An admissibility and asymptotic-preserving scheme for systems of conservation laws with source terms on 2D unstructured meshes

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Outline

1. General context and examples
2. State-of-the-art
3. Development of a new asymptotic preserving FV scheme
4. Conclusion and perspectives

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2. State-of-the-art
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Problematic

Hyperbolic systems of conservation laws with source terms:

$$\partial_t \mathbf{U} + \mathrm{div}(\mathbf{F}(\mathbf{U})) = \gamma(\mathbf{U})(\mathbf{R}(\mathbf{U}) - \mathbf{U}) \tag{1}$$

- $\mathcal{A}$: set of admissible states,
- $\mathbf{U} \in \mathcal{A} \subset \mathbb{R}^N$,
- $\mathbf{F}$: flux,
- $\gamma > 0$: controls the stiffness,
- $\mathbf{R} : \mathcal{A} \to \mathcal{A}$: smooth function with some compatibility conditions developed by C. Berthon – P.G. Le Floch – R. Turpault [3].

Problematic

Hyperbolic systems of conservation laws with source terms:

$$\partial_t \mathbf{U} + \mathrm{div}(\mathbf{F}(\mathbf{U})) = \gamma(\mathbf{U})(\mathbf{R}(\mathbf{U}) - \mathbf{U}) \qquad (1)$$

Under compatibility conditions on $\mathbf{R}$, when $\gamma t \to \infty$, (1) degenerates into a smaller parabolic system:

$$\partial_t \mathbf{u} - \mathrm{div}\big(\mathcal{M}(\mathbf{u})\nabla \mathbf{u}\big) = 0 \qquad (2)$$

- $\mathbf{u} \in \mathbb{R}^n$, linked to $\mathbf{U}$,
- $\mathcal{M}$: positive and definite matrix.

Examples: Telegraph equations

$$\left\{ \begin{array}{lcl} \partial_t v + a\partial_x v & = & \sigma(w - v) \\ \partial_t w - a\partial_x w & = & \sigma(v - w) \end{array} \right. , a, \sigma > 0$$

Formalism of (1)

- $\mathbf{U} = (v, w)^T$
- $\mathbf{F}(\mathbf{U}) = (av, -aw)^T$
- $\mathbf{R}(\mathbf{U}) = (w, v)^T$
- $\gamma(\mathbf{U}) = \sigma$

Limit diffusion equation: *heat equation* on $(v + w)$

$$\partial_t(v + w) - \partial_x \left(\frac{a^2}{2\sigma} \partial_x(v + w) \right) = 0$$

Examples: isentropic Euler with friction

$$\begin{cases} \partial_t \rho + \partial_x \rho u + \partial_y \rho v &= 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p(\rho)) + \partial_y \rho u v &= -\kappa \rho u \\ \partial_t \rho v + \partial_x \rho u v + \partial_y(\rho v^2 + p(\rho)) &= -\kappa \rho v \end{cases}, \text{ with: } p'(\rho) > 0,\ \kappa > 0$$

$$\mathcal{A} = \{(\rho, \rho u, \rho v)^T \in \mathbb{R}^3 / \rho > 0\}$$

Formalism of (1)

- $\mathbf{U} = (\rho, \rho u, \rho v)^T$
- $\mathbf{R}(\mathbf{U}) = (\rho, 0, 0)^T$
- $\mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho u, & \rho u^2 + p & , \rho u v \\ \rho v, & \rho u v & , \rho v^2 + p \end{pmatrix}^T$
- $\gamma(\mathbf{U}) = \kappa$

Limit diffusion equation

$$\partial_t \rho - \operatorname{div}\left(\frac{1}{\kappa} \nabla p(\rho)\right) = 0$$

Examples: M1 model for radiative transfer

$$\left\{\begin{array}{rcl} \partial_t E + \partial_x F_x + \partial_y F_y & = & c\sigma^e a T^4 - c\sigma^a E \\ \partial_t F_x + c^2\partial_x P_{xx}(E,F) + c^2\partial_y P_{xy}(E,F) & = & -c\sigma^f F_x \\ \partial_t F_y + c^2\partial_x P_{yx}(E,F) + c^2\partial_y P_{yy}(E,F) & = & -c\sigma^f F_y \\ \rho C_v \partial_t T & = & c\sigma^a E - c\sigma^e a T^4 \end{array}\right.$$

$$\sigma = \sigma(E, F_x, F_y, T)$$

$$\mathcal{A} = \{(E, F_x, F_y, T) \in \mathbb{R}^4 / E > 0, T > 0, \sqrt{F_x^2 + F_y^2} < cE\}$$

