $p = 2$, $X = H^1_0(\Omega)$, $B = L^2(\Omega)$
One wants to take $v_n = \varphi(\bar{u}_n)$.
Everything is ok, except that there is no X-space...

Modified Compactness Lemma

B is a banach space ($B = L^2(Q)$)
X_n normed vector spaces ($X_n = X_{\mathcal{D}_n,0}$, $\|u\|_{X_n} = \||\nabla_{\mathcal{D}_n} u|\|_{L^2}$)
T_n a linear operator from X_n to B ($T_n(u) = \bar{u}$)
The hypothesis $X \subset B$ with compact embedding is replaced by "$u_n \in X_n$, if the sequence $(\|u_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then the sequence $(T_n(u_n))_{n\in\mathbb{N}}$ is relatively compact in B".
With this hypothesis, let $T > 0$, $1 \leq p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that $v_n \in L^p((0,T), X_n)$ for all n. Assume that

- There exists C such that $\|v_n\|_{L^p((0,T),X_n)} \leq C$ for all $n \in \mathbb{N}$
- $\|T_n(v_n)(\cdot + h) - T_n(v_n)\|_{L^p((0,T-h),B)} \to 0$, as $h \to 0_+$, uniformly w.r.t. n.

Then there exists $g \in L^p((0,T), B)$ such that, up to a subsequence, $T_n(v_n) \to g$ in $L^p((0,T), B)$.

$p = 2$, $v_n = \varphi(u_n)$. With this Compactness Lemma, one obtains that $\varphi(\bar{u}_n) \to g$ in $L^2(Q)$

Minty trick (simple version)

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of approximate solutions. One has, as $n \to \infty$,

$$\bar{u}_n \to u \text{ weakly in } L^2(Q),$$

$$\varphi(\bar{u}_n) \to g \text{ in } L^2(Q).$$

Then, the Minty trick (since φ is nondecreasing) gives $g = \varphi(u)$:
Let $w \in L^2(\Omega)$, $0 \leq \int_Q (\varphi(\bar{u}_n) - \varphi(w))(\bar{u}_n - w)dxdt$ gives, as $n \to \infty$,

$$0 \leq \int_Q (g - \varphi(w))(u - w)dxdt.$$

Taking $w = u + \varepsilon\psi$, with $\psi \in C_c^\infty(Q)$ and letting $\varepsilon \to 0^\pm$ leads to

$$\int_Q (g - \varphi(u))\psi dxdt = 0.$$

Then $g = \varphi(u)$ a.e.

Passing to the limit in the equation

It remains to pass to the limit in the approximate equation. This is possible thanks to some natural properties of consistency. That is to say, for any regular function ψ, as $\mathrm{size}(\mathcal{D}) \to 0$,

1. $\min_{v \in X_{\mathcal{D},0}} \|\bar{v} - \psi\|_{L^2(\Omega)} \to 0$
2. $\min_{v \in X_{\mathcal{D},0}} \||\nabla_{\mathcal{D}} v - \nabla\psi|\|_{L^2(\Omega)} \to 0$
3. $\max_{u \in X_{\mathcal{D},0} \setminus \{0\}} \frac{1}{\||\nabla_{\mathcal{D}} u|\|_{L^2(\Omega)}} \left| \int_\Omega \left(\nabla_{\mathcal{D}} u \cdot \psi + \bar{u}\,\mathrm{div}\psi \right) dx \right| \to 0$

Modified Compactness Lemma

B is a banach space
X_n normed vector spaces
T_n a linear operator from X_n to B
The hypothesis $X \subset B$ with compact embedding is replaced by "$u_n \in X_n$, if the sequence $(\|u_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then the sequence $(T_n(u_n))_{n\in\mathbb{N}}$ is relatively compact in B".
With this hypothesis, let $T > 0$, $1 \leq p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that $v_n \in L^p((0,T), X_n)$ for all n. Assume that

- There exists C such that $\|v_n\|_{L^p((0,T),X_n)} \leq C$ for all $n \in \mathbb{N}$
- $\|T_n(v_n)(\cdot + h) - T_n(v_n)\|_{L^p((0,T-h),B)} \to 0$, as $h \to 0$, uniformly w.r.t. n.

Then there exists $g \in L^p((0,T), B)$ such that, up to a subsequence, $T_n(v_n) \to g$ in $L^p((0,T), B)$.

Compactness Lemma, simple case

B is a banach space
X_n normed vector spaces
The sequence X_n is compactly embeded in B
$T > 0$, $1 \leq p < +\infty$

- $(v_n)_{n\in\mathbb{N}}$ bounded in $L^p((0,T),X_n)$
- $\|v_n(\cdot + h) - v_n\|_{L^p((0,T-h),B)} \to 0$, as $h \to 0$, unif. w.r.t. n.

Then there exists $v \in L^p((0,T),B)$ such that, up to a subsequence, $v_n \to v$ in $L^p((0,T),B)$.