Very high-order finite volume scheme for the 2D linear convection-diffusion problem

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SHARING HIGHER-ORDER ADVANCED RESEARCH KNOW-HOW on FINITE VOLUME
Ofir, Portugal
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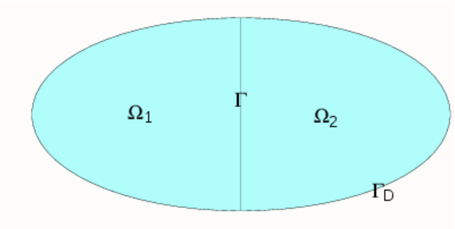
Outline

- 2D linear convection-diffusion with finite volume
- Discretization and residual formulation
- Polynomial reconstructions
- Scheme design
- Numerical tests

Model

Find $\phi = (\phi_1, \phi_2)$ on the bounded open domain Ω such that

$$
\begin{aligned}
&\nabla.(V_1\phi_1 - k_1\nabla\phi_1) = f_1, && \text{in } \Omega_1, && \text{(1a)}\\
&\nabla.(V_2\phi_2 - k_2\nabla\phi_2) = f_2, && \text{in } \Omega_2, && \text{(1b)}\\
&k_1\nabla\phi_1.n_\Gamma = k_2\nabla\phi_2.n_\Gamma, && \text{on } \Gamma, && \text{(1c)}\\
&\phi = \phi_D, && \text{on } \Gamma_D, && \text{(1d)}
\end{aligned}
$$



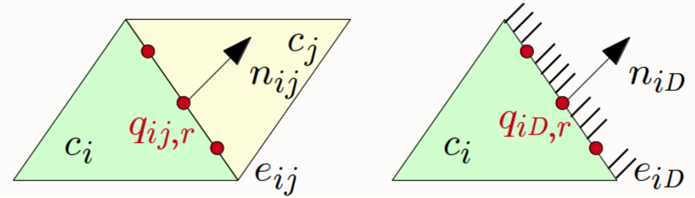
$\phi_1 = \phi_2$ on Γ

or

$k_1\nabla\phi_1.n_\Gamma = h(\phi_2 - \phi_1)$.

Discretization

$$\int_{\partial c_i} (V.n\phi - k\nabla\phi.n)ds - \int_{c_i} f\,dx = 0. \text{ (divergence theorem)}$$



$$\sum_{j\in\nu(i)} |e_{ij}| \sum_{r=1}^{R} \zeta_r \Big[V(q_{ij,r}).n_{ij}\phi(q_{ij,r}) - k(q_{ij,r})\nabla\phi(q_{ij,r}).n_{ij} \Big]$$

$$-|c_i|f_i = \mathcal{O}(h_i^{2R}). \text{ (Gauss Points)}$$

Residual formulation

Based on the previous expression: the residual formulation is

$$G_i = \sum_{j \in \nu(i)} \frac{|e_{ij}|}{|c_i|} \sum_{r=1}^{R} \zeta_r \mathcal{F}_{ij,r} - f_i,$$

where

$$\mathcal{F}_{ij,r} \approx V(q_{ij,r}).n_{ij}\phi(q_{ij,r}) - k(q_{ij,r})\nabla\phi(q_{ij,r}).n_{ij},$$

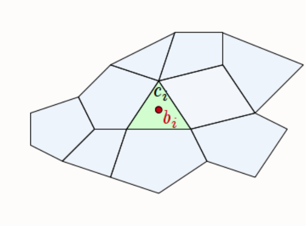
approximation of the flux at the Gauss point $q_{ij,r}$.

☞ Sixth-order approximation for $\mathcal{F}_{ij,r}$.

Polynomial reconstructions

Conservative reconstruction for cells

c_i: cell of mesh $\mathcal{T}_h$,
d: the polynomial degree,
$S(c_i, d)$: the associated stencil,
ϕ_i: mean value on c_i.



$$\widehat{\phi}_i(x; d) = \phi_i + \sum_{1 \leqslant |\alpha| \leqslant d} \mathfrak{R}_i^{d,\alpha} \left\{ (x - b_i)^\alpha - M_i^\alpha \right\}$$

$\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $x = (x_1, x_2)$, b_i the centroid of cell c_i

☞ Set $M_i^\alpha = \dfrac{1}{|c_i|} \displaystyle\int_{c_i} (x - b_i)^\alpha \, dx$ to provide the conservation.