Formalism of (1):

- $\mathbf{U} = (E, F_x, F_y, T)^T$
- $\mathbf{R}(\mathbf{U})$
- $\mathbf{F}(\mathbf{U}) = \begin{pmatrix} F_x, & c^2 P_{xx} & , c^2 P_{yx} & , 0 \\ F_y, & c^2 P_{xy} & , c^2 P_{yy} & , 0 \end{pmatrix}^T$
- $\gamma(\mathbf{U}) = c\sigma^m(\mathbf{U})$

Limit diffusion equation: *equilibrium diffusion equation*

$$\partial_t(\rho C_v T + aT^4) - \mathrm{div}\left(\frac{c}{3\sigma^r}\nabla(aT^4)\right) = 0$$

Other examples

- Euler coupled with the M1 model $\longrightarrow$ diffusion system

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Shallow water with friction

$$\left\{ \begin{array}{rcl} \partial_t h + \partial_x hv & = & 0 \\ \partial_t hv + \partial_x (hv^2 + \frac{gh^2}{2}) & = & -\kappa(h)^2 ghv|hv| \end{array} \right. , \text{ with: } \kappa(h) = \frac{\kappa_0}{h^\eta}$$

Limit diffusion equation: non linear parabolic equation

$$\partial_t h - \partial_x \left(\frac{\sqrt{h}}{\kappa(h)} \frac{\partial_x h}{\sqrt{|\partial_x h|}} \right) = 0$$

Aim of an AP scheme

Model:

$$\partial_t \mathbf{U} + \mathrm{div}(\mathbf{F}(\mathbf{U})) = \gamma(\mathbf{U})(\mathbf{R}(\mathbf{U}) - \mathbf{U})$$

$\xrightarrow{\gamma t \to \infty}$

Diffusion system:

$$\partial_t \mathbf{u} - \mathrm{div}\big(\mathcal{M}(\mathbf{u})\nabla \mathbf{u}\big) = 0$$

consistent: $\Delta t, \Delta x \to 0$ ↑

Numerical scheme $\xrightarrow[\gamma t \to \infty]{}$ Limit scheme

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consistent: $\Delta t, \Delta x \to 0$

not consistent: $\Delta t, \Delta x \to 0$

Numerical scheme

$\gamma t \to \infty$

Limit scheme

Example of a non AP scheme on Euler with friction in 1D

$$\partial_t \mathbf{U} + \partial_x(\mathbf{F}(\mathbf{U})) = \gamma(\mathbf{U})(\mathbf{R}(\mathbf{U}) - \mathbf{U})$$

$$\begin{array}{rclrcl} \mathbf{U} & = & (\rho, \rho u)^T & \mathbf{F}(\mathbf{U}) & = & (\rho u, \rho u^2 + p)^T \\ \gamma(\mathbf{U}) & = & \kappa & \mathbf{R}(\mathbf{U}) & = & (\rho, 0)^T \end{array}$$

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$$\frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} = -\frac{1}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right) + \gamma(\mathbf{U}_i^n)(\mathbf{R}(\mathbf{U}_i^n) - \mathbf{U}_i^n)$$

- $\mathcal{F}_{i+1/2}$: Rusanov flux

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Limit

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{1}{2\Delta x^2}\left(b_{i+1/2}\Delta x(\rho_{i+1}^n - \rho_i^n) - b_{i-1/2}\Delta x(\rho_i^n - \rho_{i-1}^n)\right)$$

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AP in 1D

1. controls the numerical diffusion:
   - telegraph equations: L. Gosse – G. Toscani [11],
   - M1 model: C. Buet – B. Després [7], C. Buet – S. Cordier [6] , C. Berthon – P. Charrier – B. Dubroca [2], . . .