Coefficients for $\widehat{\phi}_i$

$\mathfrak{R}_i^d$ vector gathering coefficients $\mathfrak{R}_i^{d,\alpha}$

Assume mean values ϕ_ℓ on cells c_ℓ, $\ell \in S(c_i, d)$ are known,

$\widehat{\mathfrak{R}}_i^d$ minimizes the functional

$$E_i(\mathfrak{R}_i^d; d) = \sum_{\ell \in S(c_i,d)} \left[\frac{1}{|c_\ell|} \int_{c_\ell} \widehat{\phi}_i(x; d)\, dx - \phi_\ell \right]^2,$$

☞ Lead to an over-determined linear system $\mathcal{A}_i^d \mathfrak{R}_i^d = \mathfrak{b}_i^S$ where $\mathfrak{b}_i^S$ represents the variations $\phi_\ell - \phi_i$, $\ell \in S(c_i; d)$

Preconditioning and solving

Determine the Moore-Penrose pseudo-inverse matrix for system $(\mathcal{A}_i^d \mathcal{P}_i^d)(\mathcal{P}_i^d)^{-1}\mathfrak{R}_i^d = \mathfrak{b}_i^S$ with the diagonal matrix

$$\mathcal{P}_i^d = \mathrm{diag}(|c_i|^{-|\alpha|/2})_{1 \leqslant |\alpha| \leqslant d}.$$

Motivation: the $\mathcal{A}_i^d$ matrix coefficients are

$$\frac{1}{|c_\ell|}\int_{c_\ell}(x-b_i)^\alpha dx - \frac{1}{|c_i|}\int_{c_i}(x-b_i)^\alpha dx.$$

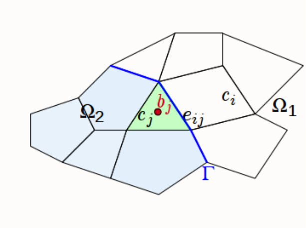
☞ Strongly reduces the effect of the power α.

We compute the pseudo inverse matrix $(\mathcal{A}_i^d \mathcal{P}_i^d)^\dagger$ and get

$$\mathfrak{R}_i^d = \mathcal{P}_i^d (\mathcal{A}_i^d \mathcal{P}_i^d)^\dagger \mathfrak{b}_i^S.$$

Conservative reconstruction for Γ

$e_{ij} \subset \Gamma$: edge on the interface,
c_j the cell on the Ω_2 side,
d: the polynomial degree,
$S(c_j, d)$: the associated stencil.



$$\check{\phi}_j(x; d) = \phi_j + \sum_{1 \leqslant |\alpha| \leqslant d} \mathfrak{R}_j^{d,\alpha} \Big\{ (x - b_j)^\alpha - M_j^\alpha \Big\}.$$

b_j the centroid of cell c_j.

☞ Set $M_j^\alpha = \dfrac{1}{|c_j|} \displaystyle\int_{c_j} (x - b_j)^\alpha \, dx$ to provide the conservation.

☞ Only use the cells on the "j" side.

Coefficients for $\check{\phi}_j$

$\mathfrak{R}_j^d$ vector gathering coefficients $\mathfrak{R}_j^{d,\alpha}$

Assume mean values ϕ_ℓ on cells c_ℓ, $\ell \in S(e_{ij}, d)$ and ϕ_{ij} the mean value on e_{ij} are known,

$\check{\mathfrak{R}}_j^d$ minimizes the functional

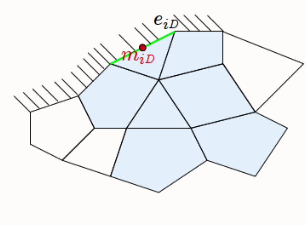
$$E_j(\mathfrak{R}_j^d; d) = \sum_{\ell \in S(c_j, d)} \Big[\frac{1}{|c_\ell|} \int_{c_\ell} \check{\phi}_j(x; d)\, dx - \phi_\ell\Big]^2$$

$$+\omega_{ij}\Big[\frac{1}{|e_{ij}|} \int_{e_{ij}} \check{\phi}_j(x; d)\, ds - \phi_{ij}\Big]^2$$

with ω_{ij} a positive weight.

Conservative reconstruction for Γ_D

e_{iD}: edge on the boundary Γ_D,
d: the polynomial degree,
$S(e_{iD}, d)$: the associated stencil,
$\phi_{iD} = \dfrac{1}{|e_{iD}|} \displaystyle\int_{e_{iD}} \phi_D(s)\, ds$.



$$\widehat{\phi}_{iD}(x; d) = \phi_{iD} + \sum_{1 \leqslant |\alpha| \leqslant d} \mathfrak{R}_{iD}^{d,\alpha} \left\{ (x - m_{iD})^\alpha - M_{iD}^\alpha \right\},$$

m_{iD} the centroid of edge e_{iD}

☞ Set $M_{iD}^\alpha = \dfrac{1}{|e_{iD}|} \displaystyle\int_{e_{iD}} (x - m_{iD})^\alpha \, ds$ to provide the conservation.

Coefficients for $\widehat{\phi}_{iD}$

$\mathfrak{R}_{iD}^{d}$ vector gathering coefficients $\mathfrak{R}_{iD}^{d,\alpha}$

Assume mean values ϕ_ℓ on cells c_ℓ, $\ell \in S(e_{iD}, d)$ are known,

$\widehat{\mathfrak{R}}_{iD}^{d}$ minimizes the functional

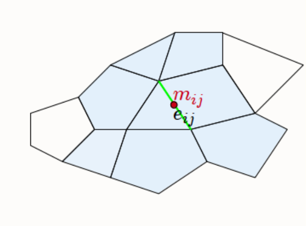
$$E_{iD}(\mathfrak{R}_{iD}^{d}; d) = \sum_{\ell \in S(e_{iD}, d)} \omega_{iD,\ell} \left[\frac{1}{|c_\ell|} \int_{c_\ell} \widehat{\phi}_{iD}(x; d)\, dx - \phi_\ell \right]^2,$$

with $\omega_{iD,\ell}$ positive weights.

☞ Coefficients $\omega_{iD,\ell}$ are very important to provide ”good” properties.

Non conservative reconstruction for inner edges

e_{ij}: inner edge of the mesh,
d: the polynomial degree,
$S(e_{ij}, d)$: the associated stencil,
no value associated to e_{ij}.



$$\tilde{\phi}_{ij}(x; d) = \sum_{0 \leqslant |\alpha| \leqslant d} \mathfrak{R}_{ij}^{d,\alpha} (x - m_{ij})^{\alpha}$$

m_{ij} the centroid of edge e_{ij}

☞ Coefficient $\mathfrak{R}_{ij}^{d,\alpha}$ for $|\alpha| = 0$ is also unknown.

Coefficients for $\widetilde{\phi}_{ij}$

$\mathfrak{R}_{ij}^{d}$ vector gathering coefficients $\mathfrak{R}_{ij}^{d,\alpha}$

Assume mean values ϕ_ℓ on cells c_ℓ, $\ell \in S(e_{ij}, d)$ are known,

$\widetilde{\mathfrak{R}}_{ij}^{d}$ minimizes the functional

$$E_{ij}(\mathfrak{R}_{ij}^{d}; d) = \sum_{\ell \in S(e_{ij}, d)} \omega_{ij,\ell} \Big[\frac{1}{|c_\ell|} \int_{c_\ell} \widetilde{\phi}_{ij}(x; d)\, dx - \phi_\ell \Big]^2,$$

where $\omega_{ij,\ell}$ are positive weights.

☞ One more time: coefficients $\omega_{ij,\ell}$ are very important.