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   - used to have AP properties
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4. generalization of L. Gosse – G. Toscani:
   - C. Berthon – P. Le Floch – R. Turpault [3]

AP in 2D

- cartesian and admissible meshes $\Longrightarrow$ 1D

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- unstructured meshes:
  1. MPFA based scheme:
     C. Buet – B. Després – E. Frank [8]
  2. using the diamond scheme (Y. Coudière – J.P. Vila – P. Villedieu [9]) for the limit scheme:
     C. Berthon – G. Moebs - C. Sarazin-Desbois – R. Turpault [4]

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- under a 'hyperbolic' CFL:

$$\max_{\substack{K \in \mathcal{M} \\ i \in \mathcal{E}_K}} \left(b_{K,i} \frac{\Delta t}{\Delta x} \right) \leq \frac{1}{2}. \tag{3}$$

Aim of the development

- for any 2D unstructured meshes,
- for any system of conservation laws which could be written as (1),
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  - stability,

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- for any system of conservation laws which could be written as (1),
- under a 'hyperbolic' CFL:
    - stability,
    - preservation of $\mathcal{A}$,
    - asymptotic preserving,

$$\max_{\substack{K\in\mathcal{M}\\ i\in\mathcal{E}_K}} \left(b_{K,i} \frac{\Delta t}{\Delta x} \right) \leq \frac{1}{2}. \qquad (3)$$

Choice of the limit scheme

FV scheme to discretize elliptic equations:

$$\mathrm{div}(\mathcal{M}\nabla \mathbf{u}) = 0$$

$$\begin{cases} \mathbf{q} &= \mathcal{M}\nabla \mathbf{u} \\ \mathrm{div}(\mathbf{q}) &= 0 \end{cases}$$

Choice:

Scheme developed by J. Droniou and C. Le Potier [10]:

- conservative and consistent,
- preserves $\mathcal{A}$,
- second order,
- nonlinear.

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On admissible meshes this scheme is equivalent to the FV4 scheme.

Presentation of the DLP scheme

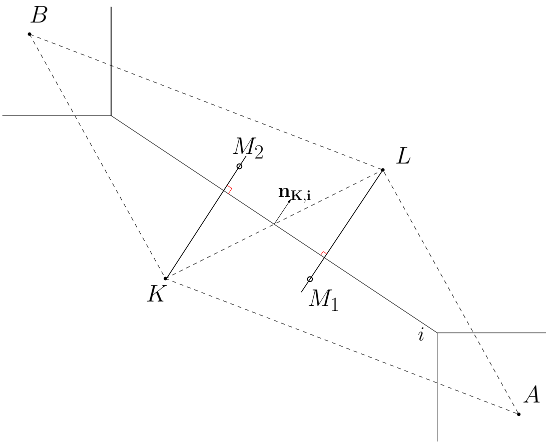
Approximation $\mathbf{F}_{K,i}$ of the flux $\mathbf{q}.\mathbf{n}_{K,i}$ with DLP scheme:

$$\begin{cases} \mathbf{q} & = \mathcal{M}\nabla\mathbf{u} \\ \mathrm{div}(\mathbf{q}) & = 0 \end{cases}$$

$$\mathbf{F}_{K,i}(\mathbf{u}) = \sum_{j \in S_{K,i}} \nu_{K,i,j}(\mathbf{u})(\mathbf{u}_J - \mathbf{u}_K)$$

- $S_{K,i}$ the set of points used for the reconstruction on edges i of cell K
- $\nu_{K,i,j}(\mathbf{u}) > 0$

Example with four points for the DLP scheme



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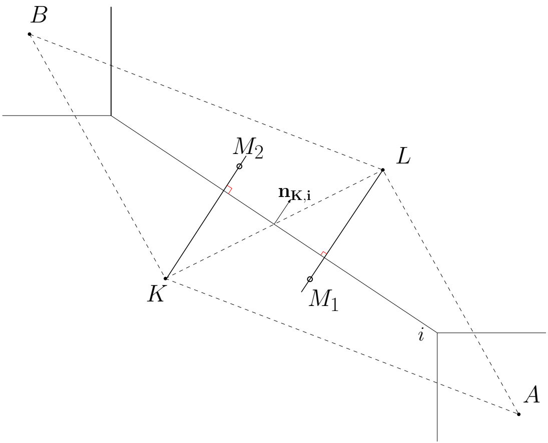


$$M_1 = \sum_{j \in S_{L,i}} a_{i,j} \; X_j = a_{i,1} \; L + a_{i,2} \; K + a_{i,3} \; A + a_{i,4} \; B$$
$$M_2 = \sum_{j \in S_{K,i}} a'_{i,j} \; X_j = a'_{i,1} \; K + a'_{i,2} \; L + a'_{i,3} \; A + a'_{i,4} \; B$$

Scheme for the hyperbolic part

- homogeneous hyperbolic system:

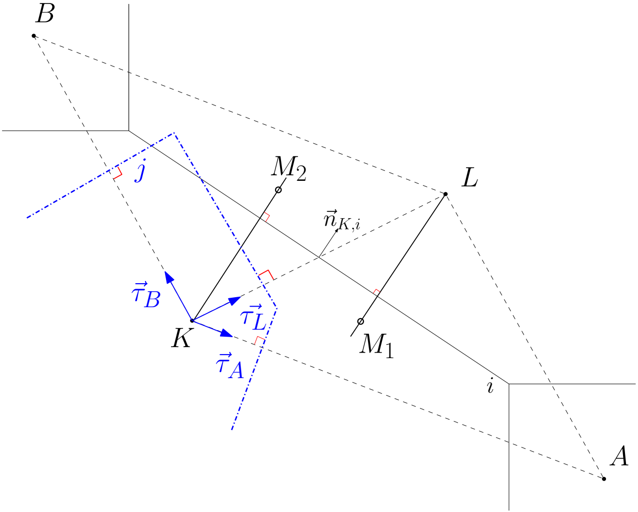
$$\partial_t \mathbf{U} + \operatorname{div}(\mathbf{F}(\mathbf{U})) = 0$$

- Rusanov-like flux:

$$\mathcal{F}_{K,i}^n \cdot \mathbf{n}_i = \bar{\mathbf{F}}_{K,i}^n \cdot \mathbf{n}_i - \frac{b_i \theta_i}{2} \nabla_i \mathbf{U}^n \cdot \mathbf{n}_i, \qquad (4)$$

- $\bar{\mathbf{F}}_{K,i}^n$: approximation of $\mathbf{F}(\mathbf{U})$,
- b_i: characteristic speed on the interface i
- $\theta_i > 0$: characteristic length,
- $\nabla_i \mathbf{U}^n \cdot \mathbf{n}_i$: approximation of the normal gradient.

Scheme for the hyperbolic part



Scheme for the hyperbolic part

$$\mathcal{F}_{K,i}^{n} \cdot \mathbf{n}_i = \bar{\mathbf{F}}_{K,i}^{n} \cdot \mathbf{n}_i - \frac{b_i \theta_i}{2} \nabla_i \mathbf{U}^n \cdot \mathbf{n}_i, \tag{4}$$

Assumptions on $\bar{\mathbf{F}}_{K,i}^{n}$:

1. Consistency:
   if $\forall K \in \mathcal{M},\ \mathbf{U}_K^n = \mathbf{U}$ then $\forall K \in \mathcal{M},\ \forall e_i \in \mathcal{E}_K,\ \bar{\mathbf{F}}_{K,i}^{n} \cdot \mathbf{n}_i = \mathbf{F}(\mathbf{U}) \cdot \mathbf{n}_i,$

Scheme for the hyperbolic part

$$\mathcal{F}^n_{K,i} \cdot \mathbf{n}_i = \bar{\mathbf{F}}^n_{K,i} \cdot \mathbf{n}_i - \frac{b_i \theta_i}{2} \nabla_i \mathbf{U}^n \cdot \mathbf{n}_i, \tag{4}$$

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2. Conservativity: if $\varepsilon_i = K \cap L$ then $\bar{\mathbf{F}}^n_{K,i} \cdot \mathbf{n}_i = -\bar{\mathbf{F}}^n_{L,i} \cdot \mathbf{n}_i$,