Polynomial Reconstruction Operators

- $\Phi = (\phi_i)_{i \in \mathcal{C}}$ the mean values vector.
- Operators $\Phi \to \widehat{\phi}_i$, $\check{\phi}_j$, $\widehat{\phi}_{iD}$, and $\widetilde{\phi}_{ij}$ are linear.
- ϕ polynomial function of degree d, $\overline{\phi}_i$ the exact mean values.

d-exact reconstruction if

$$\widehat{\phi}_i(x; d) = \check{\phi}_j(x; d) = \widehat{\phi}_{iD}(x; d) = \widetilde{\phi}_{ij}(x; d) = \phi(x), \; x \in \mathbb{R}^2.$$

☞ The finite volume method associated to the polynomial reconstruction is a $d + 1^{th}$-order method.

The flux on edge (except Γ)

1. e_{ij} is an inner edge (not on Γ),

$$\mathcal{F}_{ij,r} = [V(q_{ij,r}).n_{ij}]^{+}\widehat{\phi}_i(q_{ij,r};d) + [V(q_{ij,r}).n_{ij}]^{-}\widehat{\phi}_j(q_{ij,r};d)$$

$$-k(q_{ij,r})\nabla\widetilde{\phi}_{ij}(q_{ij,r};d).n_{ij}.$$

2. e_{iD} belongs to Γ_D,

$$\mathcal{F}_{iD,r} = [V(q_{iD,r}).n_{iD}]^{+}\widehat{\phi}_i(q_{iD,r};d) + [V(q_{iD,r}).n_{iD}]^{-}\phi_D(q_{iD,r})$$

$$-k(q_{iD,r})\nabla\widehat{\phi}_{iD}(q_{iD,r};d).n_{iD}.$$

The flux on edge of $e_{ij} \subset \Gamma$

- Transfer condition: $\mathcal{F}_{ij,r} = h(q_{ij,r})[\widehat{\phi}_i(q_{ij,r};d) - \widehat{\phi}_j(q_{ij,r};d)]$.

- Continuity condition: we perform three steps

Step 1: compute the reconstructions $\widehat{\phi}_i(x;d)$ for all $c_i \subset \Omega_1$.

Step 2: compute $\phi_{ij} = \dfrac{1}{|e_{ij}|}\displaystyle\int_{e_{ij}} \widehat{\phi}_i(x;d)ds$ for all $e_{ij} \subset \Gamma$.

Step 3: $\mathcal{F}_{ij,r} = k_2(q_{ij,r})\nabla\check{\phi}_j(q_{ij,r};d).n_{ij}$.

Resolution

① The polynomial reconstruction operators are linear.
② The flux computations are linear.
③ The residual expression is linear: $\Phi \to G_i(\Phi)$.

We get a linear operator $\Phi \to G(\Phi) = (G_1(\Phi), ..., G_I(\Phi))$.

Problem: Find $\overline{\Phi}$ such that $G(\overline{\Phi}) = 0$.

☛ Matrix-free problem: use GMRES method.

Preconditioning is very very important: P preconditioning matrix

substitute $\Phi \to G(\Phi)$ by $\Phi \to PG(\Phi)$.

Preconditioning matrix

☞ $G(\Phi) = A\Phi - b$ but we do not have matrix A: ILU not possible.

Diagonal preconditioning matrix $P = D_P^{-1}$ with

$$D_P(i,i) = \frac{1}{|c_i|} \sum_{j \in \nu(i)} |e_{ij}| \left[\frac{k(b_i)}{|b_i b_j|} + [V(m_{ij}).n_{ij}]^+ \right].$$

More sophisticated preconditioning matrix, $A_P = D_P$ for the diagonal coefficients and

$$A_P(i,j) = \frac{|e_{ij}|}{|c_i|} \left[-\frac{k(m_{ij})}{|b_i b_j|} + [V(m_{ij}).n_{ij}]^- \right],\ j \in \nu(i).$$

Incomplete inverse of A_P

Preconditioning matrix is supposed to be $P = A_P^{-1}$

Substitute with the incomplete inverse $A_P^{\dagger}$ with the same non-null entries of A_P.