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3. Admissibility of $\bar{\mathbf{F}}$: $\forall K \in \mathcal{M},\ \forall e_i \in \mathcal{E}_K,\ \exists\ \overline{\nu}_{i,j}(\mathbf{U}) \geq 0$ such that:

Scheme for the hyperbolic part

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Scheme for the hyperbolic part

$$\mathcal{F}^n_{K,i} \cdot \mathbf{n}_i = \bar{\mathbf{F}}^n_{K,i} \cdot \mathbf{n}_i - \frac{b_i \theta_i}{2} \nabla_i \mathbf{U}^n \cdot \mathbf{n}_i, \quad (4)$$

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   b. For any constant vector V, $\sum_{i \in \mathcal{E}_K} |e_i| \sum_{j \in \mathcal{S}_i} \overline{\nu}_{i,j} V \cdot \boldsymbol{\tau}_j = 0$.

Scheme for the hyperbolic part

$$\mathbf{U}_K^{n+1} = \mathbf{U}_K^n - \frac{\Delta t}{|K|} \sum_{e_i \in \mathcal{E}_K} |e_i| \mathcal{F}_{K,i}^n \cdot \mathbf{n}_i \tag{5}$$

Theorem

Under the previous assumptions on $\bar{\mathbf{F}}_{K,i}^n$, the numerical scheme (5) is stable, consistent, conservative and preserves the set of admissible states $\mathcal{A}$ as soon as the following CFL condition is satisfied :

$$\max_{\substack{K \in \mathcal{M} \\ j \in \mathcal{E}_K}} \left(\mu_j \frac{\Delta t}{\delta_j^K} \right) \leq \frac{1}{2}. \tag{6}$$

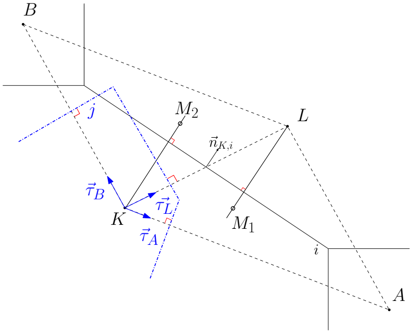
- $\mu_j = \mu_j(b_i, \theta_i)$
- δ_j^K: characteristic length

Scheme for the hyperbolic part

Idea of the proof

Rewrite the scheme (5) as a convex combination of 1D-Rusanov scheme on each interface j of normal τ_j :

$$\mathbf{U}_K^{n+1} = \sum_{j \in \overline{\mathcal{E}_K}} \omega_j \left(\mathbf{U}_K^n - \frac{\Delta t}{\delta_j^K} \left(\frac{F(\mathbf{U}_K^n) + F(\mathbf{U}_j^n)}{2} \cdot \boldsymbol{\tau}_j - \frac{\mu_j}{2} (\mathbf{U}_j - \mathbf{U}_K) \right) \right)$$



Scheme for the complete model

- complete hyperbolic system: $\partial_t \mathbf{U} + \mathrm{div}(\mathbf{F}(\mathbf{U})) = \gamma(\mathbf{U})(\mathbf{R}(\mathbf{U}) - \mathbf{U})$ (1)

$$\mathbf{U}_K^{n+1} = \mathbf{U}_K^n - \sum_{j \in \overline{\mathcal{E}_K}} \omega_j \alpha_j \left(\frac{\Delta t}{\delta_j^K} \left(\frac{F(\mathbf{U}_K^n) + F(\mathbf{U}_j^n)}{2} \cdot \boldsymbol{\tau}_j - \frac{\mu_j}{2} (\mathbf{U}_j - \mathbf{U}_K) \right) \right) + \sum_{j \in \overline{\mathcal{E}_K}} \omega_j (1 - \alpha_j) \left(\frac{\Delta t}{\delta_j^K} S_j(\mathbf{U}) \right) \quad (7)$$

- $\alpha_j = \frac{2\mu_j}{2\mu_j + \gamma_j d_j} \in [0, 1]$,
  - d_j: length of the j^{th} interface on the reconstructed cell,
  - γ_j: discretization of $\gamma(\mathbf{U})$,
- $S_j(\mathbf{U})$: representative of the discretization of the source term

Scheme for the complete model

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  $\Rightarrow$ generally not ...