Taking advantage of the structure of A_P and $A_P^{\dagger}$ provides explicit construction of $A_P^{\dagger}$

$$A_P^{\dagger}(i,j) = -A_P(i,j)\frac{A_P^{\dagger}(i,i)}{A_P(j,j)},\ j \in \nu(i)$$

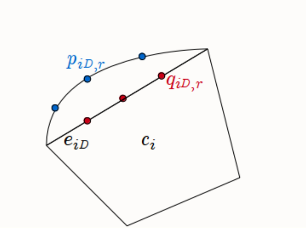
with

$$A_P^{\dagger}(i,i) = \frac{1}{A_P(i,i) - \sum_{j\in\nu(i)} \frac{A_P(i,j)A_P(j,i)}{A_P(j,j)}}.$$

Curved boundary treatment

☞ Problem: we set the boundary condition on the edge while it is prescribed on the curve.

- $q_{iD,r}$ Gauss points on e_{iD},
- $p_{iD,r}$ Gauss points on the curve,
- $\widehat{\phi}_{iD}$ is evaluated using $\phi_D(q_{iD,r})$ and not $\phi_D(p_{iD,r})$.



☞ Idea: modify the mean value $\phi_{iD} = \sum_{r=1}^{R} \zeta_r \phi_D(q_{iD,r})\, ds$ but ϕ_{iD} is still associated to edge e_{iD}.

Curved boundary treatment

Find ϕ_{iD} approximation on edge $e_{iD} \subset \Gamma_D$ such that

$$\phi_D(p_{iD,r}) - \widehat{\phi}_{iD}(p_{iD,r}) \text{ is minimal.}$$

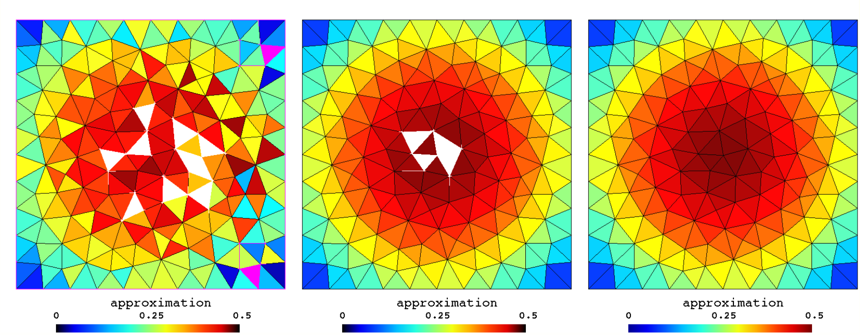
1. Initialize $\phi^0_{iD} = \sum_{r=1}^{R} \zeta_r \phi_D(q_{iD,r})$ and evaluate $\widehat{\phi}^0_{iD}$
2. Do
   - Compute $\delta^k_{iD,r} = \phi_D(p_{iD,r}) - \widehat{\phi}^k_{iD}(p_{iD,r})$ (boundary default),
   - Update ϕ_{iD} with $\phi^{k+1}_{iD} := \phi^k_{iD} + \sum\limits_{r=1}^{R} \zeta_r \delta^k_{iD,r}$,
   - Evaluate $\widehat{\phi}^{k+1}_{iD}$ with new mean value ϕ^{k+1}_{iD},

   While($|\phi^{k+1}_{iD} - \phi^k_{iD}| < Tol$)
3. Compute the flux on boundary with the update $\widehat{\phi}_{iD}$.

Examples and numerical simulations

M-matrix

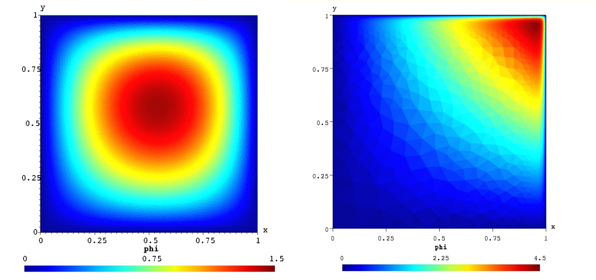
- The underlying matrix A must be an M-matrix (stability and positivity preserving).
- Number of non negative coefficients: $w = 2$ gives 46%, $w = 2.5$ gives 25%, $w = 3.$ gives 0%.

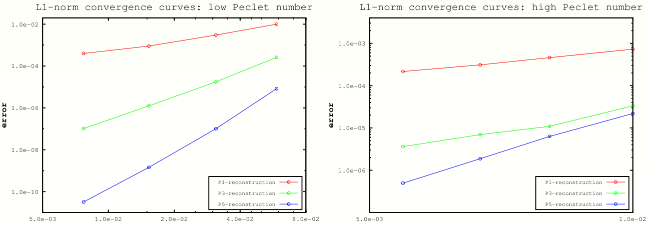


☞ I have to check if $a_{ij} \leqslant 0$ with $i \neq j$.

Convection Diffusion

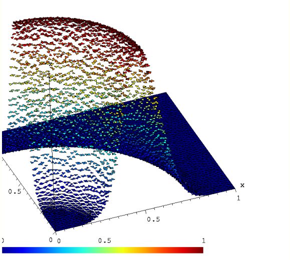
Smooth solution with low and high Peclet number.

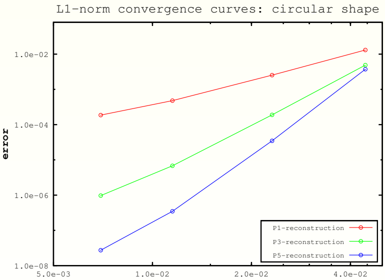




Pure convection

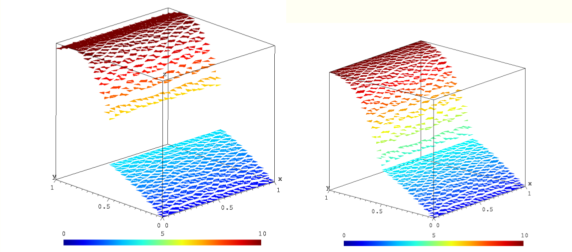
Revolution of a smooth pattern.
Velocity $V = (-y, x)$
Dirichlet condition for the inflow boundary.

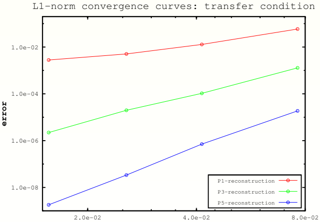


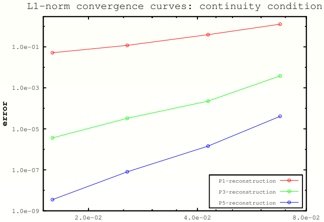


Pure diffusion with discontinuous coefficients

$$k_1 \nabla \phi_1 = h(\phi_2 - \phi_1) \qquad \phi_1 = \phi_2$$

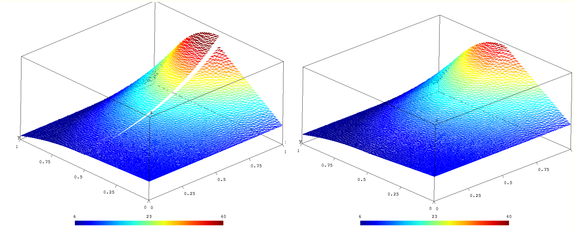


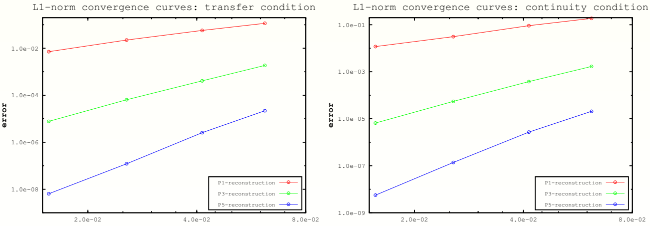




Convection diffusion with discontinous coefficients

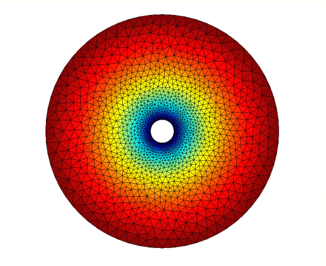
$$k_1 \nabla \phi_1 = h(\phi_2 - \phi_1) \qquad \phi_1 = \phi_2$$