Scheme for the complete model

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  $\Rightarrow$ generally not ...

Equivalent formulation

Rewrite (1) into :

$$\partial_t \mathbf{U} + \mathrm{div}(\mathbf{F}(\mathbf{U})) = (\gamma(\mathbf{U}) + \bar{\gamma}(\mathbf{U}))(\bar{\mathbf{R}}(\mathbf{U}) - \mathbf{U}) \tag{8}$$

with:

- $\gamma(\mathbf{U}) + \bar{\gamma}(\mathbf{U}) > 0$
- $\bar{\mathbf{R}}(\mathbf{U}) = \frac{\gamma \mathbf{R}(U) + \bar{\gamma}\mathbf{U}}{\gamma + \bar{\gamma}}$

Example

Reformulation of Euler with friction

$$\partial_t \mathbf{U} + \mathrm{div}(\mathbf{F}(\mathbf{U})) = (\gamma(\mathbf{U}) + \bar{\gamma}(\mathbf{U}))(\bar{\mathbf{R}}(\mathbf{U}) - \mathbf{U})$$

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Limit

$$\frac{\rho_K^{n+1} - \rho_K^n}{\Delta t} - \sum_{i \in \mathcal{E}_K} \frac{|e_i|}{|K|} \sum_{j \in S_i} \nu_{ij}(\rho)(\rho_j - \rho_K) \frac{\mu_j b_i \theta_i}{d_j(\kappa + \bar{\kappa}_j)} = 0$$

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$$\text{with: } (\kappa + \bar{\kappa}_j) = \kappa \frac{\rho_j - \rho_K}{p_j - p_K} \frac{\mu_j b_i \theta_i}{d_j}$$

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$$\frac{\rho_K^{n+1} - \rho_K^n}{\Delta t} - \sum_{i \in \mathcal{E}_K} \frac{|e_i|}{|K|} \sum_{j \in S_i} \nu_{ij} \frac{(p_j(\rho) - p_K(\rho))}{\kappa} = 0$$

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$$\Rightarrow \partial_t \rho - \mathrm{div}\left(\frac{1}{\kappa} \nabla p(\rho) \right) = 0$$

Scheme for the complete model

$$\mathbf{U}_K^{n+1} = \mathbf{U}_K^n - \sum_{j \in \overline{\mathcal{E}_K}} \omega_j \alpha_j \left(\frac{\Delta t}{\delta_j^K} \left(\frac{F(\mathbf{U}_K^n) + F(\mathbf{U}_j^n)}{2} \cdot \boldsymbol{\tau}_j - \frac{\mu_j}{2}(\mathbf{U}_j - \mathbf{U}_K) \right) \right)$$
$$+ \sum_{j \in \overline{\mathcal{E}_K}} \omega_j (1 - \alpha_j) \left(\frac{\Delta t}{\delta_j^K} S_j(\mathbf{U}) \right) \tag{7}$$

Theorem

Under the previous assumptions on $\bar{\mathbf{F}}_{K,i}^n$, the numerical scheme (7) is stable, consistent, conservative and preserves the set of admissible states $\mathcal{A}$ as soon as the following CFL condition is satisfied :

$$\max_{\substack{K \in \mathcal{M} \\ j \in \overline{\mathcal{E}_K}}} \left(\mu_j \frac{\Delta t}{\delta_j^K} \right) \leq \frac{1}{2}. \tag{6}$$

Outline

1. General context and examples
2. State-of-the-art
3. Development of a new asymptotic preserving FV scheme
4. Conclusion and perspectives

Conclusion and perspectives

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- generic theory for various hyperbolic problems with asymptotic behaviours,
- first order scheme that preserve $\mathcal{A}$ and the asymptotic

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Perspectives

- complete the numerical part,
- change the limit scheme (DLP), and the expression of numerical flux (Rusanov),
- high-order techniques applied on the 1D convex combination.

Thank you for your attention.

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Admissible mesh

Definition of an admissible mesh

A mesh is said to be admissible as soon as all the interfaces are orthogonal to the lines which joins the cells' centroids.