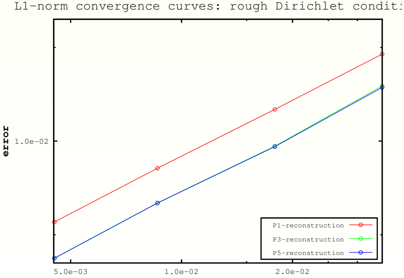


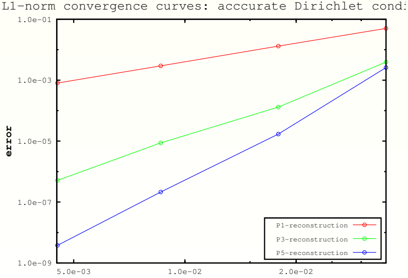


The ring problem

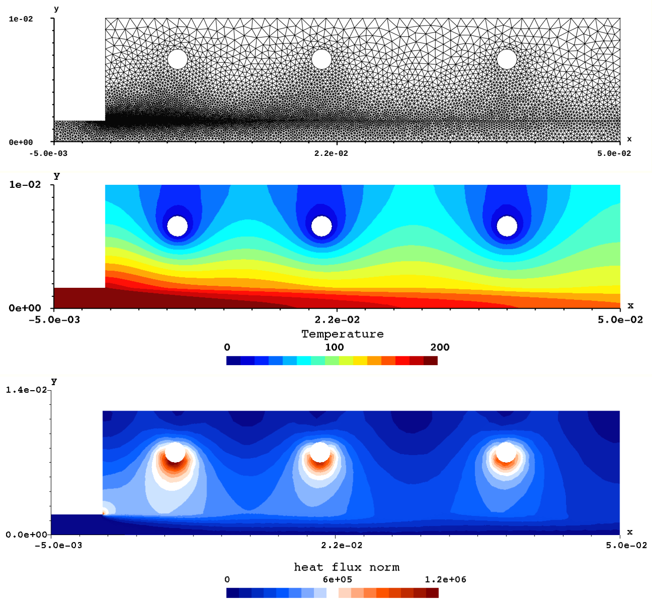
$\Delta\phi = 0,$
$\phi = 25$ at $R = 1000e - 09,$
$\phi = 0$ at $100.e - 09,$
Exact solution $a + b\ln(r)$.







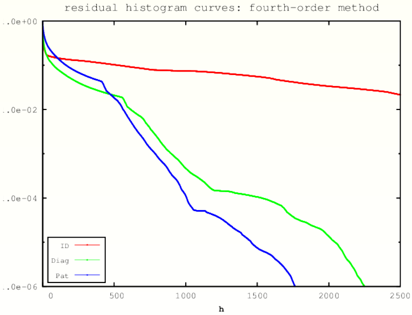
The cooler

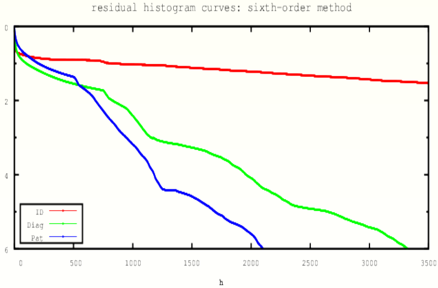

y
1e-02
0e+00
x
-5.0e-03
2.2e-02
5.0e-02
y
1e-02
0e+00
x
-5.0e-03
2.2e-02
5.0e-02
Temperature
0
100
200
y
1.4e-02
0.0e+00
x
-5.0e-03
2.2e-02
5.0e-02
heat flux norm
0
6e+05
1.2e+06

Preconditioning

Residual histogram for the three preconditioning matrices versus the order method.

Simulations of the cooler with a 23616 triangles mesh .





Conclusions

- Very high-order is achieved even for discontinuous coefficients or discontinuous function at the interface.
- Control of the M-matrix via the weights
- no spurious oscillation, high Peclet number.
- Curved boundaries seems resolved.
- 3D is in progress (curved boundary?)
- Non-stationary problem in progress.
- Next steps: non linear case, Stokes, Navier-Stokes (MOOD)